THE HILBERT SCHEME

Many important moduli spaces can be constructed as quotients of the Hilbert scheme by a group action. For example, to construct the moduli space of smooth curves of genus $g \geq 2$, we can first embed all smooth curves of genus g in $\mathbb{P}^{n(2g-2)-g}$ by a sufficiently large multiple of their canonical bundle K_C^n. Any automorphism of a variety preserves the canonical bundle. Hence, two n-canonically embedded curves are isomorphic if and only if they are projectively equivalent. If we had a parameter space for n-canonically embedded curves, then the moduli space of curves would be a quotient of this parameter space by the projective linear group. The Hilbert scheme parameterizes subschemes of projective space with a fixed Hilbert polynomial, thus provides the starting point for all such constructions.

We will take up the construction of the moduli space of curves in the next section. In this section, we sketch a construction of the Hilbert scheme and give many explicit examples.

1. The construction of the Hilbert scheme

We assume that all our schemes are Noetherian over an algebraically closed field k. Let $X \to S$ be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed polynomial. Recall that the Hilbert functor

$$Hilb_P(X/S) : \{\text{Schemes}/S\}^o \to \{\text{sets}\}$$

associates to an S scheme Y the subschemes of $X \times_S Y$ which are proper and flat over Y and have the Hilbert polynomial P.

A fundamental theorem of Grothendieck asserts that the Hilbert functor $Hilb_P(X/S)$ is representable by a projective scheme.

Theorem 1.1. *Let X/S be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed polynomial. The functor $Hilb_P(X/S)$ is represented by a morphism*

$$u : U_P(X/S) \to Hilb_P(X/S).$$

$Hilb_P(X/S)$ is projective over S.

I will now explain the main ideas of the proof. There are many excellent accounts of Theorem 1.2. My presentation will follow closely [Gr], [Mum2], [K] and [Se].

1.1. **The strategy.** Let us first concentrate on the case $X = \mathbb{P}^n$ and $S = Spec(k)$. A subscheme of projective space is determined by its equations. The polynomials in $k[x_0, \dots, x_n]$ that vanish on a subscheme form an infinite-dimensional subvector space of $k[x_0, \dots, x_n]$. Of course, each subscheme Z is defined by finitely many polynomials and Z is determined by the polynomials of degree d vanishing on Z for a sufficiently large degree. This is now a finite dimensional vector subspace of the vector space of polynomials of degree d. Suppose the degree d of the polynomials needed to determine Z were bounded depending only on the Hilbert polynomial of Z. Further suppose that the ideal sheaf $I_Z(k)$ did not have any higher cohomology for $k \geq k_0$ for some k_0 again depending only on the Hilbert polynomial of Z. Then the schemes with Hilbert polynomial P would all be determined by a finite dimensional vector space of polynomials and furthermore all these vector spaces would have the same dimension. We would thus get an injection of the schemes with Hilbert polynomial P into a Grassmannian. We have already seen that the Grassmannian (together with its tautological bundle) represents the functor classifying subspaces of a vector space. Assuming the image in the Grassmannian is an algebraic subscheme Y, we can use Y and the restriction of the tautological bundle to represent the Hilbert functor. This is exactly the strategy we will follow.

1.2. **Bounding the regularity of an ideal sheaf and constructing the Hilbert scheme as a subset of a Grassmannian.** Given a proper subscheme Y of $\mathbb{P}^n$ and a coherent sheaf F on Y, by Serre's Theorem, the higher cohomology $H^i(Y, F(m))$, $i > 0$, vanishes for m sufficiently large. The finiteness that we are looking for comes from the fact that if we restrict ourselves to ideal sheaves of subschemes with a fixed Hilbert polynomial P, we can find an integer m depending only on P (and not on the subscheme) that works simultaneously for the ideal sheaf of every subscheme with Hilbert polynomial P.

Theorem 1.2. *For every polynomial P, there exists an integer m_P depending only on P such that for every subsheaf $I \subset \mathcal{O}_{\mathbb{P}^n}$ with Hilbert polynomial P and every integer $k \geq m_P$*

1. $h^i(\mathbb{P}^n, I(k)) = 0$ *for* $i > 0$*;*
2. $I(k)$ *is generated by global sections;*
3. $H^0(\mathbb{P}^n, I(k)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, I(k+1))$ *is surjective.*

Exercise 1.3. For each $a \geq 0$, let $E_a = \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ be a vector bundle of rank 2 and degree 0 on $\mathbb{P}^1$. Show that there does not exist

k such that $h^1(\mathbb{P}^1, E_a(k)) = 0$ for all $a \geq 0$. Show that there does not exist k such that $E_a(k)$ is globally generated for all $a \geq 0$. This example shows that we will need to use the assumption that I is an ideal sheaf in the proof of Theorem 1.3. Show that E_a cannot be a subbundle of a fixed vector bundle E for all $a \geq 0$.

