Atmospheric CO_2 Levels and Rates of Change

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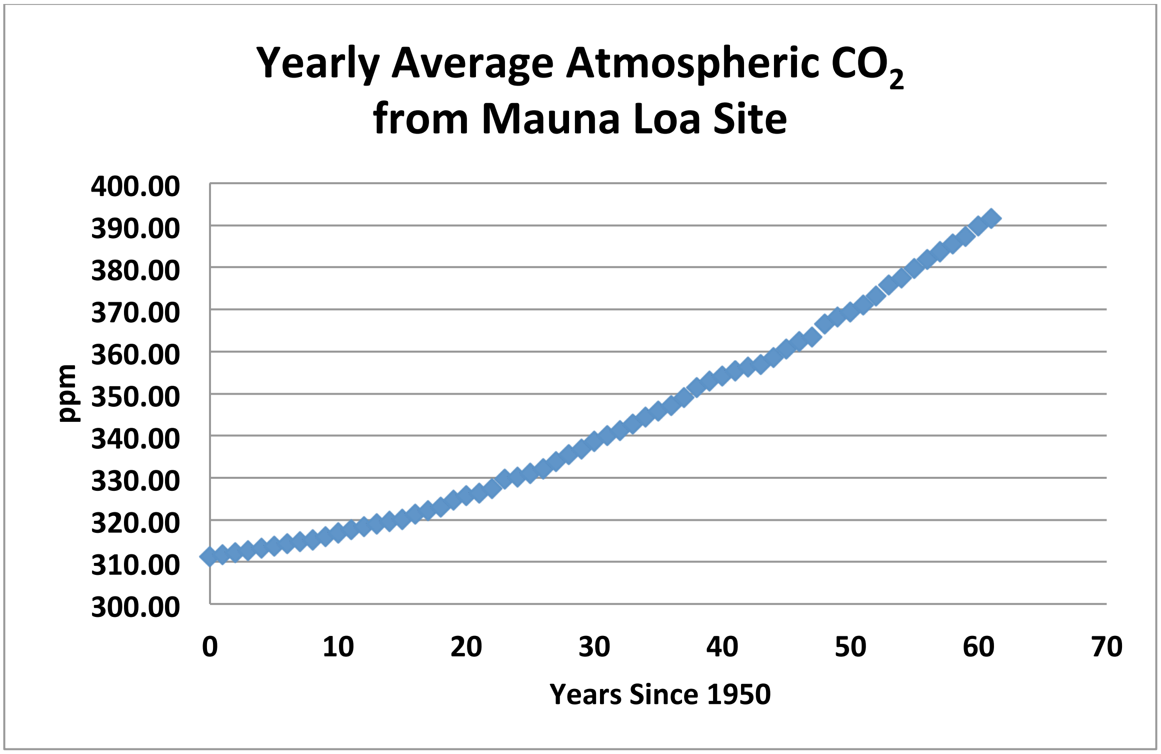

Figure 1. Atmospheric CO_2 data, 1950-2011, from the Mauna Loa site, ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt, with a fitted curve.

Lesson Overview: Data from the Mauna Loa Observatory [1, 2] (see Figure 1) indicates that CO_2 levels in the atmosphere are rising. Students will use a quadratic best fit curve to make a mathematical model of this data and then use the mathematical model to predict future CO_2 levels. They will also calculate the yearly rate of change of CO_2 levels and predict future CO_2 level growth assuming a fixed rate of change.

Level: Calculus, although the lesson could also be used in a pre-calculus class if one approximates the instantaneous rate of change (derivative) by an average rate of change. By focusing on finding the curve of best fit, it could also be used in a statistics class.

Common Core State Standards for Mathematics [3]: Standard for Mathematical Practice 4 - Model with mathematics.

AP Calculus Goals [4]: Students should:

- Be able to work with functions represented in a variety of ways: graphical, numerical, analytical, or verbal. They should understand the connections among these representations.

- Understand the meaning of the derivative in terms of a rate of change and local linear approximation, and should be able to use derivatives to solve a variety of problems.
- Be able to communicate mathematics and explain solutions to problems both verbally and in written sentences.
- Be able to use technology to help solve problems, experiment, interpret results, and support conclusions.

