

1. Let  $X$  and  $Y$  be spaces and  $p : X \times Y \rightarrow X$  and  $q : X \times Y \rightarrow Y$  be the projections; also, let  $x \in X$  and  $y \in Y$  be basepoints. Show that the natural homomorphism  $(p_{\sharp}, q_{\sharp}) : \pi_1(X \times Y, (x, y)) \rightarrow \pi_1(X, x) \times \pi_1(Y, y)$  is an isomorphism of groups. (8 pts)

2. Suppose  $M$  is a topological monoid (which we write multiplicatively) with neutral element  $b$ , that is,  $M$  is a space with an associative binary operation  $(m, n) \mapsto mn$  that is continuous and for which  $b \in M$  is a neutral element. For example,  $M$  could be a topological group. (12 pts)

a) There is a binary operation  $([\gamma], [\delta]) \mapsto [\gamma \cdot \delta]$  on  $\pi_1(M, b)$  defined by pointwise multiplication  $\gamma \cdot \delta(t) = \gamma(t)\delta(t)$ . Show this operation is well-defined.

b) Prove that  $[\gamma] \cdot [\delta] = [\gamma * \delta]$  for all  $[\gamma]$  and  $[\delta]$  in  $\pi_1(M, b)$ . (Hint: use naturality of  $*$ .)

c) Prove that  $\pi_1(M, b)$  is abelian.

3. Let  $F, G : \mathbf{C} \rightarrow \mathbf{D}$  be functors between some categories. A *natural transformation*  $\phi : F \rightarrow G$  is a collection of morphisms  $\phi_c : F(c) \rightarrow G(c)$  in  $\mathbf{D}$ , one for each object  $c \in \text{ob}\mathbf{C}$  such that for any  $f \in \text{Mor}_{\mathbf{C}}(c, c')$  we have  $\phi_{c'}F(f) = G(f)\phi_c$ . Such a natural transformation is called a *natural isomorphism* if each  $\phi_c$  is an isomorphism in  $\mathbf{D}$ . (20 pts)

a) Let  $f : X \rightarrow Y$  be a map of spaces. Show that  $x \mapsto f(x)$  and  $[\gamma] \mapsto [f \circ \gamma]$  defines a functor  $f_{\sharp} : \Pi X \rightarrow \Pi Y$  on fundamental groupoids. Verify that  $(fh)_{\sharp} = f_{\sharp}h_{\sharp}$  for composable maps  $f$  and  $h$ .

b) Let  $g : X \rightarrow Y$  be another map, and suppose  $H : X \times I \rightarrow Y$  is a homotopy from  $f$  to  $g$ . Show that  $H$  induces a natural transformation  $\phi : f_{\sharp} \rightarrow g_{\sharp}$  where  $\phi_x$  is the homotopy class of the path  $t \mapsto H(x, t)$  from  $f(x)$  to  $g(x)$ .

c) Show that the natural transformation  $\phi$  defined in b) is a natural isomorphism. (Hint: find an inverse natural transformation.)

d) Using c), conclude that if  $f : X \rightarrow Y$  is a homotopy equivalence, then  $f_{\sharp} : \pi_1(X, x) \rightarrow \pi_1(Y, y)$  is an isomorphism for all  $x \in X$ .

4. Let  $\mathbf{C}$  be groupoid. We say  $\mathbf{C}$  is connected if for all objects  $c, d \in \text{ob}\mathbf{C}$ , there is a morphism  $c \rightarrow d$ . (10 pts)

a) Show that in a connected groupoid,  $\text{Aut}(c) \cong \text{Aut}(d)$  for any pair of objects  $c$  and  $d$ . Here  $\text{Aut}(x)$  is the group of automorphisms in of  $x$  in  $\mathbf{C}$ , which in a groupoid is the same as  $\text{Mor}_{\mathbf{C}}(x, x)$ .

b) Let  $X$  be a path-connected space. Show that  $\Pi X$  is a connected groupoid. Conclude that the isomorphism type of the fundamental group of  $X$  does not depend on the chosen base point.