HOMEWORK 1

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that \( k \) is an algebraically closed field and \( R \) is a commutative ring with unit.

Problem 0.1. Show that the following conditions on a ring \( R \) are equivalent.

1. Every ascending chain of ideals \( I_1 \subseteq I_2 \subseteq \cdots \) stabilizes.
2. Every ideal is finitely generated.
3. Every non-empty set of ideals contains a maximal element.

Rings satisfying these conditions are called Noetherian rings. Show that the ring of continuous real valued functions on the unit interval is not Noetherian.

Problem 0.2. Prove that if a ring \( R \) is Noetherian, then the formal power series ring \( R[[x]] \) over \( R \) is also Noetherian.

Problem 0.3. (1) Show that the union of the coordinate axes in \( \mathbb{A}^3_k \) is a closed algebraic set. Determine generators for its ideal.

(2) Consider the curve in \( \mathbb{A}^3_k \) given in parametric form \( C = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\} \). Determine generators for the ideal of \( C \).

(3) Consider the set \( \{(0,0), (1,1), (0,1), (1,0)\} \) of four points in \( \mathbb{A}^2_k \). Find generators for its ideal.

(4) Consider the set \( \{(0,0), (1,1), (2,2), (1,0)\} \) of four points in \( \mathbb{A}^2_k \). Find generators for its ideal. How does your answer differ from the previous part? What is special about these four points?

Problem 0.4. Consider the set \( V = \{(t^3, t^4, t^5) \mid t \in k\} \) in \( \mathbb{A}^3_k \). Show that \( V \) is an affine variety. Find generators of its ideal. How many generators do you need? Can \( V \) be described as the zero locus of two polynomials?

Problem 0.5. Let \( f \) be a polynomial in \( k[x_1, \ldots, x_n] \). Show that \( \mathbb{A}^n - V(f) \) can be realized as the affine variety \( V(x_{n+1}f - 1) \) in \( \mathbb{A}^{n+1}_k \). Conclude that the general linear group \( GL(n, k) \) (invertible \( n \times n \) matrices with entries in \( k \) under usual matrix multiplication) can be realized as an affine variety in \( \mathbb{A}^{n^2+1}_k \).

Problem 0.6. Let \( S = \mathbb{A}^2_k - \{(0,0)\} \) be the complement of the origin in \( \mathbb{A}^2_k \). Find \( I(S) \), the set of polynomials vanishing on \( S \). What is \( V(I(S)) \)? Can \( S \) be an affine variety?

Problem 0.7. Let \( X \subset \mathbb{A}^n \) and \( Y \subset \mathbb{A}^m \) be two affine varieties. Prove that \( X \times Y \subset \mathbb{A}^{n+m} \) is an affine variety.