MATH 516 MIDTERM

This is the take-home midterm for Math 516. This is an open book exam. You have a week to do it. You may discuss the problems with people in the class and you may consult books. However, the final write-up must be yours. You may not collaborate while writing the solutions and you may not use anything other than the text book and your course notes.

Problem 1. List all the conjugacy classes in the group $\mathfrak{S}_6$. Determine the order of each conjugacy class. Determine the orders of all the conjugacy classes of the alternating group A_6. Prove that A_6 is simple.

Problem 2. Classify all abelian groups of order $81000 = 2^3 3^4 5^3$. Find the number of subgroups of

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$$

that are isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Problem 3. Let $p < q < r$ be three prime numbers such that $r \neq 1 \mod q$ and $q, r, qr \neq 1 \mod p$. Prove that a group of order pqr is cyclic. Conclude that any group of order $595 = 5 \cdot 7 \cdot 17$ or $1235 = 5 \cdot 13 \cdot 19$ is cyclic.

Problem 4. Let p be a prime number. Let G be a non-abelian group of order p^3.

- Determine the possible orders of the center $Z(G)$ of G?
- Classify all possible quotient groups $G/Z(G)$.
- Show that there can be non-isomorphic non-abelian groups of order p^3.

Problem 5. Let G be a finite group. Suppose that every conjugacy class in G has representatives that are commuting. Prove that G is abelian.

Problem 6. Let G be a non-trivial finite group. Suppose that given any two non-identity elements $a, b \in G$ there exists an automorphism ϕ of G such that $\phi(a) = b$.

- Prove that $G \cong \oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z}$ for some positive integer n and some prime number p.
- Determine the order of $\mathrm{Aut}(\oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z})$.

Problem 7. Let $GL_n(\mathbb{Z}/p\mathbb{Z})$ denote $n \times n$ invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ for a prime number p. $GL_n(\mathbb{Z}/p\mathbb{Z})$ is a group under matrix multiplication and is an example of a finite group of Lie type.

- Determine the order of $GL_n(\mathbb{Z}/p\mathbb{Z})$.
- Show that $GL_n(\mathbb{Z}/p\mathbb{Z}) \cong Aut_{Ab}(\oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z})$.
- In particular, deduce that $GL_2(\mathbb{Z}/2\mathbb{Z}) \cong \mathfrak{S}_3$.

Problem 8. Classify all groups of order 30 and 44.

Problem 9. Let R be a commutative ring with unit. Prove that if R is Noetherian, then $R[[x]]$ the formal power series ring over R is also Noetherian.

Problem 10. Do problem V.1.10 on page 250

Problem 11. Do problem V.1.17 on page 251

Problem 12. Do problem V.2.14 on page 259

Problem 13. Do problem V.2.18 on page 260

Problem 14. Do problem V.2.19 on page 260

Problem 15. Do problem V.2.20 on page 260

Problem 16. Do problem V.2.25 on page 260