The Hall of 20,000 Ceiling Lights

A Car Talk Problem

http://www.cartalk.com/content/hall-20000-ceiling-lights
(No fair asking Ray for hints!)

In case you don't have internet access, here's the synopsis from one of their radio shows.

RAY: This next puzzler is from my "ceiling light" series.

Imagine, if you will, that you have a long, long corridor that stretches out as far as the eye can see. In that corridor, attached to the ceiling are lights that are operated with a pull cord.

There are gazillions of them, as far as the eye can see. Let's say there are 20,000 lights in a row.

They're all off. Somebody comes along and pulls on each of the chains, turning on each one of the lights. Another person comes right behind, and pulls the chain on every second light.

TOM: Thereby turning off lights 2, 4, 6, 8 and so on.

RAY: Right. Now, a third person comes along and pulls the cord on every third light. That is, lights number 3, 6, 9, 12, 15, etcetera. Another person comes along and pulls the cord on lights number 4, 8, 12, 16 and so on. Of course, each person is turning on some lights and turning other lights off.

If there are 20,000 lights, at some point someone is going to come skipping along and pull every 20,000th chain.

When that happens, some lights will be on, and some will be off. Can you predict which ones will be on? Think you know?

Induction: Scientific or Mathematical?

In the following problems, find a pattern, state a general theorem summarizing the pattern you observe, and prove it.

Some Odd Sums

$$\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
1+3+5+7+9+11 &= 36\\
1+3+5+7+9+11+13 &= 49\\
\ldots & \quad \ldots\\
1+3+5+7+9+11+13+\cdots+(2n-1) &= ??
\end{aligned}$$

Sums of Cubes

$$\begin{aligned}
1^3 &= 1\\
1^3+2^3 &= 9\\
1^3+2^3+3^3 &= 36\\
1^3+2^3+3^3+4^3 &= 100\\
1^3+2^3+3^3+4^3+5^3 &= 225\\
1^3+2^3+3^3+4^3+5^3+6^3 &= ??\\
\ldots & \quad \ldots\\
1^3+2^3+3^3+4^3+5^3+6^3+\cdots+n^3 &= (1+2+3+\cdots+n)^2
\end{aligned}$$

Decimal Expansions of Rational Numbers Every rational number has a decimal expansion that either terminates or is eventually periodic. Find the decimal expansion of each of the following rational numbers:

- $\frac{1}{3}, \frac{2}{3};$
- $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7};$
- $\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13};$
- $\frac{1}{89}, \frac{1}{9899}, \frac{1}{998999}, \ldots$

Pascal's Triangle

Pascal's Triangle: We have written down a portion of Pascal's triangle below.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

(1) What is the sum of the entries in the 7th row (i.e. the row at the bottom)? the 6th row? the 5th row? Any conjectures?

(2) Repeat the above experiment but this time take the alternating sum of each row. What do you see now?

(3) How many of the entries in the 7th row are odd? How many in the 6th row? the 5th row? Can you guess how many entries of the 100th row of Pascal's triangle are odd? How many entries in the 100th row do you think will be prime to 3? to 5? What's going on?

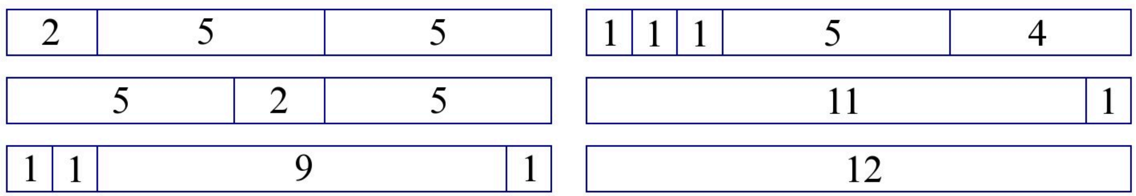
A Problem from George Polya

Abracadabra: in how many ways can one spell out “ABRACADABRA” by traversing the following diamond, always going from one letter to an adjacent one?

$$
\begin{array}{ccccccccccc}
 & & & & & A & & & & & \\
 & & & & B & & B & & & & \\
 & & & R & & R & & R & & & \\
 & & A & & A & & A & & A & & \\
 & C & & C & & C & & C & & C & \\
A & & A & & A & & A & & A & & A \\
 & D & & D & & D & & D & & D & \\
 & & A & & A & & A & & A & & \\
 & & & B & & B & & B & & & \\
 & & & & R & & R & & & & \\
 & & & & & A & & & & &
\end{array}
$$

No Rods of Length One

You can use “rods” of positive integer lengths to build “trains” that all have a common length. For instance, a “train of length 12” is a row of rods whose combined length is 12. Here are six examples:



Notice that the 2-5-5 train and the 5-2-5 train contain the same rods but are listed separately. Trains built from the same rods but in different orders are considered to be separate trains.

How many trains of length 12 are there that use no rods of length 1?