

### HOMEWORK 3

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding.

**Problem 0.1.** Let  $V$  and  $W$  be two linear spaces of dimension  $k$  and  $m$ , respectively, in  $\mathbb{P}^n$ . Prove that if  $k + m \geq n$ , then  $V \cap W$  have non-empty intersection.

**Problem 0.2.** Using problem one, show that given

$$k \leq \binom{n+d}{d} - 1$$

points in  $\mathbb{P}^n$ , there exists a non-zero homogeneous polynomial of degree  $d$  in  $n+1$  variables vanishing on all  $k$  points. We say that the points impose independent conditions on polynomials of degree  $d$  in  $n+1$  variables if the codimension of the vector space of homogeneous polynomials of degree  $d$  vanishing on the  $k$  points is  $\min(k, \binom{n+d}{d})$ . Prove the following statements.

- (1)  $k \leq n+1$  points impose independent conditions on linear polynomials if and only if the points are in general linear position.
- (2) 4 points fail to impose independent conditions on degree two polynomials in three variables if and only if they lie on a line. (What is the relation to the problems in week 1?)
- (3) Let  $p_1, \dots, p_9$  be the points of intersection of two irreducible cubic polynomials in  $\mathbb{P}^2$ . Show that any cubic polynomial containing 8 of the points contains the ninth as well.

**Problem 0.3.** Prove the following statements.

- (1) Show that 4 points in  $\mathbb{P}^3$  fail to impose independent conditions on quadrics (degree two polynomials) if and only if they lie on a line. Does the same hold for  $\mathbb{P}^n$ ?
- (2) Show that 6 points that lie on a conic in  $\mathbb{P}^3$  fail to impose independent conditions on quadrics.
- (3) Show that 8 points that lie on a twisted cubic fail to impose independent conditions on homogeneous polynomials on quadrics. More generally, show that  $2n+2$  points that lie on a rational normal curve of degree  $n$  in  $\mathbb{P}^n$  fail to impose independent conditions on quadrics (degree two polynomials in  $n+1$  variables).

**Problem 0.4.** Let  $X = \nu_2(\mathbb{P}^2)$ , the second Veronese embedding of  $\mathbb{P}^2$ . Show that the hyperplane sections of  $X$  are either rational normal curves of degree 4 or the union of two conics intersecting at a point or a (double) conic. (When do we get a rational normal curve? When do we get a union of two conics/a (double) conic?)

**Problem 0.5.** Find the equations of the projection of the standard twisted cubic  $[x_0, x_1] \mapsto [x_0^3, x_0^2x_1, x_0x_1^2, x_1^3]$  from the points  $[1, 0, 0, 1]$  and  $[0, 1, 0, 0]$ . Harder: Show that any projection of a twisted cubic in  $\mathbb{P}^3$  from a point outside the twisted cubic is projectively equivalent to one of these two. Challenge: Show that the rational normal quartic in  $\mathbb{P}^4$  has smooth projections to  $\mathbb{P}^3$  (from points outside the curve) that are not projectively equivalent.