Theorem 1.3 enables us to construct a subset of a Grassmannian that parameterizes all the ideal sheaves with Hilbert polynomial P. Let $Y \subset \mathbb{P}^n$ be a closed subscheme with Hilbert polynomial P. Choose $k \geq m_P$. By Theorem 1.3 (2), $I_Y(k)$ is generated by global sections. Consider the exact sequence

$$0 \to I_Y(k) \to \mathcal{O}_{\mathbb{P}^n}(k) \to \mathcal{O}_Y(k) \to 0.$$

This realizes $H^0(\mathbb{P}^n, I_Y(k))$ as a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. Since $I_Y(k)$ is globally generated, this subspace determines $I_Y(k)$ and hence I_Y) and the subscheme Y. Furthermore, by Theorem 1.3 (1), the higher cohomology of $I_Y(k)$ vanishes. Therefore, $h^0(\mathbb{P}^n, I_Y(k)) = P(k)$. Since k depends only on the Hilbert polynomial, we can find a subset of $G(P(k), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ parameterizing subschemes of $\mathbb{P}^n$ with Hilbert polynomial P. Of course, we will need to put a scheme structure on this subset and show that it represents the Hilbert functor, but we begin by proving Theorem 1.3.

We first introduce a definition due to Mumford that streamlines the cohomology calculations.

Definition 1.4. A coherent sheaf F on $\mathbb{P}^n$ is called *Castelnuovo-Mumford m-regular* or simply *m-regular* if $H^i(\mathbb{P}^n, F(m-i)) = 0$ for all $i > 0$.

Exercise 1.5. Show that $\mathcal{O}_{\mathbb{P}^n}(a)$ is m-regular for all $m \geq -a$.

Exercise 1.6. Show that the ideal sheaf of a rational normal curve is m-regular for all $m \geq 1$. Show that the ideal sheaf of a smooth, elliptic curve of degree d in $\mathbb{P}^{d-1}$ is m-regular for all $m \geq 3$.

Exercise 1.7. Let C be a canonically embedded smooth curve of genus g in $\mathbb{P}^{g-1}$. Show that the ideal sheaf of C is m-regular for $m \geq 4$.

The following proposition is the main technical tool in the proof of Theorem 1.3.

Proposition 1.8. *If F is an m-regular coherent sheaf on $\mathbb{P}^n$, then*

(1) *$h^i(\mathbb{P}^n, F(k)) = 0$ for $i > 0$ and $k + i \geq m$.*
(2) *$F(k)$ is generated by global sections if $k \geq m$.*

(3) $H^0(\mathbb{P}^n, F(k)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, F(k+1))$ *is surjective if* $k \geq m$.

Proof. The proof proceeds by induction on the dimension n. When $n = 0$, the result is clear. Take a general hyperplane H defined by a global section s of $\mathcal{O}_{\mathbb{P}^n}(1)$ and consider the natural exact sequence

$$0 \to F(k-1) \to F(k) \to F_H(k) \to 0$$

obtained by multiplication by s. When $k = m - i$, the associated long exact sequence of cohomology gives that

$$H^i(F(m-i)) \to H^i(F_H(m-i)) \to H^{i+1}(F(m-i-1)).$$

In particular, if F is m-regular on $\mathbb{P}^n$, then both $H^i(F(m-i)) = H^{i+1}(F(m-i-1)) = 0$. We conclude that the sheaf F_H on $\mathbb{P}^{n-1}$ is also m-regular. Now we can prove (1) by induction on k. Consider the similar long exact sequence

$$H^{i+1}(F(m-i-1) \to H^{i+1}(F(m-i)) \to H^{i+1}(F_H(m-i)).$$

$H^{i+1}(F(m-i-1) = 0$ for $i > 0$ by the assumption that F is m-regular. $H^{i+1}(F_H(m-i)) = 0$ for $i > 0$ by the induction hypothesis.We conclude that F is $m+1$ regular. By induction on k, F is k-regular for all $k > m$. This proves (1).

