Elementary Embeddings and Relativization

Remark. These notes are written by Thomas Dean, originally to provide some comments on the last homework in the Math 502 course taught by Dima Sinapova at UIC in Fall 2017. These hopefully provide a further reference for those interested in becoming more familiar with the types of arguments involving large cardinals and elementary embeddings. Feel free to contact me if you find any errors!

Let $j\colon V \to M$ be your favorite elementary embedding witnessing that κ is measurable. We first collect some facts about how j interacts with sets. Then we prove some of the recent HW problems about measurable cardinals. For the sake of making this document more legible, I may sometimes write jx for $j(x)$.

Suppose for each set x we're able to define another set $F(x)$. In other words, suppose we have a formula $\phi(v, w)$ such that, for all x, there's a unique y such that $\phi(x, y)$ holds. Further, $\phi(x, y)$ holds iff $F(x) = y$.

From this, we may conclude that $V \models (\forall x)(\exists! y)\phi(x, y)$, and so by elementarity we have $M \models (\forall x)(\exists! y)\phi(x, y)$. In a similar way, denote the unique $y \in M$ satisfying $M \models \phi(x, y)$ by $F^M(x)$.

Now, it's not true in general that $j(F(x)) = F(jx)$. Instead, we have the following:

Fact 1. $j(F(x)) = F^M(jx)$

Why? We know $V \models \phi(x, F(x))$. Then, $M \models \phi(jx, j(Fx))$ by elementarity. By definition, we have $j(F(x)) = F^M(jx)$. ∎

The above fact generalizes in reasonable looking ways if our definition is for n-tuples of sets or if we have parameters. And fortunately, if the property $\phi(v, w)$ is absolute for transitive models, we do get $j(F(x)) = F(jx)$.

Indeed, we'd have that $M \models \phi(jx, j(Fx))$ iff $\phi(jx, j(Fx))$ holds, implying $j(F(x)) = F(jx)$. For example, this holds if ϕ is any Δ_0 formula. In these cases, this makes it look like j commutes with the symbols used to represent the set defined by ϕ.

In practice, recognizing when a property is Δ_0 amounts to seeing if we can figure out if the property holds by "looking inside of" the sets we're given. For example, the formula $x = (y, z)$ is Δ_0 because $(y, z) = \{\{y\}, \{y, z\}\}$ by definition, and so seeing if x is equal to this simply requires "looking inside of" x and checking if x has the form $\{\{y\}, \{y, z\}\}$. So, for this example we'd have that $j((y, z)) = (jy, jz)$.

Example 1. Being a successor ordinal is Δ_0. Let $\phi(\beta, \alpha)$ be the formula that says α and β are ordinals, $\beta < \alpha$ and $(\forall \gamma < \alpha)\gamma \leqslant \beta$. So, $F(\beta) = \alpha = \beta + 1$. Although this technically isn't defined for all sets, we may define $F(x) = 0$ if x isn't an ordinal. Either way, it follows by the discussion above that $j(\beta+1) = j(F(\beta)) = F(j\beta) = j(\beta) + 1$.

Example 2. Being the successor cardinal of a cardinal λ is **not** absolute for transitive models. However, we do always get that $(\lambda^+)^M \leqslant \lambda^+$. So, in general, all we can say is that $j(\lambda^+) = (j(\lambda)^+)^M$. In other words, $j(\lambda^+)$ is what M believes is the successor of $j(\lambda)$. In the case where $j(\lambda) = \lambda$, it seems we do get equality, as $j(\lambda^+) = (\lambda^+)^M \leqslant \lambda^+ \leqslant j(\lambda^+)$. It's left as an exercise that, for any ordinal γ, $j(\gamma) \geqslant \gamma$.

Example 3. For a cardinal λ, 2^λ is **not** absolute for transitive models. Recall, $2^\lambda = |\mathcal{P}(\lambda)|$. It follows then that $(2^\lambda)^M = |\mathcal{P}^M(\lambda)|^M$. So, $j(2^\lambda) = (2^{j(\lambda)})^M$. Even if V and M agree on $\mathcal{P}(\lambda)$, they could compute cardinalities differently. In general, for $X \in M$, we have that $|X| \leqslant |X|^M$. Therefore, in this particular case we'd have that $2^\lambda \leqslant (2^\lambda)^M$. Notice that this inequality is going in the opposite direction of the one in Example 2.

Now, normally one tends not to mention when properties are Δ_0 (and hence absolute), but for the following solutions I'll make more of an effort to do that so we can practice recognizing when they occur.

Problem 1. Show that every measurable cardinal is inaccessible.

