Critical Orbit Structure of $f_c(z) = z^2 + c$ over $\mathbb{C}_p$

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Question

Fix a prime $p > 2$ and consider $f_c(z) = z^2 + c$ with $c \in \mathbb{C}_p$. How is the critical portrait of f_c (over $\mathbb{C}_p$) related to the critical portrait of the reduction map $\bar{f}_c$ (over the residue field $k = \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_p = \overline{\mathbb{F}}_p$)?

Theorem: Periodic Critical Orbits

If 0 has exact period n for $f_c : \mathbb{C}_p \to \mathbb{C}_p$, then 0 has exact period n for $\bar{f}_c : k \to k$.

Note, this result also appears in [3]. We give a different proof.

Remark: In general, the critical portrait may not be preserved. For example, consider $f(z) = z^2 - 2$, $p = 2$. The critical portrait is changed considerably by reduction, with $\bar{f}(z) = z^2$:

For $f(z)$: $\ast \to \cdot \to \cdot \circlearrowleft$, $0 \to -2 \to 2 \circlearrowleft$ For $\bar{f}(z)$: $\ast \circlearrowleft$, $0 \circlearrowleft$

Setting

Let K be a finite extension of $\mathbb{Q}_p$. Let $|\cdot|_v$ be a non-Archimedean absolute value which extends a p-adic absolute value. It satisfies the strong triangle inequality:

$$|\alpha + \beta|_v \leq \max\{|\alpha|_v, |\beta|_v\} \text{ for all } \alpha, \beta \in K.$$

Define

$$\begin{aligned} \mathcal{O}_K &= \{\alpha \in K : |\alpha|_v \leq 1\}, \text{ the valuation ring of } K. \\ \mathfrak{m}_v &= \{\alpha \in K : |\alpha|_v < 1\}, \text{ the maximal ideal of } \mathcal{O}_K. \\ \mathcal{O}_K^* &= \{\alpha \in K : |\alpha|_v = 1\}, \text{ the group of units of } \mathcal{O}_K. \\ k &= \mathcal{O}_K/\mathfrak{m}_v, \text{ the residue field of } \mathcal{O}_K. \end{aligned}$$

Note $|\cdot|_v = \pi^{-v(\alpha)}$ for some $\pi > 1$, where $v(\alpha) = \text{ord}_p(\alpha)$ = the exponent of the highest power of p dividing α.

Newton Polygon for $p_n(c)$

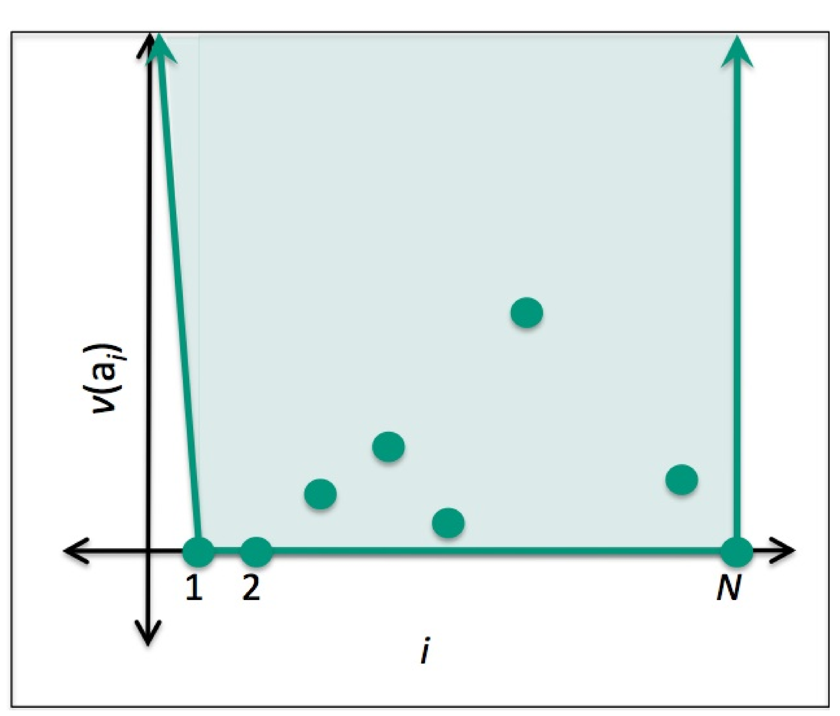

Figure 1: The Newton Polygon for $p_n(c) = c^N + a_{N-1}c^{N-1} + \cdots + c^2 + c = 0$. The slope from $i = 0$ to $i = 1$ is $-\infty$, so one root is such that $|c|_p = p^{-\infty} = 0$. The slope from $i = 1$ to $i = N$ is 0, so the remaining $N - 1$ roots are such that $|c|_p = p^0 = 1$.

Intermediate Results

Basic Lemma

If 0 has exact period $n > 1$ for $f_c(z) = z^2 + c \in \mathbb{C}_p[z]$, then $|c|_p = 1$.

For fixed period n, $f_c^n(0)$ is a polynomial in c of degree $N = 2^{n-1}$, whose roots are the c values for which 0 has period $m = n$ or $m|n$ for $f_c(z)$. Write $f_c^n(0) = p_n(c)$.

