Math 294 Week 9

3/5/2019 or 3/7/2019

This worksheet will focus on modular arithmetic. If we have to numbers n and m, we only care about the remainder when one of them is divided by another. The definition of congruence (modulo n) is what happens when you give a name to "having the same remainder" when divided by n.

Definition. For $a, b, n \in \mathbb{Z}$, we say $a \equiv b(\text{mod } n)$, (read: a is congruent to b mod n), iff $n|(a-b)$. This is equivalent to saying that $a = nk + b$ for some $k \in \mathbb{Z}$.

Claim 0.1. $\equiv (\text{mod } n)$ is an equivalence relation on $\mathbb{Z}$.

Claim 0.2. If $a \equiv b(\text{mod } n)$ and $c \equiv d(\text{mod } n)$, then $a + c \equiv b + d(\text{mod } n)$ and $ac \equiv bd(\text{mod } n)$

Theorem. n is divisible by 3 iff the sum of its digits is divisible by 3.

Problem 1. Prove that n is divisible by 4 if and only if the last two digits of n is divisible by 4. (E.g. 140 is divisible by 4 because 40 is.)

Problem 2. Prove that n is divisible by 9 if and only if the sum of the digits of n is divisible by 9.

Problem 3. Prove that n is divisible by 11 if and only if the alternating sum of the digits of n is divisible by 11. (E.g. 84535 is divisible by 11 because 8-4+5-3+5 is divisible by 11.)

Problem 4. Show that if there is an $x \in \mathbb{Z}$ such that $ax \equiv 1(\text{mod } m)$ then a and m are relatively prime (i.e. their greatest common divisor is 1.

Problem 5. Find the remainder when 54^{124} is divided by 17.

Fun Problem (for submission if you'd like):

Problem 6. There's a jolly band of pirates that just recently "acquired" a treasure chest with some gold coins. However, when the pirates divided the coins into equal piles, 3 coins were left over. A brawl ensued to determine who should get the extra coins, and one of the pirates was slain as a result. When the remaining pirates divided the coins into equal piles, 10 coins were left over. Again, the pirates fought over who should get the extra coins, and another pirate was slain. When they divided the coins in equal piles a third time, no coins were left over. What's the least number of gold coins a band of 17 pirates could have stolen?