Associated Materials: The spreadsheet CO2 Data contains the data used to create the graph as well as the linear, quadratic and exponential fit to the data.

Mathematical Content:

1. Graphs. Graphs provide information about the real world. In this case they show the amount of the CO_2 in the atmosphere (measured in parts per million, ppm) as a function of time (measured in years). Graphs can be generated by data that is given in a table.
2. Curve of best fit. Given a set of data points, one can find a curve that fits the data as well as mathematically possible.
3. Mathematical modeling. The curve of best fit is described by a formula or function. This function provides what mathematicians call a mathematical model of atmospheric CO_2 levels. One can use a mathematical model to predict the future – in this case the future level of CO_2 in the atmosphere.
4. Properties of graphs. Graphs can be increasing or decreasing and be concave up or down. These properties have important contextual implications.
5. Rate of change of a function. The instantaneous rate of change is given by the slope of the tangent line to the curve which in calculus terms is the derivative. One can approximate this instantaneous rate of change by taking an average rate of change over a small time interval. This average is sometimes called a finite difference quotient. Henceforth, when we use the term rate of change, we will always mean instantaneous rate of change. When we mean average rate of change, we will explicitly include the term average.
6. Units. Units are important in the mathematics of real world problems and keeping track of units can help one better understand mathematical concepts. The units of rate of change in this problem are ppm per year = ppm/year which have a crucial meaning in the context of the problem.
7. Solving quadratic equations. The quadratic formula (or other methods of finding the roots of quadratic equations) can provide useful information about the real world.

Sustainability Content:

Scientists predict that rising levels of atmospheric CO_2 levels will impact the Earth's environment and climate in a wide variety of ways. Rising CO_2 concentrations have direct impacts, such as causing the oceans to become more acidic (decreasing pH), which threaten to kill off the world's coral reefs [5, 6]. CO_2 is a greenhouse gas that traps heat that would otherwise escape from the atmosphere and go into space. Thus rising CO_2 concentrations are linked to increases in average global temperatures. Some of the consequences of increasing temperatures have already been observed such as increased melting of glaciers and Arctic sea ice and rising sea levels due to the heat induced expansion of water. Scientists predict that other outcomes of the climate change associated with rising greenhouse gas levels will include more extreme weather events, including more powerful storms, as well as drought in many parts of the world [7, 8]. While it is true that CO_2 levels oscillate naturally, as can be seen in a stunning video showing the CO_2 levels over the past 800,000 years [9], that video also shows that over that long time period, CO_2 levels have never been as high as they are now.

In spite of clear evidence of the changes already under way, and strong scientific evidence of the harmful effects of potential future changes, both our national and international political systems has so far been unable to take decisive action to address rising CO_2 levels. There are no easy answers to the complex challenges posed by climate change. Society will need to draw upon the best thinking of knowledgeable experts from many different fields of study to figure out what to do. And society will need an educated citizenry who can make informed decisions about what course of action to take. Mathematics has an important role to play in this work. More examples of ways that mathematics contributes to creating a sustainable society can be found at www.mathaware.org [10].

