Math 294 Week 8

3/5/2019 or 3/7/2019

This worksheet will emphasize partial orderings!

Definition. A set $\mathbb{P}$ with a relation $\leq$ is called a *partial ordering (poset)* if the following hold:

1. **Reflexive**: For all $p \in \mathbb{P}$, $p \leq p$.
2. **Antisymmetric**: For all $p, q \in \mathbb{P}$, if $p \leq q$ and $q \leq p$, then $p = q$.
3. **Transitive**: For all $p, q, r \in \mathbb{P}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.

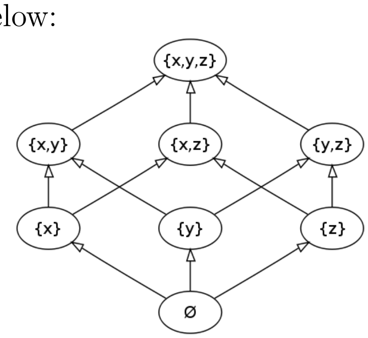
Definition. A poset $(\mathbb{P}, \leq)$ is called a *linear* order if it satisfies this additional condition:

1. **Comparability**: For each $p, q \in \mathbb{P}$, either $p \leq q$ or $q \leq p$.

We say that $\leq$ is an *ordering* and that $\leq$ *orders* $\mathbb{P}$.

Definition. Given a poset $(\mathbb{P}, \leq)$, $p \in \mathbb{P}$ is called a *minimal* element if there's no element q such that $q < p$. It is called *maximal* if there's no element q such that $p < q$.

Frequently, a poset with only finitely many elements can be represented by a Hasse Diagram, like below:



Problem 1. Let $\mathbb{N}^*$ be the set of natural numbers greater than 1 and let's define a relation $\preceq$ on $\mathbb{N}^*$ by $n \preceq m$ iff n divides m. Prove that $(\mathbb{N}^*, \preceq)$ is a partial ordering. What are its minimal elements? Draw the Hasse Diagram for the divisors of 120 ordered with $\preceq$.

Problem 2. Consider the posets $(\mathcal{P}(\{a, b, c\}), \subseteq)$ and $(\{0, 1\}^3, \leq)$ with the ordering $(a, b, c) \leq (x, y, z)$ iff $a \leq x$ and $b \leq y$ and $c \leq z$. Draw the Hasse Diagrams for both of this posets. What are the minimal/maximal elements? How are they similar?

Problem 3. There are two frequently used orderings that we can define on $\mathbb{N}^2$. They are defined as follows:

Dictionary ordering: $(a, b) \leq_1 (c, d) \Leftrightarrow a \leq c$ or $(a = c$ and $b \leq d)$

Product ordering: $(a, b) \leq_2 (c, d) \Leftrightarrow a \leq c$ and $b \leq d$

First, how can $\mathbb{N}^2$ be thought of as a subset of the Cartesian plane? Next, prove that both of these are partial orderings. Figure out which is a linear order and prove your answer. How can you represent these orderings with pictures?

Problem 4. Let Σ be the set of binary sequences of length ≤ 4 including the empty string. (Recall, a binary sequence is a sequence of 0's and 1's.) Order Σ by $u \leq v$ iff u is an initial segment of v. (For example, $010 \leq 0101$ but $010 \not\leq 0001$). Draw the Hasse Diagram for Σ, and prove that for any $u \in \Sigma$, the set $D = \{v \in \Sigma \colon v \leq u\}$ is linearly ordered.

Problem 5. Consider the set $\mathbb{R} \times \mathbb{Z}$ ordered with the ordering $\leq_1$ from Problem 3. How can you think of this as a subset of the cartesian plane? Then, prove that, for every element $(x, n) \in \mathbb{R} \times \mathbb{Z}$, $(x, n + 1)$ is the least element bigger than (x, n).

Problem 6. Consider the set $\mathbb{Z} \times \mathbb{R}$ ordered with the ordering $\leq_1$ from Problem 3. How can you think of this as a subset of the cartesian plane? Then, prove that between any two elements $(n_1, x_1) <_1 (n_2, x_2)$, there's an element (n, x) such that $(n_1, x_1) <_1 (n, x) <_1 (n_2, x_2)$