1. MATH 494: Homework 3

This problem set is due Wednesday September 22. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding.

Problem 1.1. Determine which of the following affine varieties in $\mathbb{A}^2_{\mathbb{C}}$ are isomorphic. You must justify your answers.

(1) $V(y - x)$
(2) $V(y - x^3)$
(3) $V(y^2 - x^2)$
(4) $V(y^2 - x^3 - x^2)$.

Problem 1.2. Show that $\mathbb{A}^1 - \{0\}$ is an affine variety by showing that it is isomorphic to $xy = 1$ in $\mathbb{A}^2$. More generally, if $V(f)$ is the hypersurface in $\mathbb{A}^n$ defined by the polynomial f, show that $\mathbb{A}^n - V(f)$ is an affine variety. (Hint: Show that $\mathbb{A}^n - V(f)$ is isomorphic to $x_{n+1}f = 1$.) Conclude that the set of invertible $n \times n$ matrices is an affine variety.

Problem 1.3. Show that $X = \mathbb{A}^2 - \{(0,0)\} = \mathbb{A}^2 - V(x, y)$ is not isomorphic to an affine variety by carrying out the following steps.

(1) Show that any polynomial vanishing on X vanishes on $\mathbb{A}^2$.
(2) Conclude that the coordinate rings of X and $\mathbb{A}^2$ are isomorphic with isomorphism induced by inclusion.
(3) Using the correspondence between morphisms between k-algebras and morphisms between affine varieties, obtain a contradiction.

Problem 1.4. Find the irreducible components of $V(xyz)$ in $\mathbb{A}^3_{\mathbb{C}}$. Find the irreducible components of $V(xy, xz, yz)$ in $\mathbb{A}^3_{\mathbb{C}}$.

Problem 1.5. Let $f : X \to Y$ be a surjective morphism of affine varieties. Show that if X is irreducible, then Y is irreducible. Use this fact to check that $V(y - x^2, z - xy)$ in $\mathbb{A}^3_{\mathbb{C}}$ is irreducible (Hint: Think about Homework 2 Problem 2).