MIDTERM MATH 320

There are 4 problems and two pages on this exam. You may use the book and your course notes, but no other resources. In order to receive credit, you must show all your work and give complete proofs of all your assertions.
GOOD LUCK!

Problem 1. (16 points) Find all solutions of the following system of equations.

\[ \begin{align*}
2x + 2y + 2z - w &= 2 \\
3x + 3y + z + 2w &= 4 \\
12x + 12y + 8z + w &= 14
\end{align*} \]

Problem 2. (20 points) Let \( h : \mathbb{R}^4 \to \mathbb{R}^3 \) be the homomorphism determined by the following matrix

\[
\begin{pmatrix}
3 & 2 & -2 & 1 \\
0 & 3 & -1 & -1 \\
6 & 3 & 3 & 3
\end{pmatrix}
\]

(1) Find a basis for the kernel of \( h \).
(2) Find a basis for the image of \( h \).
(3) Calculate the rank of \( h \).
(4) Calculate the nullity of \( h \).

Problem 3. (24 points) Determine with proof whether the following subsets \( S \) of \( P_3 \), the space of polynomials \( p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \) of degree at most 3 with real coefficients, are subspaces. If \( S \) is a subspace, find a basis and determine the dimension of \( S \).

(1) \( S = \{ p(x) \in P_3 \mid p(1) = 3p(0) \} \).

(2) \( S = \{ p(x) \in P_3 \mid \frac{d^2 p(x)}{dx^2} + \frac{dp(x)}{dx} = 0 \} \).

PLEASE TURN THE PAGE OVER
Problem 4. (40 points) For each of the following four statements, determine whether the statement is TRUE or FALSE. If the statement is true, give a proof. If the statement is false, give a counterexample. You will receive NO credit if you do not give a proof or a counterexample.

(1) A matrix $M$ is invertible if and only if its transpose $M^T$ is invertible.

(2) Let $M$ be an $n \times n$ diagonal matrix. Then the dimension of the space of solutions of a homogeneous linear system of equations with coefficient matrix $M$ is equal to the number of diagonal entries of $M$ that are equal to zero.

(3) Let $N$ be a $k \times n$ matrix of rank $r$. Let $M$ be an $m \times k$ matrix. The matrix $MN$ has rank $r$ if and only if $\text{NullSpace}(M) \cap \text{ColumnSpace}(N) = \{0\}$.

(4) A sequence

$$0 \longrightarrow V_1 \xrightarrow{h_1} V_2 \xrightarrow{h_2} V_3 \longrightarrow 0$$

is called a short exact sequence of finite dimensional vector spaces if $V_i$ are finite dimensional vector spaces and $h_i$ are homomorphisms such that $h_1 : V_1 \rightarrow V_2$ is one-to-one (injective), $h_2 : V_2 \rightarrow V_3$ is onto (surjective) and the kernel of $h_2$ equals the image of $h_1$. For a short exact sequence of finite dimensional vector spaces,

$$\dim(V_1) - \dim(V_2) + \dim(V_3) = 0.$$