HOMEWORK 9

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding.

Problem 0.1. *Calculate the degree of the Grassmannian of lines in* $\mathbb{P}^n$ *in its Plücker embedding. (Hint: You may want to use Pieri's formula and learn about Catalan numbers.)*

Problem 0.2. *Show that a variety* X *and a general hyperplane section* $X \cap H$ *have the same degree. Calculate the degree of the surface scroll in* $\mathbb{P}^4$ *defined by the rank one locus of the matrix*

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

Problem 0.3. *Let* $a \leq b$ *be two positive integers. The surface scroll* $S_{a,b}$ *is constructed as follows. Pick two disjoint linear spaces* $\mathbb{P}^a$ *and* $\mathbb{P}^b$ *in* $\mathbb{P}^{a+b+1}$*. Fix rational normal curves* C_a *and* C_b *of degrees* a *and* b *in the* $\mathbb{P}^a$ *and* $\mathbb{P}^b$*, respectively, and an isomorphism* $\phi : C_a \to C_b$*.* $S_{a,b}$ *is the surface obtained by taking the union of the lines joining* $p \in C_a$ *with* $\phi(p)$ *in* C_b*. Prove that* $S_{a,b}$ *is an algebraic surface. Describe in detail* $S_{1,1}$*. Show that the surface in the previous exercise is* $S_{1,2}$*. We can also allow* $a = 0$ *and* ϕ *to be the constant map. In that case the surface is a cone over a rational normal curve. Calculate the degree of the surface scroll* $S_{a,b}$*.*

Problem 0.4. *Let* X *be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree* d *and dimension* k *in* $\mathbb{P}^n$*. Show that* $d \geq n - k + 1$*. Show that for rational normal curves, the Veronese surface* $v_2(\mathbb{P}^2)$ *in* $\mathbb{P}^5$ *and surface scrolls* $S_{a,b}$ *equality holds. Challenge: Classify the varieties where equality holds.*

Problem 0.5. *Prove that the every automorphism of projective space* $\mathbb{P}^n$ *is induced by an automorphism* $\phi \in GL(n+1)$ *of* k^{n+1}*. In other words, the automorphism group of* $\mathbb{P}^n$ *is* $\mathbb{P}GL(n+1)$*.*