Abstract

In complex dynamics, Hubbard trees offer a combinatorial description of the dynamics of post-critically finite (PCF) polynomials. What are the analogous objects in a non-Archimedean setting: what is a p-adic Hubbard tree? We explore this question by studying the critical orbit trees associated to quadratic maps $f_c(z) = z^2 + c$, with $c \in \mathbb{Z}_p$ (for $p > 2$).

Complex Setting

The Hubbard tree $H_f$ of a PCF polynomial $f$ is a finite tree in the Julia set connecting all points contained in the critical orbits. The action of $f$ on $H_f$ and the embedding of $H_f$ in the complex plane (up to isotopy class) captures all of the important information of the global dynamical system $f : \mathbb{C} \to \mathbb{C}$ (see [2], [8]).

Non-Archimedean Setting

We view $\mathbb{Z}_p$ as a subtree of the Berkovich project line over $\mathbb{Q}_p$, endowed with the $p$-adic metric. Each vertex of the tree is a disk with rational radius of the form $p^{-m}$, denoted $D(a, r)$. Two vertices are connected by an edge (branch) if one disk is contained in the other, so each point has $p$ edges branching off of it. $\mathbb{Z}_p$ is exactly the set of ends of the branches, and the top of the tree is the Gauss point $D(0, 1)$.

Figure 1: An example of a Hubbard tree from [3], an invariant subset of the Julia set.

Figure 2: The top of the $\mathbb{Z}_p$ tree.

For $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$, we suggest the following Hubbard tree analogy:

Definition: The critical orbit tree for $f_c$ is the convex hull of the critical orbit in the $\mathbb{Z}_p$ tree, together with the induced action of $f_c$. The critical orbit tree (mod $p$) is the subtree consisting of residue classes (mod $p$).

We say that a point $\alpha$ has orbit type $(m, n)$ if $m$ and $n$ are the least integers such that $f^{m+n}(\alpha) = f^m(\alpha)$. Then $m$ is the tail length and $n$ is the cycle length of the orbit of $\alpha$.

Main Results

To understand the critical orbit tree structures for $f_c$, it is necessary to study the behavior of the critical orbit (mod $p$). The periodic case may be deduced from ([1], [4], [6] and [9]), based on the existence of an attracting n-cycle:

Theorem 1: Periodic (mod $p$)

Let $p \geq 3$ and consider the critical orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$. If $0$ has orbit type $(0, n)$ (mod $p$), then either $0$ is periodic of exact period $n$ or $0$ has infinite orbit, with orbit type $(m, n)$ (mod $p$) for all $n \geq 1$.


Theorem 2: Pre-periodic (mod $p$)

Let $p \geq 3$ and consider the critical orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$. If $0$ is strictly pre-periodic with orbit type $(m, n)$ (mod $p$), then either $0$ has orbit type $(m, n)$ over $\mathbb{Z}_p$, or there exists some $k \geq 1$ in $\mathbb{Z}$ such that

- $0$ has orbit type $(m, n)$ (mod $p$) for all $n \geq k$, and
- $0$ has orbit type $(m, r - n)$ (mod $p$) for all $j \geq k$, with $r|p - 1$.

Otherwise, $0$ has infinite orbit, with orbit type $(m, n)$ (mod $p$) for all $i \geq 1$.

A key piece of the proof is the fact that if $0$ has tail length $m$ (mod $p$), then the tail length is fixed at $m$ when the critical orbit is calculated modulo higher powers of $p$, even if the orbit of $0$ is infinite over $\mathbb{Z}_p$.

Remark: There is a finite number of PCF parameters in $\mathbb{Z}_p$. The work of [7] and [10] gives a uniform bound on the total number for given prime $p$.

Theorems 1 and 2 also have the following implications on the tree structures of finite critical orbits in $\mathbb{Z}_p$.

Theorem 3: Critical Orbit Trees in $\mathbb{Z}_p$

Again let $p \geq 3$ and suppose $0$ has finite orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$.

- If $0$ is periodic of exact period $n$, the critical orbit tree coincides with the critical orbit tree (mod $p$). It is a finite tree with a single vertex of degree $n$, and $f_c(z)$ acts on the $n$ end points by a cyclic permutation.
- If $0$ is strictly pre-periodic with orbit type $(m, n)$, the critical orbit tree either coincides exactly with the critical orbit tree (mod $p$) or it differs by one instance of branching.

Remark: Linearization gives a bound on how far into the $\mathbb{Z}_p$ tree the branching can occur, for given prime $p$, and consequently there is a finite number of possible critical orbit trees for PCF parameters in $\mathbb{Z}_p$.

Examples

In light of Theorem 3, it is straightforward to calculate all possible finite critical orbit trees for a given $p$. We give the complete list for $p = 3$ and $p = 5$.

Figure 3: The 4 distinct finite critical orbit trees in $\mathbb{Z}_p$.

Note that the $(2,3)$ tree matches the $(2,1)$ tree (mod 3) and then branches once below (mod 3$^2$). For $p = 5$ we have an example of 2 distinct critical orbit trees that correspond to orbit type $(2,2)$.

Figure 4: The 7 distinct finite critical orbit trees in $\mathbb{Z}_p$.

Remark: Going beyond the finite orbit trees for $p = 3$, we can give a complete description of the orbit trees for all $c \in \mathbb{Z}_p$ by exploiting the existence of linearization disks near periodic cycles, as detailed in [3] and [5].

References