Homework Week IV Due Wednesday October 6

Problems to turn in:

Problem 1: (i) A map $\phi : G \to H$ between two groups G and H is called a *homomorphism* if for any two elements g_1, g_2 of G, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. Prove that if ϕ is a homomorphism of groups and e is the identity element of G, then $\phi(e)$ is the identity element of H. Prove that $\phi(g^{-1}) = \phi(g)^{-1}$, where g^{-1} denotes the inverse of g. More generally, prove that $\phi(g^n) = \phi(g)^n$ for any integer n.

(ii) A homomorphism $\phi : G \to H$ is called an *isomorphism* if there exists a group homomorphism $\psi : H \to G$ such that $\phi \circ \psi$ is the identity on H and $\psi \circ \phi$ is the identity on G. Show that a group homomorphism is an isomorphism if and only if it is one-to-one and onto. Two groups are called *isomorphic* if there exists an isomorphism between them.

Problem 2: We denote the group of integers modulo m under addition by $\mathbb{Z}/m\mathbb{Z}$. We denote the group of reduced residue classes modulo m under multiplication by $(\mathbb{Z}/m\mathbb{Z})^*$. A group G is called *cyclic* if there exists an element $g \in G$ such that every element of G is of the form g^i for some integer i.

(i) Prove that a cyclic group of order $m < \infty$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.
(ii) Prove that any group whose order is a prime number is cyclic.
(iii) Show that $(\mathbb{Z}/8\mathbb{Z})^*$ is not cyclic.
(iv) Prove that any group of order 4 is isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/8\mathbb{Z})^*$.
(v) Show that if p is a prime, then $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group of order $p-1$.

page 91 section 2.6 problems: 3, 4, 9, 10
page 96 section 2.7 problems: 3, 4
page 106 section 2.8 problems: 2, 3, 6, 14, 22, 37

Additional suggested problems:
page 91 section 2.6 problems: 8, 11
page 106 section 2.8 problems: 7, 9, 15, 21
page 119 section 2.10 problem 1
page 126 section 2.11 problems: 2, 7