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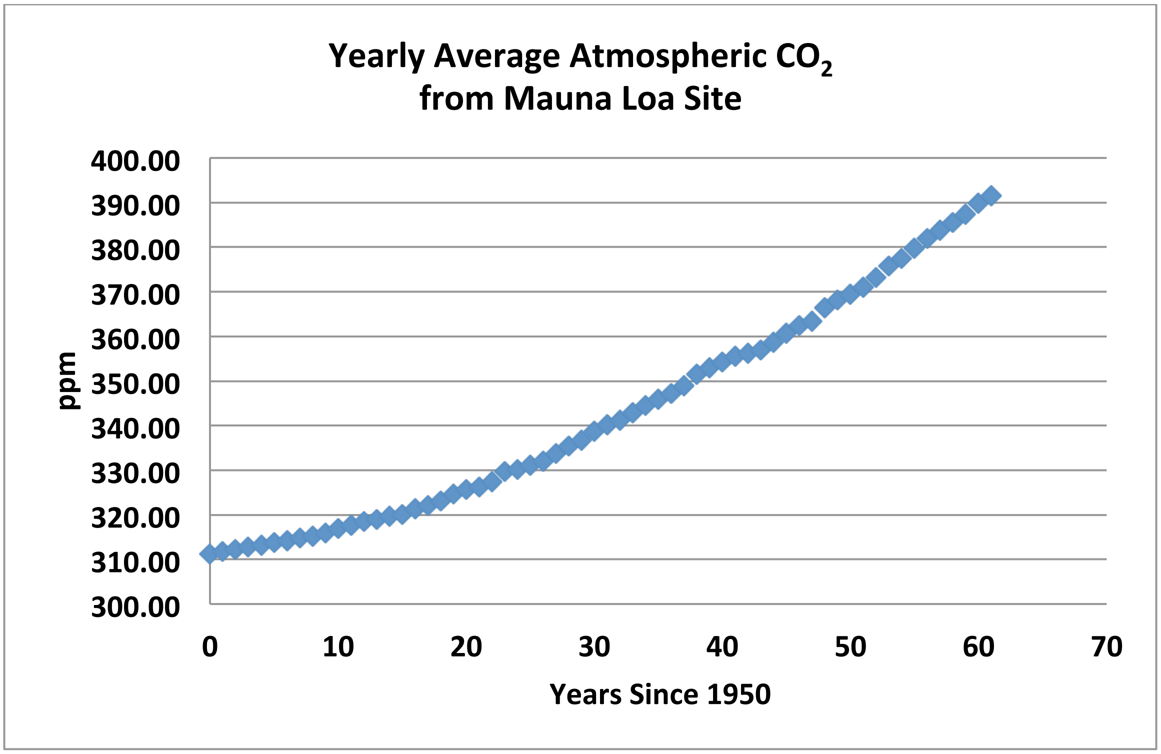

Figure 1. Atmospheric CO_2 data, 1950-2011, from the Mauna Loa site,
ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt

Goal: Predict the future – use mathematical modeling to estimate the level of CO_2 in the atmosphere in future years.

The above graph shows the atmospheric CO_2 levels as measured at the Mauna Loa Observatory as a function of time. The units for time are years counted starting from 1950. The units for CO_2 concentration are parts per million (ppm).

1. What year corresponds to t = 50 in the above graph?
2. What is the atmospheric CO_2 concentration in that year? What do the units **parts per million** (**ppm**) mean?
3. Which of the following terms apply to the graph: increasing, decreasing, concave up, concave down?
4. The curve is made by plotting data points that can be found at the above website. We wish to create a function that will provide a good fit to the data and then use that function to predict the future. Given the shape of the graph, what type of function do you think will provide a better fit: linear, quadratic or exponential. On Figure 1, sketch by hand a smooth curve that fits the data.

