Math 547 **Homework 4** **Due October 15, 2007**

General Remarks: This homework is on homological algebra. Throughout, we fix a commutative ring R and work with R-modules. A *chain complex* C_* of such modules is a sequence $\{C_n | n \in \mathbb{Z}\}$ of R-modules together with homomorphisms (called the differentials of the chain complex) $\partial = \partial_n = \partial_n^C : C_n \to C_{n-1}$ such that $\partial^2 = 0$. If the modules are only specified for some bounded subset of integers (say, only for $n \geq 0$, or for n in some interval), we view this as a chain complex by setting all the other modules equal to zero. As we will discuss in class, chain complexes of R-modules form a category, where a homomorphism of chain complexes $f : C_* \to D_*$ is a sequence $\{f_n : C_n \to D_n\}$ of homomorphisms of R-modules such that $\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C$ (from now, we'll write $\partial f = f \partial$ for this relation). Given a complex C_*, its homology is the graded module $H_*(C)$ where $H_n(C) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$. Finally, let 0 denote the chain complex all of whose modules and differentials are 0.

1. Suppose $f : C_* \to D_*$ is a homomorphism of complexes. Show that f induces a homomorphism $f_* = H_*(f) : H_*(C) \to H_*(D)$ of graded modules. Show that the assignment $C_* \mapsto H_*(C)$ and $f \mapsto f_*$ is a functor from the category of chain complexes of R-modules to the category of graded R-modules. (10 pts)

2. Suppose $f : C_* \to D_*$ is a homomorphism of chain complexes. Let $\ker(f)$ ($\mathrm{im}(f)$, $\mathrm{coker}(f)$, resp.) be the sequence of modules $\{\ker(f_n)|n\}$ ($\{\mathrm{im}(f_n)|n\}$, $\{\mathrm{coker}(f_n)|n\}$, resp.). Show that the differentials of C_* (resp., D_*) make these into chain complexes such that the natural inclusion $\ker(f) \to C_*$ (the inclusion $\mathrm{im}(f) \to D_*$, the quotient map $D_* \to \mathrm{coker}(f)$, resp.) are homomorphisms of complexes. (10 pts)

3. Let $f : X \to Y$ be a map of spaces and $f_n : C_n(X, R) \to C_n(Y, R)$ be the homomorphism such that $f_n(\sigma) = f \circ \sigma$ for a singular n-simplex $\sigma : \Delta^n \to X$. Show that the sequence $\{f_n\}$ forms a homomorphism $f_* : C_*(X, R) \to C_*(Y, R)$ of complexes. Conclude that f induces a homomorphism $f_* : H_*(X, R) \to H_*(Y, R)$ on homology and that the assignment $X \mapsto H_*(X, R)$ and $f \mapsto f_*$ is a functor from the category of spaces to the category of graded R-modules. (15 pts)

4. A sequence of chain complexes and homomorphisms $0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$ is called a short exact sequence if for each n, the sequence of modules $0 \to C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \to 0$ is a short exact sequence of modules. Given such a short exact sequence of chain complexes, show that there exists a sequence of natural homomorphisms $\delta_n : H_n(E) \to H_{n-1}(C)$ (which we'll simply write δ for) such that the sequence

$$\cdots \xrightarrow{\delta} H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\delta} H_{n-1}(C) \xrightarrow{f_*} \cdots$$

is a long exact sequence (that is, at each place, the kernel of the outgoing homomorphism is equal to the image of the incoming one). Hint: the proof is similar to the proof of the snake lemma. (20 pts)

5. Suppose C_* is a bounded complex of finite dimensional k-vector spaces (that is, $R = k$, and $C_n = 0$ for all sufficiently small or large n). Let $\Xi_C = \sum_n (-1)^n \dim_k(C_n)$ and $\chi_C = \sum_n (-1)^n \dim_k(H_n(C))$. (This number is called the Euler characteristic of the complex). Show that $\Xi_C = \chi_C$.