Math 294 Week 7b

2/26/2019 or 2/28/2019

This worksheet will emphasize equivalence relations and partitions. These seemingly unrelated ideas are actually intimately connected. Recall the definition of a relation on a set A:

Definition. A relation R on a set A is *any* subset of ordered pairs of elements of A. In symbols, $R \subseteq A \times A$. If $(a, b) \in R$, we write aRb.

Definition. An equivalence relation on A is some relation $\equiv$ that satisfies three additional properties:

1. **Reflexive**: For all $a \in A$, $a \equiv a$.
2. **Symmetric**: For all $a, b \in A$, $a \equiv b$ implies $b \equiv a$.
3. **Transitive**: For all $a, b, c \in A$, $a \equiv b$ and $b \equiv c$ implies $a \equiv c$. (This is normally expressed by saying $a \equiv b \equiv c$ implies $a \equiv c$.

Definition. A partition of a set A is a collection $\mathcal{P}$ of subsets of A such that the following hold:

1. For all $X, Y \in \mathcal{P}$, $X \cap Y = \varnothing$.
2. For all $X \in \mathcal{P}$, X is nonempty.
3. For all $a \in A$, there's some $X \in \mathcal{P}$ such that $a \in X$.

If $a \in A$, then the equivalence class of a modulo $\equiv$ is the set $[a]_\equiv = \{b \in A \colon a \equiv b\}$. Normally, we denote the set of all such equivalence classes as $A/\equiv$, called the **quotient** of A by $\equiv$.

Theorem. The Fundamental Theorem of Partitions/Equivalence Relations: Every partition is induced by an equivalence relation, and every equivalence relation is induces a partition.

Problem 1. Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 + y^2$ is even. Prove R is an equivalence relation. Describe its equivalence classes. Describe the partition *induced* by R.

Problem 2. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x - 5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes. Describe the partition *induced* by R.

Problem 3. Prove or disprove: If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.

Problem 4. Suppose P is a partition of a set A. Define a relation R on A by declaring xRy if and only if $x, y \in X$ for some $X \in P$. Prove R is an equivalence relation on A. Then prove that P is the set of equivalence classes of R.

Problem 5. Recall the combination formula ${}_nC_r$ which is the number of ways to choose r-many elements from a set of size n, where the order doesn't matter. Prove, using equivalence relations, that ${}_nC_r = \frac{n!}{r!(n-r)!}$.

Problem 6. Suppose R is a reflexive and symmetric relation on a finite set A. Define a relation S on A by declaring xSy if and only if for some $n \in \mathbb{N}$ there are elements $x_1, x_2, \ldots, x_n \in A$ satisfying $xRx_1, x_1Rx_2, x_2Rx_3, x_3Rx_4, \ldots, x_{n-1}Rx_n$, and x_nRy. Show that S is an equivalence relation and $R \subseteq S$. Prove that S is the unique smallest equivalence relation on A containing R. (S is called the transitive closure of R.)