MATH 417: MIDTERM I

This midterm is due Wednesday October 13, 2020 before class. You may use your class notes and the course text book. You may not use any other materials, including other text books, the web, question centers, etc. The work should be yours and yours alone. Please do not collaborate. There are 10 problems each worth 10 points.

Problem 1. Find all the solutions of the equation $z^8 = i$.

Problem 2. Find all the values of the following expressions and determine their principal values.

(1) $(8i)^{4i}$

(2) $\log\left(\frac{e^2}{\sqrt{2}} + \frac{e^2 i}{\sqrt{2}}\right)$

Problem 3. Determine whether the function $e^{-x}\cos(y) + x^5 - 10x^3y^2 + 5xy^4$ can be the real part of an analytic function. If so, find all analytic functions that have it as their real parts.

Problem 4. Calculate the integral

$$\int_C (\sin(z)e^{\sin(z^2)} + e^{z^2}\cos^3(z^2) - 3\overline{z})dz$$

where C is the positively oriented boundary of a rectangle with vertices $-2-2i, 2-2i, 2+2i, -2+2i$.

Problem 5. Estimate the integral

$$\int_{C_R} \frac{(z^4+7)\mathrm{Log}(z)}{z^6 - 3z^4 + 3z^2 - 1}dz$$

where C_R is the positively oriented half circle $Re^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$ (assume $R \gg 0$). Compute the limit of the integral as R tends to infinity.

Problem 6. Calculate the integral

$$\int_C \frac{e^{z^2}\cos(z)}{z^2}dz,$$

where C is the circle $|z| = 3$ positively oriented.

Problem 7. Calculate the integral

$$\int_C \frac{z^5+1}{(z+1)(z-i)(z-4i)}dz,$$

where C is the circle $|z| = 3$ positively oriented.

Problem 8. Let $f(z) = u(x,y) + iv(x,y)$ be an entire function with real and imaginary parts $u(x,y)$ and $v(x,y)$. Assume that the imaginary part is bounded $v(x,y) < M$ for every $z = x+iy$. Prove that f is a constant.

Problem 9. Assume that $f(z) : D \to \mathbb{C}$ is an analytic function in a domain D (caution: do not assume that f is entire). Suppose that the image of f in $\mathbb{C}$ is purely imaginary. Show that f must be constant.

Problem 10. Let $f(z)$ be an entire function. Assume that there exists a complex number z_0 and a positive number ϵ such that $|f(z) - z_0| > \epsilon > 0$ for every $z \in \mathbb{C}$ (in other words, the image of $f(z)$ misses a neighborhood of z_0). Show that $f(z)$ is a constant. In other words, the image of a nonconstant entire function comes arbitrarily close to any complex value.