MATH 320 FINAL EXAM

This is the final examination for MATH 320. You may use the textbook and your class notes but no other materials. To receive full credit you must show all your work and you must justify your claims.

Problem 1. (10 points) Find all the solutions of the system of linear equations

$$\begin{aligned} x - y + z - w &= 3 \\ x + y - z + w &= 3 \\ x + y + z - w &= 1 \end{aligned}$$

Problem 2. (15 points) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be the homomorphism given by the following matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & -8 & 3 & -1 \end{pmatrix}$$

(1) Find a basis for the kernel of h and determine the nullity of h
(2) Find a basis for the image of h and determine the rank of h.

Problem 3.(10 points) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ defined by

$$S = \{(x_1, x_2, x_3, x_4) \mid x_1 - x_2 + x_3 - x_4 = 0\}$$

under the usual inner product.

Problem 4. (10 points) Calculate the determinant of the following matrix.

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Determine whether A is singular.

Problem 5. (25 points) Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find a matrix that conjugates this matrix to a diagonal matrix and calculate A^{70}.

Problem 6. (10 points) Suppose the characteristic polynomial of a matrix is $(\lambda - 1)^2(\lambda - 2)^2$. Find all possible Jordan canonical forms up to conjugation. Determine the minimal polynomial of each Jordan canonical form.

Problem 7. (20 points) Decide whether the following statements are TRUE or FALSE. If the statement is true, provide a proof. If the statement is false, provide a counterexample. You will receive no credit if you do not provide justification for your answer. Let A^T denote the transpose of A.

(1) Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \neq \lambda_j$ if $i \neq j$) and corresponding eigenvectors $v_1, \ldots, v_n$. If B is an $n \times n$ matrix such that $AB = BA$, then $v_1, \ldots, v_n$ are also eigenvectors for B.
(2) The eigenvalues of A and A^T are equal.
(3) Let A be an 5×5 matrix with characteristic polynomial $\lambda^3(\lambda^2 - 1)$. A is diagonalizable if and only if the nullity of A is 3.
(4) Let A be an $n \times n$ special orthogonal matrix, i.e. $AA^T = I_n$ and $det(A) = 1$. If n is odd, then 1 is an eigenvalue of A.