Consider the commutative diagram

$$\begin{array}{ccc} H^0(F(k-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{u} & H^0(F_H(k-1)) \otimes H^0(\mathcal{O}_H(1)) \\ \downarrow g & & \downarrow f \\ H^0(F(k-1)) \xrightarrow{t} H^0(F(k)) & \xrightarrow{v} & H^0(F_H(k)) \end{array}$$

When $k = m+1$, by the regularity assumption, the map u is surjective. The map f is surjective by induction on dimension. It follows that $v \circ g$ is also surjective. Notice that the image of $H^0(F(k-1))$ under the map t is contained in the image of g. It follows that g has to be surjective and we conclude (3).

Part (2) of the theorem is a straightforward consequence of (3). $\square$

Proof of Theorem 1.3. By Proposition **??**, Theorem 1.3 will be proved if we can show that the ideal sheaves of proper subchemes of $\mathbb{P}^n$ with Hilbert polynomial P are m_P-regular for an integer m_P depending only on P. We prove this claim by induction on the dimension n. Choose a general hyperplane H and consider the exact sequence

$$(*) \quad 0 \to I(m) \to I(m+1) \to I_H(m+1) \to 0.$$

I_H is a sheaf of ideals so we may use induction on dimension.

Assume the Hilbert polynomial P is given by

$$P(m) = \sum_{i=0}^{n} a_i \binom{m}{i}.$$

We then have

$$\begin{aligned} \chi(I_H(m+1)) &= \chi(I(m+1)) - \chi(I(m)) \\ &= \sum_{i=0}^{n} a_i \left(\binom{m+1}{i} - \binom{m}{i} \right) \\ &= \sum_{i=0}^{n-1} a_{i+1} \binom{m}{i} \end{aligned}$$

By the induction hypothesis, there exists an integer m_1 depending only on the Hilbert polynomial of I_H such that I is m_1-regular for every ideal sheaf with the same Hilbert polynomial. The integer m_1 depends only on the coefficients $a_1, \dots, a_n$ of the Hilbert polynomial P. The long exact sequence associated to the short exact sequence $(*)$

$$H^{i-1}(I_H(m+1)) \to H^i(I(m)) \to H^i(I(m+1)) \to H^i(I_H(m+1))$$

shows that if $m \geq m_1 - i$ and $i \geq 2$, then $H^i(\mathbb{P}^n, I(m)) \cong H^i(\mathbb{P}^n, I(m+1))$. By Serre's theorem, $H^i(\mathbb{P}^n, I(k)) = 0$ for $i > 0$ and k sufficiently large. We conclude that $H^i(\mathbb{P}^n, I(k)) = 0$ for $k \geq m_1 - i$ and $i \geq 2$ since these groups are all isomorphic in this range and vanish for k sufficiently large. When $i = 1$, we can only conclude that the map

$$H^1(\mathbb{P}^n, I(m)) \to H^1(\mathbb{P}^n, I(m+1))$$

is surjective for $m \geq m_1 - 1$. If this map is an isomorphism for some m', then it follows that it is an isomorphism for $m \geq m'$ and by the same argument as for $i \geq 2$ the groups vanish. Hence, we can assume that $h^1(I(m))$ is a strictly decreasing function for $m \geq m_1 - 1$. Hence $H^1(\mathbb{P}^n, I(m)) = 0$ for $m \geq h^1(\mathbb{P}^n, I(m_1 - 1))$. We conclude that I is $m_1 + h^1(I(m_1 - 1))$-regular. So far our argument did not use that I is an ideal sheaf. We need this assumption to bound $h^1(I(m_1 - 1))$. Using the exact sequence

$$0 \to I(m) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_Z(m) \to 0,$$

we get

$$\begin{aligned} h^1(I(m_1 - 1)) &= h^0(I(m_1 - 1)) - \chi(I(m_1 - 1)) \\ &\leq h^0(\mathcal{O}_{\mathbb{P}^n}(m_1 - 1)) - \chi(I(m_1 - 1)). \end{aligned}$$

This bound clearly depends only on the Hilbert polynomial P; hence concludes the proof of Theorem 1.3. ☐

1.3. **Endowing the Hilbert scheme with a scheme structure.** In the previous subsection, we have given an injection from the set of subshemes of $\mathbb{P}^n$ with Hilbert polynomial P to the Grassmannian $G(P(m), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$ for any $m \geq m_P$ by sending the subscheme to the $P(m)$-dimensional subspace $H^0(\mathbb{P}^n, I(m))$ of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$. This subspace uniquely determines the subscheme. We now need to show that the image has a natural scheme structure and that this subscheme represents the Hilbert functor. For this purpose we will use flattening stratifications.