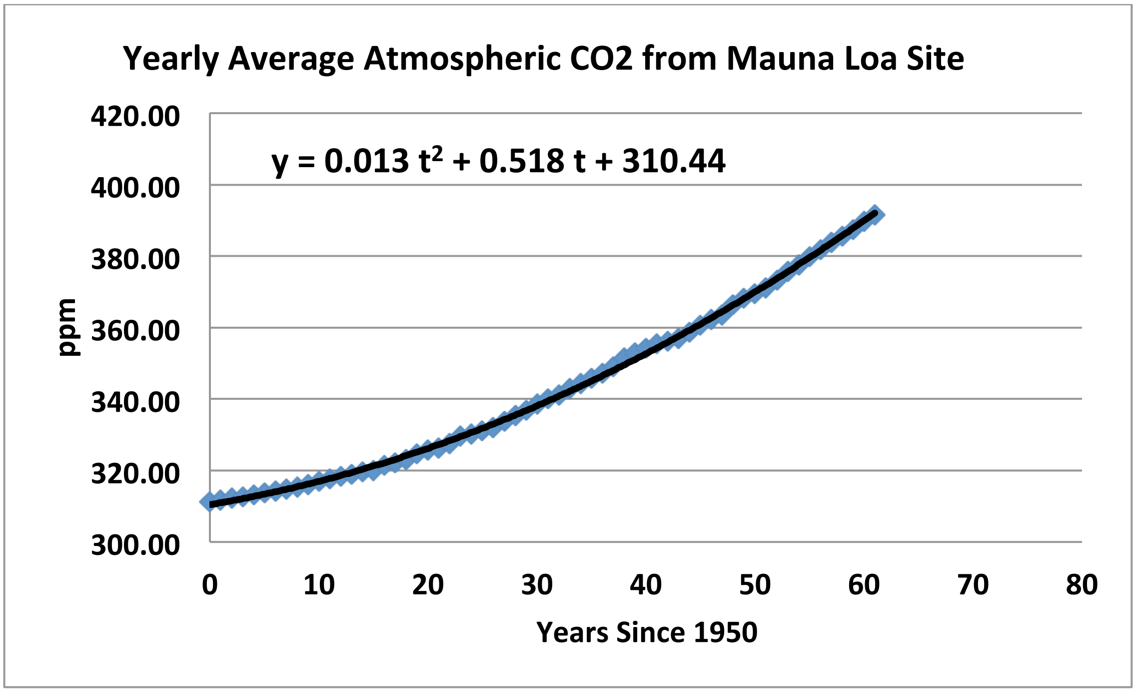

Figure 2. Atmospheric CO_2 data, 1950-2011, from the Mauna Loa site with quadratic fit. ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt, with a fitted curve.

The quadratic function is a better fit than either a linear function (see Figure 2), due to the concavity of the graph, or an exponential function. Answer the following questions using this quadratic model.

5. According to the model what will CO_2 levels be in 2050?

Rising CO_2 levels are already causing changes to the Earth's environment and climate and are predicted to have even more extensive impacts in the future. To head off the worst of these changes, society will need to prevent CO_2 levels from rising too high. Scientists have suggested that 450 ppm is an important threshold that we do not want to cross.

6. According to our quadratic model, in what year do we reach a CO_2 level of 450 ppm?
7. Using the model, determine the rate of change of CO_2 in 2012. What are the units for this rate of change? What is the percentage rate of change?
8. On Figure 2, sketch the line that has this constant rate of change (slope) and is tangent to the curve at time 2012.
9. This line corresponds to the assumption that beginning in 2012, the CO_2 level grows at a constant rate given by 2012 rate of change. With this constant growth assumption, in what year would we reach a CO_2 level of 450 ppm?
10. In order to avoid reaching 450 ppm of atmospheric CO_2 what would have to happen to the shape of the CO_2 curve? Illustrate by making a sketch. What math terms describe your picture?
11. Which of the two scenarios, continued quadratic growth or a switch to linear growth with a fixed rate of change, do you think is most likely to represent what will really happen in the future? Why?
12. Formulate a question based on your work here. It could be a mathematical question. It could be a question about the real world implications of rising CO_2 levels.
13. Discussion Question: What are actions that one can take personally or that society can take to reduce CO_2 emissions?

Lesson Plan Notes

This lesson can be taught in a variety of ways ranging from a teacher centered presentation to having students working through the assignment on their own or in groups and then presenting their findings to the class.

0. Lesson launch. Show the students a short video about CO_2 levels. Engage students in a discussion of CO_2 levels in the atmosphere and climate change. What do students know about this? What questions do they have?

   One could have students give presentations about this issue or partner with a social studies/government teacher to have a discussion of the political and governmental aspects, both nationally and internationally, of this issue.

1. The Graph. Understanding and paying attention to the units is important. Often in math problems, there are no units. So students do not always pay attention when there are units. The horizontal axis measures time. To simplify the calculations, we choose to measure time starting from 1950 which is the start of our data. If one uses the actual year (ex. 2013) for the time axis, then the numbers become so large that calculations are difficult. When answering the questions in the lesson, the students should always include the units.

2. The vertical axis gives the concentration of CO_2 in the atmosphere. The units are parts per million (ppm). If the concentration is 370 ppm, then in a typical collection of 1 million gas molecules in atmosphere, there will be 370 CO_2 molecules.

   One way to help students understand this concept is to have 100 pennies (or red candies). Then replace 20 of them with nickels (or blue candies). The nickels then have a concentration of 20 parts per hundred.

