MATH 553: FINAL PROBLEM SET

For the rest of the semester, we will devote Fridays to solving problems. These problems collectively will constitute the final problem set of the course. Please email me a written copy of your solutions by Wednesday April 29 at noon. Throughout assume that all the curves and surfaces are defined over an algebraically closed field of characteristic 0 or sufficiently large characteristic.

Please do the following problems by Friday April 3.

Problem 1. Compute the genus of the unique nonsingular projective model of the following curves:

(1) $y^2 = x^7 - 1$
(2) $y^3 = x^5 - 1$

Problem 2. Compute the genus of a smooth, complete intersection curve of degrees $d_1, \dots, d_{n-1}$ in $\mathbb{P}^n$. Deduce that a smooth curve of genus 8 cannot be realized as a complete intersection in projective space.

Problem 3. Prove that a genus 2 curve cannot be realized as a smooth, complete intersection curve in projective space.

Problem 4. Prove that there does not exist a nonsingular curve of degree 9 and genus 11 in $\mathbb{P}^3$.

Please do the following problems by Friday April 10.

Problem 5. Show that a line, a conic, a twisted cubic curve and an elliptic quartic curve in $\mathbb{P}^3$ have no multi-secant lines. Prove that every other smooth, irreducible, projective curve in $\mathbb{P}^3$ has infinitely many multisecant lines.

Problem 6. Show that any two smooth, nondegenerate rational curves of degree 4 in $\mathbb{P}^4$ are projectively equivalent. Show that a rational curve of degree 4 in $\mathbb{P}^3$ is always contained in a unique smooth quadric surface. The class of the curve on the quadric is $(1,3)$. Show that two such curves need not be projectively equivalent (Hint: Think of the cross-ratio of the branch points of the degree 3 projection to $\mathbb{P}^1$).

Problem 7. Show that a rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface. Exhibit examples of such curves which are contained in a quadric surface and exhibit examples which are not contained in a quadric surface.

Problem 8. Show that the smallest degree embedding of a genus 2 curve has degree 5. Show that a genus 2 degree 5 curve has to lie on a quadric surface. Does the quadric surface have to be smooth?

Please do the following problems by Friday April 17.

Problem 9. Show that the Hilbert scheme of n points on $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^n$

Problem 10. Describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(\mathbb{P}^3)$ as explicitly as possible.

Problem 11. Show that the Hilbert scheme of twisted cubics $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ has at least 2 components. The general point of one component parameterizes twisted cubics. The general point of the other component parameterizes a plane cubic curve union a point. Compute the dimension of each of these components. Where do these components intersect?

Problem 12. Describe the Hilbert scheme $\mathrm{Hilb}_{3t}(\mathbb{P}^3)$ as explicitly as possible.

Please do the following problems by Friday April 23.

Problem 13. Prove that $\mathrm{Hilb}_{96}(\mathbb{C}^3)$ is reducible. (Hint: Consider ideals $I = (V, m^8)$, where m is the maximal ideal at the origin and V is a 24-dimensional vector space of homogeneous polynomials of degree 7)

Problem 14. Prove that $\mathrm{Hilb}_8(\mathbb{C}^4)$ is reducible. (Hint: Consider ideals $I = (V, m^3)$, where m is the maximal ideal at the origin and V is a 7-dimensional vector space of homogeneous polynomials. Check that the Hilbert scheme is smooth at such a point if V is spanned by $x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4+x_2x_3$.)