MATH 417: FINAL PROBLEM SET

Due Wednesday December 1 before class in Gradescope. You may use your class notes, previous homework and the course text book. You may not use any other materials, including other text books, the web, question centers, etc. The work should be yours and yours alone. Please do not collaborate. The total amount of time you spend on these problems should not exceed 96 hours.

Problem 1. (10 points) Let $u(x, y)$ and $v(x, y)$ be a conjugate pair of smooth harmonic functions in a domain D. Prove that $u^4 - 6u^2v^2 + v^4$ is a harmonic function in D. Find all possible harmonic conjugates of $u^4 - 6u^2v^2 + v^4$.

Problem 2. (10 points) Let $f(z)$ be an entire function. Suppose that for $z \neq 0$,

$$f(z) = f\left(\frac{1}{z}\right).$$

Prove that $f(z)$ is constant.

Problem 3. (10 points) Calculate the positively oriented contour integral

$$\int_{|z|=3} \frac{e^{2z}}{(z^2-4)^2} dz.$$

Problem 4. (10 points) Find all the Laurent series expansions of the function

$$\frac{1}{z(z-3)}$$

around the point $z = 1$.

Problem 5. (10 points) Find all the singularities of the following function and determine the type of singularity at each singular point

$$\frac{ze^{\frac{1}{z-2}}}{e^z - 1}.$$

Problem 6. (10 points) Calculate the following integral

$$\int_0^{2\pi} \frac{1}{4 + \sin(\theta)} d\theta$$

using residue calculus. Make sure to show all your work.

Problem 7. (10 points) Calculate the integral

$$\int_0^{\infty} \frac{\sqrt[4]{x}}{(x^2+4)(x^2+9)} dx$$

using residue calculus. Make sure to show all your work.

Problem 8. (10 points) Calculate the integral

$$\int_0^{\infty} \frac{\cos(x) - 1}{x^2(x^2+1)} dx$$

using residue calculus. Make sure to show all your work.

Problem 9. (10 points) Find the number of zeros (counting with multiplicity) of the polynomial

$$z^{10} + z^8 - 5z^3 + 1$$

in the annulus $1 \le |z| < 2$.

Problem 10. (10 points) Find a linear fractional transformation that takes the points $1, 2, 3$ to the points $0, 1, -1$, respectively.