3-4. Finding a Fit to the Curve. Engage the students in a discussion about what type of function will provide a good fit. Students are so accustomed to using straight lines and linear functions, that they have a tendency to think that one ***always*** uses a linear fit. Draw out in the discussion that the curve is concave up. A linear fit will not capture this essential feature of the curve and so will probably not give the best fit.

   In Excel, we used the Chart – Add Trendline feature and found that a quadratic function gives a better fit than linear or an exponential. See these results in the spreadsheet CO2 Data.

5. Use Mathematics to Predict the Future: Using the model (i.e. the quadratic function), have the students predict what the CO_2 levels will be in future years. We have asked the students to calculate the level in 2050 which corresponds to t = 100. You could ask them to calculate the values in other years too such as 2100 (t = 150). These calculations are best undertaken using technology. As an extension, one can ask students to graph the function up to some future date.

6. Students will need to set the quadratic function equal to 450, i.e.

$$0.0134*t^2 + 0.5202*t + 310.43 = 450,$$

do algebra to rearrange the equation into the form: new quadratic expression = 0. Then they solve this, for example by using the quadratic equation. The students should quickly realize that this equation is too hard to solve by factoring. Students are often tempted to immediately use the quadratic applied to $0.0134*t^2 + 0.5202*t + 310.43$ without taking the 450 into account.

Note: While we use 450 ppm in this unit as target level for CO_2 concentration, some scientists argue that 450 ppm is too high and we should try to keep CO_2 concentration at lower levels, for instance 350 ppm.

7. Rate of change. The (instantaneous) rate of change is gotten by taking the derivative of the quadratic function and then evaluating the derivative at the specific time in question (here t = 62). In a pre-calculus class, one can estimate the (instantaneous) rate of change of the model by determining an average rate of change using the values of the quadratic function near t = 62.

   The percentage of rate of change, which has units 1/year, is the (rate of change, ppm/year) divided by (value of the function, ppm).

   To determine the units for the rate of change = derivative $= lim_{t \to 0} \frac{\Delta y}{\Delta t}$, one keeps track of the units that arise in the difference quotient: $\frac{\Delta y}{\Delta t} = \frac{\text{ppm}}{\text{years}}$. So the units of rate of change are ppm per year. If one has a rate of change of 2 ppm per year, that means the CO_2 concentration will increase by 2 ppm each year.

   One can help students understand the meaning of this rate of change by using the coins. Each minute, replace two of the pennies with nickels. Then the rate of change of concentration of nickels is 2 parts per hundred per minute = 2 pph/min.

8. This straight line is the tangent line to the curve at t = 62.

9. If the rate of change is constant at 2 $\frac{\text{ppm}}{\text{year}}$, then after T additional years, the CO_2 level will have increased by 2 $\frac{\text{ppm}}{\text{year}} \times \text{T years} = \text{2T ppm}$. The students will need to solve the equation:

   CO_2 level in 2012 as given by the model + (rate of change in 2012) x T = 450,

   where T= 2050 – 2012 = 38. The curve corresponding to this constant rate of change is the straight line that is tangent to the curve at t = 62.

10. If the curve were to switch concavity at an inflection point and then become concave down, it could avoid passing 450 ppm. The curve might become asymptotic to the line y = 450 ppm or it might reach a maximum at 450 ppm or less and then decrease. From an environmental impact perspective, this latter outcome would be the most desirable.

11. At present, society has not made much progress in changing its pattern of CO_2 emissions. If present trends continue, the quadratic model would be a good predictor. If the nations of the world develop a more effective system to reduce the amount of CO_2 being emitted, then one could argue that a linear model might become more accurate. It will take yet more extensive changes in

international CO_2 emissions practices before we could expect a concave downward curve to be an accurate model.

12. Encourage the students to use their creativity and thoughts to come up with their own question. Any question that connects in any way to the lesson (the math or CO_2 levels or sustainability) is fine. Have the students share their questions with each other either in small groups or in a whole class discussion.

