HOMEWORK 8

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In this problem set assume that all the varieties are defined over the complex numbers.

Problem 0.1. *Consider the closed algebraic set X in $\mathbb{G}(1,3) \times \mathbb{G}(1,3) \times \mathbb{G}(1,3)$ consisting of triples of lines in $\mathbb{P}^3$ any two of which intersect:*

$$X := \{(l_1, l_2, l_3) \in \mathbb{G}(1,3) \times \mathbb{G}(1,3) \times \mathbb{G}(1,3) \mid l_i \cap l_j \neq \emptyset\}$$

Determine the number of irreducible components of X and the dimension of each of the irreducible components. Is X smooth? Is X normal?

Problem 0.2. *Prove that a smooth, projective variety of dimension k admits an embedding in $\mathbb{P}^{2k+1}$.*

Problem 0.3. *Let $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the Veronese surface in $\mathbb{P}^5$ given by*

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

Consider the projection of X from the point $(0,0,0,0,0,1)$ to $\mathbb{P}^5$ given by

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2).$$

Is the image of this projection normal? (Remark: Zariski's Main Theorem asserts that if $f : X \to Y$ is a regular and birational map between projective varieties and Y is normal, then the fibers $f^{-1}(y)$ of f are connected. What is the relevance to this exercise?)

Problem 0.4. *Prove that the quadric cone $x_1^2 + \cdots + x_n^2$ in $\mathbb{A}^n$ $(n \geq 3)$ is normal.*

Problem 0.5. *A projective variety $X \subset \mathbb{P}^n$ is called projectively normal if its homogeneous coordinate ring $S(Y)$ is integrally closed. Show that if Y is projectively normal, then Y is normal. Show that the converse does not in general hold by considering the quartic curve C given as the image of*

$$(t, u) \mapsto (t^4, t^3u, tu^3, u^4)$$

in $\mathbb{P}^3$. Show that C is normal, but not projectively normal.