Solution. To show that κ is regular, assume instead that there's a sequence of ordinals $\{\kappa_\alpha\}_{\alpha<\lambda}$ for some $\lambda < \kappa$ such that $\bigcup_{\alpha<\lambda} \kappa_\alpha = \kappa$. To be more precise, what we have is a function (a sequence) f with $\mathrm{dom}(f) = \lambda$ such that $\mathrm{ran}(f) \subseteq \kappa$ and $\bigcup \mathrm{ran}(f) = \kappa$. Since everything in the above sentence is absolute for transitive models, we get that $j(f)$ is a sequence of length $j(\lambda) = \lambda$, and that $j(\kappa) = j(\bigcup \mathrm{ran}(f)) = \bigcup \mathrm{ran}(j(f))$. But, for all $\alpha < \lambda$, $f(\alpha) = j(f(\alpha)) = j(f)(j(\alpha)) = j(f)(\alpha)$. This follows because being the image of an element under f is absolute for transitive models. So, $j(f) = f$. But then $j(\kappa) = \bigcup \mathrm{ran}(f) = \kappa$, a contradiction.

To show κ is strong limit, assume instead that $\lambda < \kappa$ and $2^\lambda \geqslant \kappa$. As described in Example 3, 2^λ may certainly change if we switch between models. As such it's better to work with functions that witness the statement $2^\lambda \geqslant \kappa$ as these functions will be absolute. In other words, fix a surjection $f\colon \mathcal{P}(\lambda) \to \kappa$. By elementarity and since being an onto function is absolute, $j(f)\colon j(\mathcal{P}(\lambda)) \to j(\kappa)$ is onto.

Our first goal is to show that $j(\mathcal{P}(\lambda)) = \mathcal{P}(\lambda)$. We do this by showing $\mathcal{P}(\lambda) \in V_\kappa$. To see this, first notice that κ is a limit ordinal, as it's a cardinal. Now, $\lambda \subseteq V_\lambda$. Given any $X \subseteq \lambda$, $X \subseteq V_\lambda$, implying $X \in V_{\lambda+1}$ by definition. So, $\mathcal{P}(\lambda) \subseteq V_{\lambda+1}$, implying $\mathcal{P}(\lambda) \in V_{\lambda+2}$. Since $\lambda + 2 < \kappa$, $\mathcal{P}(\lambda) \in V_\kappa$, and we win.

So, $j(f)\colon \mathcal{P}(\lambda) \to j(\kappa)$ is onto. Next, given any $X \in \mathcal{P}(\lambda)$, $j(X) = X$ by the previous paragraph. Also, we have that $f(X) < \kappa$. So, $f(X) = j(f(X)) = j(f)(j(X)) = j(f)(X)$. This implies that $j(f) = f$. But, this is a contradiction because f can't be both onto κ and $j(\kappa)$, as $\kappa < j(\kappa)$. So, κ is indeed strong limit, and thus is inaccessible as desired. ■

Before we prove the next HW problem, let's collect some more facts about κ.

Fact 2. $(\kappa^+)^M = \kappa^+$

Solution. We know from Example 2 that $(\kappa^+)^M \leqslant \kappa^+$. If $(\kappa^+)^M < \kappa^+$, then by definition of κ^+, there's an onto function $f\colon \kappa \to (\kappa^+)^M$. Since M contains its κ sequences, it follows that $f \in M$. By absoluteness of everything that we care about, $M \models$ "$f\colon \kappa \to (\kappa^+)^M$ is onto." But then this contradicts that $(\kappa^+)^M$ is what M believes is κ^+. ■

Problem 2. Suppose that κ is measurable and GCH holds below κ. Show that $2^\kappa = \kappa^+$

Solution. Observe that it's enough to show that $2^\kappa \leqslant \kappa^+$. Towards this end, notice that the hypothesis implies $V \models (\forall \alpha < \kappa)(2^\alpha = \alpha^+)$. By elementarity, $M \models (\forall \alpha < j(\kappa))(2^\alpha = \alpha^+)$. Since $\kappa < j(\kappa)$, we have that $M \models (2^\kappa = \kappa^+)$. Equivalently, $(2^\kappa)^M = (\kappa^+)^M$. So, then Example 3 and Fact 2 imply that $2^\kappa \leqslant (2^\kappa)^M = (\kappa^+)^M = \kappa^+$, as desired. ■

Warning! Just because M is closed under κ sequences, we can't conclude that $j(2^\kappa) = 2^\kappa$. In fact, this is always false. To see this, $j(\kappa) < j(2^\kappa)$ by elementarity and absoluteness of $<$ for ordinals. Further, $(2^\kappa)^M < j(\kappa)$ because $j(\kappa)$ is inaccessible in M. So, $2^\kappa \leqslant (2^\kappa)^M < j(\kappa) < j(2^\kappa)$. ■