13. Finale. We do not want to have the students leave the lesson being discouraged about the implications of their mathematical findings. Finish by giving them the opportunity to think of ways that they can take action to improve the situation and make a difference.

Lesson Feedback

Gather feedback from the students, either in written form or orally, about their reaction to the lesson. One could have a short classroom discussion asking the students what they thought about this lesson. Giving students the opportunity to reflect on their learning experience will help strengthen their learning while also giving the instructor useful information for future lessons.

Extensions

The students could graph Figure 1 themselves using the data in the spreadsheet CO2 Data and the graphing features of the spreadsheet. In that case, one should delete Figure 1 from the handout.

A statistics class could focus on finding the best fit to the curve by trying different types of functions and seeing which provides the best fit.

In Figure 3, one sees a graph of CO_2 levels over the last 1000 years determined from ice core samples at Law Dome, Antarctica [11]. This graph show that the quadratic function would only provide an accurate fit for the CO_2 values for a specific time period. One could have the students calculate over what time period in the past the quadratic approximation remains accurate.

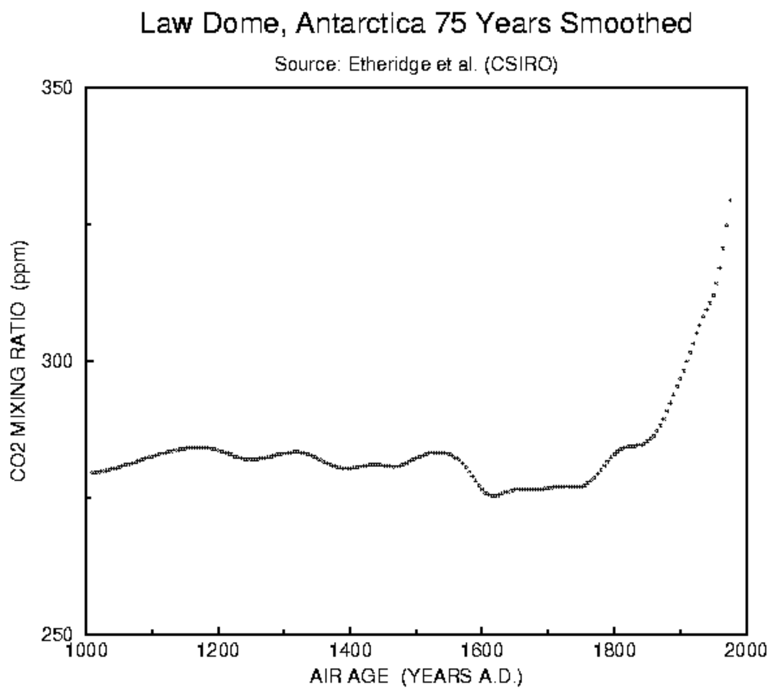

Figure 3. CO_2 Concentrations over past 1000 years from http://cdiac.ornl.gov/trends/co2/lawdome.html

One could give the students Figure 3 to examine and ask them when the character of the graph seemed to change and what might be the cause. With the start of the industrial revolution in the late 1700s and the associated increase in coal use, the CO_2 levels started to rise.

There is lots of data on CO_2 levels at http://www.esrl.noaa.gov/gmd/ccgg/trends/mlo.html including a fascinating video [9] of CO_2 levels over the past 800,000 years. Have the students watch the video and look for local and global maximum of CO_2 levels. Not counting the present era, what was the highest CO_2 levels over this long time period?

If one plots the monthly CO_2 levels, rather than just the yearly values, one sees an interesting periodic oscillation [12]. One can use trigonometric functions to model this data. Take the quadratic fit and add on a suitably adjusted sine function of the form A sin((t – B) C).

References:

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Acknowledgements:

This lesson is based on Tom Pfaff's units: Atmosphere: CO_2 and Temperature at http://www.ithaca.edu/tpfaff/sustainabilityC.html and A Look at Atmospheric CO_2 at http://www.sustainabilitymath.org/Materials.html. The lesson was developed in conjunction with the MAA PREP workshop: Educating with Math for a Sustainable Future workshop. Jeff Tecosky-Feldman, Lynn Foshee Reed and Robert Styer read and provide valuable feedback on the lesson.