HOMEWORK 2

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. *Consider the following five closed affine sets in $\mathbb{A}^2$. Give a thorough discussion of which among them are isomorphic.*

(1) $X_1 = \{(x, y) \in \mathbb{A}^2 \mid x = y\}$
(2) $X_2 = \{(x, y) \in \mathbb{A}^2 \mid x = y^{17}\}$
(3) $X_3 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2\}$
(4) $X_4 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^3\}$

Problem 0.2. *Prove the following statements:*

(1) *If X is an affine variety, then any non-empty Zariski open subset U of X is dense in X.*
(2) *If X is an affine variety, then any non-empty Zariski open subset U of X is irreducible.*
(3) *Let $f : X \to Y$ be a regular, surjective map of closed affine sets. If X is irreducible, then Y is irreducible.*

Problem 0.3. *Show that any two ordered sets of $n + 2$ points in general position in $\mathbb{P}^n$ are projectively equivalent. Show that two sets of four points in $\mathbb{P}^1$ are projectively equivalent if and only if their cross-ratios are equal. Harder: Characterize when $n + 3$ points in general linear position in $\mathbb{P}^n$ are projectively equivalent.*

Problem 0.4. *Let Γ be a set of points in $\mathbb{P}^n$ of cardinality d. Show that Γ can be expressed as the zero locus of polynomials of degree at most d. Show that if all the points in Γ do not lie on a line, then in fact Γ can be expressed as the zero locus of polynomials of degree $d - 1$ or less.*

Problem 0.5.

(1) *Show that the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric hypersurface in $\mathbb{P}^3$.*
(2) *Let L, M and N be three pairwise skew lines in $\mathbb{P}^3$. Show that union of all the lines in $\mathbb{P}^3$ intersecting L, M and N is isomorphic to the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$.*
(3) *How many lines in $\mathbb{P}^3$ intersect the four lines $L_1 = (z_1 = z_2 = 0)$, $L_2 = (z_3 = z_4 = 0)$, $L_3 = (z_1 = z_2, z_3 = z_4)$ and $L_4 = (z_1 + 2z_2 = z_3 + z_4, z_1 + 2z_4 = z_2 + z_3)$?*

Problem 0.6. *Recall that the twisted cubic curve C is the image of the map $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ given by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$*

(1) *Show that the homogeneous ideal is generated by $Q_1 : z_0 z_2 = z_1^2, Q_2 : z_1 z_3 = z_2^2, Q_3 : z_0 z_3 = z_1 z_2$.*
(2) *Show that an alternative way to describe the twisted cubic is as the rank one locus of the matrix*
$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix}$$
(3) *Show that $Q_i \cap Q_j$ for $i \neq j$ is C union a line.*