Math 294 Week 7

2/26/2019 or 2/28/2019

This worksheet will emphasize equivalence relations and partitions. These seemingly unrelated ideas are actually intimately connected. Recall the definition of a relation on a set A:

Definition. A relation R on a set A is *any* subset of ordered pairs of elements of A. In symbols, $R \subseteq A \times A$. If $(a, b) \in R$, we write aRb.

Definition. An equivalence relation on A is some relation $\equiv$ that satisfies three additional properties:

1. **Reflexive**: For all $a \in A$, $a \equiv a$.
2. **Symmetric**: For all $a, b \in A$, $a \equiv b$ implies $b \equiv a$.
3. **Transitive**: For all $a, b, c \in A$, $a \equiv b$ and $b \equiv c$ implies $a \equiv c$. (This is normally expressed by saying $a \equiv b \equiv c$ implies $a \equiv c$.

Definition. A partition of a set A is a collection $\mathcal{P}$ of subsets of A such that the following hold:

1. For all $X, Y \in \mathcal{P}$, $X \cap Y = \varnothing$.
2. For all $X \in \mathcal{P}$, X is nonempty.
3. For all $a \in A$, there's some $X \in \mathcal{P}$ such that $a \in X$.

If $a \in A$, then the equivalence class of a modulo $\equiv$ is the set $[a]_{\equiv} = \{b \in A : a \equiv b\}$. Normally, we denote the set of all such equivalence classes as $A/\equiv$, called the **quotient** of A by $\equiv$.

Theorem. The Fundamental Theorem of Partitions/Equivalence Relations: Every partition is induced by an equivalence relation, and every equivalence relation is induces a partition.

Problem 1. Suppose that $f\colon A \to B$ is a function, $\{A_i\}_{i\in I}$ is a partition of A, and $\{B_j\}_{j\in J}$ is a partition of B. Prove the following statements:

1. If f is injective, then $\{f[A_i]\}_{i\in I}$ is a partition of B.
2. If f is surjective, then $\{f^{-1}[B_j]\}_{j\in J}$ is a partition of A.
3. What can you conclude if f is a bijection?

Problem 2. Suppose that $\{A_i\}_{i\in I}$ is a partition of A and $\{B_j\}_{j\in J}$ is a partition of B. Prove that the collection $\{A_i \times B_j\}_{(i,j)\in I\times J}$ is a partition of $A \times B$.

Definition. Given a function $f\colon A \to B$, we may define an equivalence relation $\equiv_f$ on A by saying $x_0 \equiv_f x_1$ iff $f(x_0) = f(x_1)$. This is called the equivalence relation induced by f. Convince yourself that this is an equivalence relation.

Problem 3. Assume $f\colon A \to A$ is a function and $\equiv_f$ is the equivalence relation induced by f. Prove that $f \circ f = f$ iff, for every $z, x \in A$, $z \in [x]_f \Rightarrow f(z) \in [x]_f$.

Problem 4. Let G and H be equivalence relations on A, and assume that $G\cup H$ is an equivalence relation on A. Prove that each equivalence class modulo $G\cup H$ is the union of an equivalence class modulo G with an equivalence class modulo H. More exactly, show that $[x]_{G\cup H} = [x]_G \cup [x]_H$ for each $x \in A$.

Problem 5. Recall the combination formula ${}_nC_r$ which is the number of ways to choose r-many elements from a set of size n, where the order doesn't matter. Prove, using equivalence relations, that ${}_nC_r = \frac{n!}{r!(n-r)!}$.