Math 294 Week 4

2/5/2019 or 2/7/2019

This worksheet will emphasize a proof technique called induction and how it's used in practice. Induction is used to prove that something is true for all natural numbers $n \in \mathbb{N}$. For example, consider the following statement: "The sum of the first n odd natural numbers equals n^2". How would we prove a statement like this? Well, one attempt would be to give examples, but as we all well know by now, proofs by example aren't *actually* proofs.

To motivative how induction works, imagine we had an infinite line of dominoes $D_1, D_2, D_3, \ldots$. Imagine further we knew that if one of the dominoes fell over, then the very next one would fall over. In symbols, if D_n fell over, then D_{n+1} would fall over. Finally, imagine we knew that D_1 did actually fall over. The question for you is: did all the dominoes fall over? Yes! Why? Well, we know that D_1 fell over, and so it must follow that D_2 fell over. But, since D_2 fell over, then it must be that D_3 fell over, and so on.

How induction works. Let $\Phi(n)$ be any mathematical statement where n is a variable for a natural number, and let's suppose we want to show that $\Phi(n)$ holds for any natural number n. In order to prove this, we must show two different things:

1. **Base Case**: We must show that $\Phi(1)$ is a true statement.
2. **Induction Step**: We assume that $\Phi(n)$ holds for an arbitrary natural number n, and prove that $\Phi(n+1)$ holds. In symbols, we must show that $\Phi(n) \Rightarrow \Phi(n+1)$ for any natural number n.

If we can show both 1 and 2, then mathematical induction let's us conclude that $\Phi(n)$ is true for any natural number n.

What's cool about induction is that it let's us prove infinitely many individual statements $(\Phi(1), \Phi(2), \Phi(3), \ldots)$ by just proving two different things.

Prove the following statements with either induction, or proof by smallest counterexample

Problem 1. Prove that $1+2+3+4+\cdots+n = \frac{n^2+n}{2}$ for every positive integer n.

Problem 2. Prove that $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.

Problem 3. Prove that $6|(n^3-n)$ for every integer $n \geq 0$.

Problem 4. Prove that $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for every $n \in \mathbb{N}$.

Problem 5. Prove that $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{N}$.

This problem will not be accepted as a submission for the end of class, but something fun to think about:

Challenge Problem: 100 prisoners were sentenced to death and were going to be executed as follows: on the day of the execution they will be lined up, so that everybody can see everybody in front of them (but not themselves). Each of the prisoners will have a red or blue hat put on him, but he won't be told which color it is (although he can see the colors of prisoners' hats in front). On command, each prisoner (one-by-one) makes a guess as to what color he thinks his hat is. Whoever guesses right, goes home free. The good news is that the prisoners thought of a plan the day before the execution, so that at most one prisoner dies. How did they do it?