This problem set is due Friday September 15. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that \(k\) is an algebraically closed field and \(R\) is a commutative ring with unit.

**Problem 0.1.** Consider the following five closed affine sets in \(A^2\). Give a thorough discussion of which among them are isomorphic.

1. \(X_1 = \{(x, y) \in A^2 \mid x = y\}\)
2. \(X_2 = \{(x, y) \in A^2 \mid x = y^{17}\}\)
3. \(X_3 = \{(x, y) \in A^2 \mid x^2 = y^2\}\)
4. \(X_4 = \{(x, y) \in A^2 \mid x^2 = y^3\}\)

**Problem 0.2.** Prove the following statements:

1. If \(X\) is an affine variety, then any non-empty Zariski open subset \(U\) of \(X\) is dense in \(X\).
2. If \(X\) is an affine variety, then any non-empty Zariski open subset \(U\) of \(X\) is irreducible.
3. Let \(f : X \to Y\) be a regular, surjective map of closed affine sets. If \(X\) is irreducible, then \(Y\) is irreducible.

**Problem 0.3.** Show that any two ordered sets of \(n + 2\) points in general position in \(P^n\) are projectively equivalent. Show that two sets of four points in \(P^1\) are projectively equivalent if and only if their cross-ratios are equal. Harder: Characterize when \(n + 3\) points in general linear position in \(P^n\) are projectively equivalent.

**Problem 0.4.** Let \(\Gamma\) be a set of points in \(P^n\) of cardinality \(d\). Show that \(\Gamma\) can be expressed as the zero locus of polynomials of degree at most \(d\). Show that if all the points in \(\Gamma\) do not lie on a line, then in fact \(\Gamma\) can be expressed as the zero locus of polynomials of degree \(d - 1\) or less.

**Problem 0.5.**

1. Show that the Segre image of \(P^1 \times P^1\) is a quadric hypersurface in \(P^3\).
2. Let \(L, M\) and \(N\) be three pairwise skew lines in \(P^3\). Show that union of all the lines in \(P^3\) intersecting \(L, M\) and \(N\) is isomorphic to the Segre image of \(P^1 \times P^1\).
3. How many lines in \(P^3\) intersect the four lines \(L_1 = (z_1 = z_2 = 0), L_2 = (z_3 = z_4 = 0), L_3 = (z_1 = z_2, z_3 = z_4)\) and \(L_4 = (z_1 + 2z_2 = z_3 + z_4, z_1 + 2z_4 = z_2 + z_3)\)?

**Problem 0.6.** Recall that the twisted cubic curve \(C\) is the image of the map \(\nu : P^1 \to P^3\) given by \((x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)\)

1. Show that the homogeneous ideal is generated by \(Q_1 : z_0z_2 = z_1^2, Q_2 : z_1z_3 = z_2^2, Q_3 : z_0z_3 = z_1z_2\).
2. Show that an alternative way to describe the twisted cubic is as the rank one locus of the matrix

\[
\begin{pmatrix}
z_0 & z_1 & z_2 \\
z_1 & z_2 & z_3
\end{pmatrix}
\]

3. Show that \(Q_i \cap Q_j\) for \(i \neq j\) is \(C\) union a line.