MATH 330 MIDTERM 1

This is the first midterm for Math 330. It will be handed out on Wednesday Oct 3 at the end of class. It is due on Monday Oct 8 before class. Late exams will not be accepted. You may use the class text book and your class notes. However, you may not use any other sources, such as other text books, the internet, math question centers. You may not discuss the questions with anyone or collaborate with anyone. All the work should be your own.

There are 5 problems on this exam. To receive full credit you must justify all your answers and calculations. Be sure to cite any theorem you are using.

Problem 1. (10 pts) Calculate the gcd and lcm of the integers 256 and 280.

Problem 2. (20 pts.) Let $\mathbb{Z}/n\mathbb{Z}$ denote the group of integers modulo n under addition.

i) List all the subgroups of $\mathbb{Z}/18\mathbb{Z}$ and determine the order of each of the subgroups.

ii) List all the generators of the group $\mathbb{Z}/36\mathbb{Z}$.

iii) List the left cosets of the subgroup of order 3 of $\mathbb{Z}/24\mathbb{Z}$.

Problem 3. (20 pts.) Let G be a group. We say that an element x of G is conjugate to an element y of G if there exists an element $g \in G$ such that $x = gyg^{-1}$.

i) Prove that 'being conjugate' is an equivalence relation. The equivalence classes are called conjugacy classes.

ii) Show that if x and y are conjugate, then they have the same order.

iii) Prove that a group G is abelian if and only if every conjugacy class of G consists of a unique element.

iv) Prove that an element x of G is contained in the center of G if and only if the conjugacy class of x contains only x.

Problem 4. (20 pts.) Let S_n denote the symmetric group on n letters.

i) Consider the permutations

$$a(1) = 3, a(2) = 4, a(3) = 2, a(4) = 1, a(5) = 7, a(6) = 5, a(7) = 6$$

and

$$b(1) = 7, b(2) = 6, b(3) = 4, b(4) = 2, b(5) = 3, b(6) = 1, b(7) = 5.$$

Express these permutations in cycle notation. Find the order of a and b. Discuss whether a and b are odd or even. Calculate $a \circ b$ and find its order. Is $a \circ b$ odd or even?

ii) Write down an example of an element of order 4 in S_4. How many elements of order 4 does S_4 have?

iii) Write down two distinct examples of elements of order 3 in S_6 with distinct cycle structures. How many elements of order 3 are there in S_6?

Problem 5. (30 pts.) Throughout this problem let ϕ denote the Euler ϕ function.

i) Let n be a positive integer and let p be a prime number. Calculate $\phi(p^n)$ in terms of p and n. Check that your formula holds for $n = 1$. (Hint: How many positive integers less than or equal to p^n are there? How many of these are divisible by p?)

ii) Let m and n be relatively prime integers. Suppose that

$$U(m) = \{[a_1], [a_2], \ldots, [a_{\phi(m)}]\}$$

and that

$$U(n) = \{[b_1], [b_2], \ldots, [b_{\phi(n)}]\}.$$

Show that

$$U(mn) = \{[a_i n + b_j m] | 1 \leq i \leq \phi(m) \text{ and } 1 \leq j \leq \phi(n)\}.$$

iii) Use the previous part to show that if m and n are relatively prime positive integers, then $\phi(mn) = \phi(m)\phi(n)$.

iv) Let $p_1, p_2, \ldots, p_r$ be distinct prime numbers. Let $n_1, \ldots, n_r$ be positive integers. Use the previous part and the first part to calculate $\phi(p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r})$.

v) What is the order of $U(2800)$?

vi) How many elements of order 420 are there in $\mathbb{Z}/84000\mathbb{Z}$?