Using the method of Newton Polygons, we show that there is exactly one root $c = 0$ for each $p_n(c)$, and $N - 1$ roots such that $|c|_p = 1$. See Figure 1. Note $c = 0$ corresponds to $f(z) = z^2$, with fixed point $z = 0$.

Rigidity Lemma

Fix $p \geq 2$. Let $f_c(z) = z^2 + c \in \mathbb{C}_p[z]$ be such that 0 has an infinite orbit, and $|c|_p = 1$. If k is the least integer such that $\text{ord}_p(f_c^k(0)) > 0$, then for all $n \in \mathbb{N}$,

$$\text{ord}_p(f_c^n(0)) = \begin{cases} 0 & \text{if } k \nmid n \\ \text{ord}_p(f_c^k(0)) & \text{if } k \mid n \end{cases}$$

Define a polynomial $g_n(z)$ by $f^n(z) = z \cdot g_n(z) + f^n(0)$.

- $k \mid n$: Write $n = mk$, and show by induction on m that $\text{ord}_p(f^{mk}(0))$ =$\text{ord}_p(f^k(0))$. Use the above definition of $f^n(z)$.
- $k \nmid n$: Write $n = qk + r$ and note that $\text{ord}_p(f^r(0)) = 0$. Use the previous case.

Corollary of Rigidity

The critical orbit of $f_c(z)$ over $\mathbb{C}_p$ cannot be recurrent.

Under iteration, the critical point will either land back on itself or remain a fixed distance away from 0.

Proof of Theorem

- Let K_n be the finite extension of $\mathbb{Q}_p$ containing all roots of $p_n(c)$ for fixed n. For each root c of $p_n(c)$, write
$$c = \sum_{i=m}^{\infty} a_i \pi^i.$$
- From theory on height functions: there exist only finitely many c values with finite critical orbit in K_n.
- Truncate the expansion of c to $c_k = \sum_{i=m}^{k} a_i \pi^i$ such that $|c_k|_p = 1$ and 0 has infinite orbit for f_{c_k}.
- Apply the Rigidity Lemma to $f_{c_k}(z)$ and relate back to $f_c(z)$. The result follows.

Rigidity Only Holds for $z = 0$

Let $f(z) = z^2 - 1$, $z_0 = 2$, and fix prime $p = 3$. Then z_0 has infinite orbit, $|c|_3 = 1$, and $k = 1$ is minimal such that $\text{ord}_3(f^k(2)) > 0$. However, $k = 1$ gives no additional information about the highest power of 3 that divides $f^n(2)$:

Iterates of $f(2)$

$$\begin{aligned} f^1(2) &= 3 & &\Rightarrow \text{ord}_3 f^1(2) = 1 = 2^0 \\ f^2(2) &= 8 & &\Rightarrow \text{ord}_3 f^2(2) = 0 \\ f^3(2) &= 63 & &\Rightarrow \text{ord}_3 f^3(2) = 2 = 2^1 \\ f^4(2) &= 3968 & &\Rightarrow \text{ord}_3 f^4(2) = 0 \\ f^5(2) &= 15745023 & &\Rightarrow \text{ord}_3 f^5(2) = 4 = 2^2 \end{aligned}$$

Note that the odd iterates are 3-adically getting closer to 0.

Motivation

Hubbard trees classify hyperbolic components of the Mandelbrot set in the complex setting. What are the analogous trees in Berkovich space, in the p-adic setting?

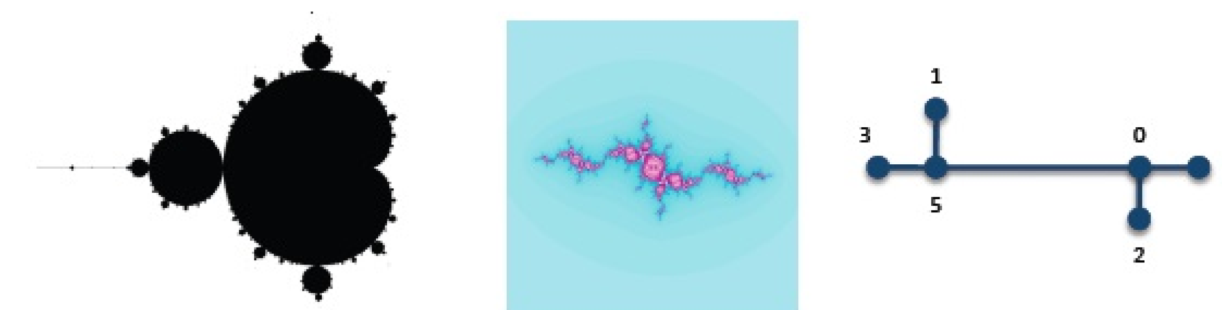

Figure 2: The complex Mandelbrot set; the Julia set for a period-6 critical orbit in the 1/2 limb; and its Hubbard tree.

The convex hull of the critical orbit in Berkovich space always has the same structure, exactly n branches surrounding the Gauss Point.

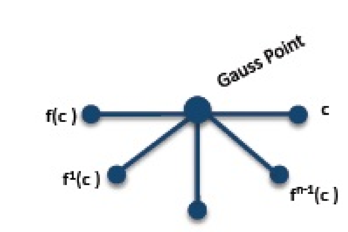

Figure 3: A tree in Berkovich space for $f_c(z)$ with periodic critical orbit.

Future study: What happens in the case of a pre-periodic critical orbit? Is it possible to have a Berkovich tree with more branches?

References

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