MATH 417 HOMEWORK 11

You may collaborate on the homework. However, the final write-up must be yours and should reflect your own understanding of the problem. Please be sure to properly cite any help you get.

Problem 1 Apply Rouché's Theorem to $f(z) = z^n$ and $g(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}$, where $n \geq 1$, on a circle of appropriately chosen radius R around the origin (Hint: What should R be?) to prove that the polynomial

$$a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$$

has precisely n zeros counting with multiplicity. Recall that we proved the Fundamental Theorem of Algebra before using Liouville's Theorem.

Problem 2 Suppose that $f(z)$ is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Suppose that f has n zeros $z_1, \ldots, z_n$ inside C with multiplicities $m_1, \ldots, m_n$, respectively. Show that

$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

Problem 3 Determine the number of zeros (counting with multiplicity) of the following polynomials contained in the unit circle $|z| = 1$

$$(a)\ z^{15} - 2z^{12} + 17z^7 - 3 \quad (b)\ z^9 - z^7 + 3z^3 - z - 12.$$

Problem 4 Determine the number of zeros (counting with multiplicity) of the following polynomials in the annulus $1 < |z| < 2$

$$(a)\ z^9 - 7z^5 + 3z - 2 \quad (b)\ z^7 - 15z^6 + 23z + 1.$$

Problem 5 Suppose c is a complex number such that $|c| > e$, show that the equation $cz^n = e^z$ has n roots inside the unit circle $|z| = 1$ counting with multiplicity. Now instead suppose $|c| < \frac{1}{e}$. How many solutions does the equation $cz^n = e^z$ have inside the unit circle $|z| = 1$?