Critical Orbit Structure of $f_c(z) = z^2 + c$ over $\mathbb{C}_p$

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Question

Fix a prime $p > 2$ and consider $f_c(z) = z^2 + c$ with $c \in \mathbb{C}_p$. What are the different possibilities for the structure of the orbit of 0 under iteration of $f_c$? A full classification of finite critical orbit trees is known in the complex setting; we will study some of the variations over $\mathbb{C}_p$ by looking at the critical portrait of the reduction map $f_i$ over the residue field $k = \Omega_{\mathbb{C}_p/m} = \mathbb{F}_p$.

Motivation

Hubbard trees classify hyperbolic components of the Mandelbrot set in the complex setting. What are the analogous trees in Berkovich space, in the $p$-adic setting?

First, we look at the classical Mandelbrot set that gives the picture above:

$$\mathcal{M} = \{c \in \mathbb{C} : \text{critical orbit of } f_c(z) = z^2 + c \text{ is finite}\}$$

And the corresponding set in the $p$-adic setting:

$$\mathcal{M}_p = \{c \in \mathbb{C}_p : \text{critical orbit of } f_c(z) = z^2 + c \text{ is bounded}\},$$

which is actually the rather bland unit disk $D = \{c \in \mathbb{C}_p : |c| \leq 1\}$.

Given the rich structure of the space $\mathcal{C}_p$, however, $\mathcal{M}_p$ is more than “just a disk”: it is an infinitely branching tree with many interesting features. These features lead to some unexpected patterns in the structure of the critical orbit for the post-critically bounded (PCB) polynomials over $\mathbb{C}_p$.

Setting

Let $\mathbb{C}_p$ be the algebraically closed (metric) completion of $\mathbb{Q}_p$. Let $| \cdot |$ be a non-Archimedean absolute value which extends a $p$-adic absolute value. It satisfies the strong triangle inequality:

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\} \text{ for all } \alpha, \beta \in \mathbb{C}_p.$$ 

Define $D = \{c \in \mathbb{C}_p : |c| \leq 1\}$, the valuation ring of $\mathbb{C}_p$, the closed unit disk, $U = \{c \in \mathbb{C}_p : |c| < 1\}$, the maximal ideal of $D$, the open unit disk. $\mathbb{F}_p = D/U$, the residue field of $D$.

$\mathbb{Z}_p = \{z \in \mathbb{Q}_p : z = \sum a_n p^n, \text{ where } a_n \in \{0, \ldots, p-1\}\}$

Note $|z|_p = p^{-\nu(p)}$ for some $\nu > 1$, where $\nu(c) = \ord_p(c) = \exp$ of the highest power of $p$ dividing $c$.

Main Results

Definition: A point $\alpha$ has orbit type $(m, n)$ if $m$ and $n$ are the least integers such that $f_{m+n}(\alpha) = f_m(\alpha)$. Then $m$ is the tail length and $n$ is the cycle length of the orbit of $\alpha$.

The following theorem applies to all polynomials of the form $f_c(z) = z^2 + c$ over $\mathbb{Z}_p$, both with finite critical orbit and infinite critical orbit.

Theorem: Critical Orbit Type

For $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$, $p > 2$, we have the following:

- Periodic Reduction: If $0$ has orbit type $(0, n)$ (mod $p$), then $0$ has orbit type $(s(k), n)$ (mod $p^l$) for all $k$, where $s(k)$ is a constant that depends on $k$.
- Pre-periodic Reduction: If $0$ has orbit type $(m, n)$ (mod $p$), then $0$ has orbit type $(m, t(k) \cdot n)$ (mod $p^l$) for all $k$, where $t(k) = t \cdot p^l$ for $t < p$ and $s < k$.

Key ingredients of the proof:

- Lemma: If $0$ is periodic of exact period $n$ under iteration of $f_c(z)$ (mod $p$), then there exists an attracting $n$-cycle of for $f_c(z)$.
- Proof of the proof relies on Henel’s Lemma. See also Corollary 1.5 in [2].

In local coordinates, away from the critical point, the reduced iterates of $f$ are linear transformations, which act bijectively on the periodic cycles.

Remark: The patterns that arise in infinite critical orbits over $\mathbb{Z}_p$ give insight about possible structures of finite critical orbits over $\mathbb{C}_p$. It is known that if $0$ is periodic of period $n$ over $\mathbb{C}_p$, then $0$ will have exact period $n$ under the reduced map over $\mathbb{F}_p$. If $0$ is strictly pre-periodic over $\mathbb{C}_p$, we have the following.

Conjecture: Pre-periodic Critical Orbit

If $0$ is pre-periodic over $\mathbb{C}_p$ of orbit type $(m, n)$, it will follow the same orbit structure as that of a map with infinite critical orbit and reduced orbit type $(m, n')$, with $n = n' \cdot r^l$ for a root of unity (mod $p$) of order $r$.

Extension

Once the patterns in the parameter space and in the critical orbit structure are fully understood, a natural question to ask is: do the same patterns exist in $\mathcal{M}_p$ and the Julia set corresponding to a polynomial with parameter in $\mathcal{M}_p$?

Example: Dynamics over $\mathbb{Z}_p$

$\mathbb{Z}_p$ may be viewed as a tree inside the closed unit disk of $\mathbb{C}_p$, so any parameter in $\mathbb{Z}_p$ corresponds to a PCB polynomial. A visual of the top of the $\mathbb{Z}_p$ tree:

Structure of $\mathcal{M}_p$ over $\mathbb{Z}_p$

- $D(0, 3^{-1}) = \text{Hyperbolic Component 1: parameters with an attracting fixed point.}$
- $D(2, 3^{-1}) = \text{Hyperbolic Component 2: parameters with an attracting 2-cycle.}$
- $D(1, 3^{-1})$ contains parameters with strictly pre-periodic critical points.

In $D(1, 3^{-1}) \cap \mathbb{Z}_p$, there are only 2 possible finite critical orbits: $(2, 1)$ and $(2, 3)$. The other parameters have infinite (but bounded) critical orbits, and those orbits grow in a fixed pattern. Follow $c \equiv -2$, which has finite critical orbit type $(2, 1)$.

References