MATH 517 MIDTERM

This midterm is due Friday March 15, 2024 at 1 pm. You may use Aluffi and your class notes, but please do not collaborate or use other sources.

Problem 1. Do Problems VIII 1.20 and 1.21 from Aluffi.

Problem 2. Calculate the Galois group of the splitting field of $x^5 - 7$ over $\mathbb{Q}$ and over $\mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive fifth root of unity.

Problem 3. Calculate the Galois groups of the following polynomials over $\mathbb{Q}$

(1) $x^3 - x - 1$
(2) $x^3 - 3x - 1$
(3) $x^4 + 2x + 2$
(4) $x^4 + 8x + 12$
(5) $x^4 - 4x^2 + 2$
(6) $x^4 - 6x^2 + 7$

Problem 4. Count the number of irreducible monic polynomials of degree 4 over $\mathbb{F}_{25}$.

Problem 5. Let V be a 5 dimensional vector space over a field k with a basis $e_1, \ldots, e_5$. Determine whether

$$e_1 \wedge e_2 + e_1 \wedge e_3 + e_4 \wedge e_5$$

can be expressed as a pure wedge. If so, express it as the wedge of two vectors. Determine whether

$$e_1 \wedge e_3 + 2e_1 \wedge e_4 + e_1 \wedge e_5 + e_2 \wedge e_3 + 2e_2 \wedge e_4 + e_2 \wedge e_5 + 2e_3 \wedge e_4 + e_3 \wedge e_5$$

can be expressed as a pure wedge. If so, express it as a wedge of two vectors.

Problem 6. If $\dim(V) = 4$, show that every element of $\bigwedge^2 V$ can be expressed as a linear combination of two pure wedges. Show that if $\dim(V) \geq 8$, it is not possible to express every element of $\bigwedge^2 V$ as a linear combination of two pure wedges. (As a challenge, determine for which $4 < \dim V < 8$, one can express every element of $\bigwedge^2 V$ as a linear combination of two pure wedges.)

Problem 7. Compute the Hilbert functions of the following graded algebras (where the grading on the polynomial rings is the usual grading). In each case, find the polynomial that is equal to the Hilbert function if $m \gg 0$.

(1) $k[x, y, z]/(x, y)$
(2) $k[x, y, z]/(x, y^3 - z^3)$
(3) $k[x, y, z]/(x^2, xy, y^2)$
(4) $k[x, y, z, w]/(x^d)$, where $d > 0$ is an integer
(5) $k[x, y, z, w]/(x, y)$
(6) $k[x, y, z, w]/(xz - y^2, yw - z^2, xw - yz)$

Problem 8. Compute $\mathrm{Tor}^i_{\mathbb{Z}}(M, N)$ and $\mathrm{Ext}^i_{\mathbb{Z}}(M, N)$ for $i \geq 0$ when M and N are as follows:

(1) $M = \mathbb{Z} \oplus \mathbb{Z}$, $N = \mathbb{Z} \oplus \mathbb{Z}/2$
(2) $M = \mathbb{Z}/4$, $N = \mathbb{Z}/2$
(3) $M = \mathbb{Z}/a$, $N = \mathbb{Z}/b$

(4) $M = \mathbb{Z}/2 \oplus \mathbb{Z}/3$, $N = \mathbb{Z}/2 \oplus \mathbb{Z}/5$

Problem 9. Compute $\mathrm{Ext}^i_{\mathbb{Z}/15}(M, N)$ for all $i \geq 0$ when M and N are as follows:

(1) $M = N = \mathbb{Z}/5$
(2) $M = N = \mathbb{Z}/3$
(3) $M = \mathbb{Z}/5$, $N = \mathbb{Z}/3$.

Problem 10. This problem is optional, but I encourage you (especially those of you thinking of going into number theory or model theory) to attempt it. Let $K \subset L$ be an infinite algebraic extension. Then L is defied to be Galois over K if L is normal and separable over K. In this case, the Galois group $\mathrm{Gal}(L/K)$ is $\mathrm{Aut}_K(L)$. Assume that $K \subset L$ is Galois.

(1) Show that for any subfield $K \subset F \subset L$, the extension $F \subset L$ is Galois.
(2) Show that any finite set of elements $a_1, \ldots, a_m$ of L is contained in a finite Galois extension F of K.
(3) Assume that $K \subset F \subset L$ is a subfield such that F is Galois over K. Show that any element $\sigma \in \mathrm{Gal}(L/K)$ maps F into itself. Further show that the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K)$ obtained by restriction is surjective with kernel equal to $\mathrm{Gal}(L/F)$.
(4) Show that $L^{\mathrm{Gal}(L/K)} = K$.
(5) A subgroup H of $\mathrm{Gal}(L/K)$ is defined to be G-open if there exists a finite subextension $K \subset F \subset L$ such that $H = \mathrm{Gal}(L/F)$. Show that there exists a Hausdorff topology on $\mathrm{Gal}(L/K)$ such that the G-open subgroups are open in this topology, the multiplication map $\mathrm{Gal}(L/K) \times \mathrm{Gal}(L/K) \to \mathrm{Gal}(L/K)$ is continuous and the G-open subgroups form a fundamental system of neighborhood of the identity. Further show that the G-open subgroups are the only open subgroups and that every open subgroup is contained in a normal open subgroup.
(6) Set $G = \mathrm{Gal}(L/K)$ and for any finite subextension $K \subset F \subset L$ set $G_F = \mathrm{Gal}(F/K)$. Let $\mathcal{F}$ denote the set of finite subextensions. Show that G is isomorphic to
$$\lim_{F \in \mathcal{F}} G_F.$$
If we endow the latter with the topology induced by the product topology, show that this is a homeomorphism and conclude that G is compact (use Tychonoff's Theorem).
(7) Prove that for any subextension $K \subset F \subset L$, the group $\mathrm{Gal}(L/F)$ is a closed subgroup of $\mathrm{Gal}(L/K)$.
(8) Finally, conclude the main theorem of infinite Galois extensions: There is a one-to-one correspondence between subextensions $K \subset F \subset L$ and closed subgroups of the Galois group $\mathrm{Gal}(L/K)$.
(9) Calculate the Galois group of the extension $\mathbb{F}_p \subset \overline{\mathbb{F}}_p$. Give an example of a subgroup of the Galois group that does not correspond to a subextension.