Math 547 **Homework 5** **Due October 26, 2007**

1. Suppose

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

is a commutative diagram of abelian groups (or modules over a ring). Suppose the top and bottom row are exact. (20pts)

a) Show that if the second and fourth vertical homomorphism are injective, and the first vertical homomorphism is surjective, then the middle vertical homomorphism is injective.

b) Dually, show that if the second and fourth vertical homomorphism are surjective, and the fifth is injective, then the middle vertical homomorphism is surjective.

c) From a) and b), conclude that if the second and fourth vertical homomorphisms are isomorphisms, the first is surjective and the fifth is injective, then the middle vertical homomorphism is an isomorphism. (This statement is known as the five lemma).

2. Let X be a space. The *cone* on X is the space $CX = (X \times I)/X \times \{1\}$. If $f : Y \to X$ is a map, then the cone of f is the space $C(f) = X \cup_Y CY$, that is, the union of X and CY with y and $f(y)$ identified. (20 pts)

a) Show that for any space X, CX is contractible.

b) Let $i : A \to X$ be the inclusion of a subspace. Using excision and homotopy invariance, show that there is a natural isomorphism $H_*(X, A) \cong H_*(C(i), v)$ where v is the vertex of the cone.

c) Now let $i : X \to CX$ be the obvious inclusion of X into its cone. The cone $SX = C(i)$ is called the suspension of X. Show that for all $n \geq 0$, there is an isomorphism $\tilde{H}_n(X) \cong H_{n+1}(SX)$. This is called the suspension isomorphism.