HOMEWORK 4

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. *A polynomial is completely reducible if it is a product of linear terms. Prove that the locus of completely reducible polynomials of degree d in $n+1$ variables in $\mathbb{P}^{\binom{n+d}{d}-1}$ is a projective variety. Similarly, prove that polynomials that are d-th powers of linear forms (L^d) form a projective variety. Prove that this variety is the d-th Veronese embedding of $\mathbb{P}^n$.*

Problem 0.2. *Let the dual projective space $\mathbb{P}^{n*}$ denote the space of hyperplanes in $\mathbb{P}^n$. Show that the universal hyperplane*

$$\Gamma = \{(H, p) : p \in H\} \subset \mathbb{P}^{n*} \times \mathbb{P}^n$$

(i.e. the pairs consisting of a hyperplane and a point contained in that hyperplane) is a projective variety. Prove, in fact, that it is a hyperplane section of the Segre variety $\mathbb{P}^{n*} \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$. What is its dimension? Let X be a projective variety. Prove that the universal hyperplane section of X defined as

$$\Gamma_X = \{(H, p) : p \in H \cap X\} \subset \mathbb{P}^{n*} \times \mathbb{P}^n$$

is a projective variety. Calculate the dimension of Γ_X in terms of the dimension of X and n.

Problem 0.3. *Let $\mathbb{P}^{\binom{n+d}{d}-1}$ denote the parameter space of hypersurfaces of degree d in $n+1$ variables. Show that the universal hypersurface*

$$\Omega_d = \{(F, p) : F(p) = 0\} \subset \mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{P}^n$$

is a projective variety. What is this variety when $d = 1$? Calculate the dimension of Ω_d.

Problem 0.4. *For this problem assume that $k = \mathbb{C}$. We say that a hypersurface defined by a polynomial $F = 0$ is singular at a point p if*

$$F(p) = F_{x_0}(p) = \cdots = F_{x_n}(p) = 0$$

F and all its first order partial derivatives vanish at p. Prove that a quadratic polynomial in $n+1$ variables is singular if and only if the determinant of the associated symmetric matrix is zero.

Problem 0.5. *Show that the locus of homogeneous polynomials of degree d in $n+1$ variables that have a singular point is a projective variety. Show that it has codimension one in the space of all polynomials of degree d in $n+1$ variables. Hence, it can be described as the zero locus of a single polynomial in $\binom{n+d}{d}$ variables. Show that if $d = 2$, then the degree of this polynomial is $n+1$. Challenge: What is the degree of this polynomial for arbitrary d?*

Problem 0.6. *Show that a general hypersurface of degree $d > 2n - 3$ in $\mathbb{P}^n$ does not contain any lines. Generalize this statement from lines to linear spaces of higher dimension.*