Definition 1.9. A *stratification* of a scheme S is a finite collection $S_1, \dots, S_j$ of locally closed subschemes of S such that

$$S = S_1 \sqcup \cdots \sqcup S_j$$

is a disjoint union of these subschemes.

Theorem 1.10. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n \times S$. Let S and T be Noetherian schemes. There exists a stratification of S such that for all morphisms $f : T \to S$, $(1 \times f)^*\mathcal{F}$ to $\mathbb{P}^n \times T$ is flat over T if and only if the morphism factors through the stratification.*

The stratification produced in Theorem **??** is called the flattening stratification. We will now sketch a proof of Theorem **??** following Lecture 8 in [Mum2].

Sketch of the proof of Theorem **??**. First assume that $n = 0$. Let $\mathcal{F}$ be a sheaf on S. Since the sheaf $f^*\mathcal{F}$ is flat on T if and only if it is locally free over T, we have to find a stratification of S such that over each stratum $\mathcal{F}$ is locally free. Let $k(s)$ denote the residue field at $s \in S$. For $s \in S$, let

$$e(s) = \dim_{k(s)}(\mathcal{F}_s \otimes_{\mathcal{O}_s} k(s)).$$

Let $s \in S$ be a point with $e(s) = e$. Pick a basis of $\mathcal{F}_s \otimes_{\mathcal{O}_s} k(s)$. Then in a sufficiently small neighborhood U of s, the natural map $\mathcal{O}_S^e \xrightarrow{\phi} \mathcal{F}$ is surjective and the kernel of ϕ may be assumed to be generated by global sections over U. We thus get an exact sequence

$$\mathcal{O}_S^f \xrightarrow{\psi} \mathcal{O}_s^e \xrightarrow{\phi} \mathcal{F} \to 0.$$

The dimension function $e(s)$ is upper semi-continuous. Therefore, the sets $S_e = \{s \in S \mid e(s) = e\}$ are locally closed. Furthermore, $S_e \cap U$ is endowed with a natural scheme structure defined by requiring the entries $\psi_{i,j}$ to vanish. Now it is clear that $f^*\mathcal{F}$ is locally free on T if and only if f factors through the stratification given by S_e.

The strategy is to reduce the general case to this case. To achieve this reduction first observe that only finitely many polynomials occur as the Hilbert polynomials of $\mathcal{F}_s$. This follows from generic flatness: Let $f : X \to Y$ be a morphism of schemes with Y integral. Let $\mathcal{F}$ be a sheaf over X. Then $\mathcal{F}$ is flat over a Zariski open set $U \subset Y$. By Noetherian induction, it follows that Y can be stratified into finitely many locally closed sets so that over each set $\mathcal{F}$ is flat. Hence only finitely many polynomials occur as Hilbert polynomials of $\mathcal{F}_s$. In particular, by Theorem 1.3, we can choose a uniform m such that for all $k \geq m$ the higher cohomology of $\mathcal{F}_s(k)$ vanishes.

Exercise 1.11. Prove generic flatness by carrying out the following steps.

(1) Reduce to the case when $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, where B is an A-algebra and $\mathcal{F} = \tilde{M}$ for a B module M.
(2) Prove that it suffices to show that there is an element $f \in A$ such that M_f is a free A_f-module. (Note, in fact, that this argument will show generic local freeness and not just generic flatness.) In the rest of the exercise, prove this statement by a series of reductions.
(3) By using composition series, reduce to the case $M = B$ and B is an integral domain.
(4) Pass to the fraction fields of A and B and use induction on the transcendence degree to conclude the proof.

Consider the diagram

$$\begin{array}{ccc} \mathbb{P}^n \times T & \stackrel{1\times f}{\to} & \mathbb{P}^n \times S \\ \downarrow q & & \downarrow p \\ T & \stackrel{f}{\to} & S \end{array}$$

Notice that $(1 \times f)^*\mathcal{F}$ is flat over T if and only if $q_*(1 \times f)^*\mathcal{F}(k)$ is locally free for $k \geq m$. We have thus reduced checking the flatness to checking that a collection of sheaves is locally free.

By the first step of the proof, we know how to stratify S so that each of the sheaves $\mathcal{F}(k)$ are locally free over S. The apparent problem now is that we have to stratify S so that infinitely many sheaves are simultaneously locally free. Given two stratifications $S = \sqcup Y_i = \sqcup Z_j$, we can define a common refinement of the two stratifications $S = \sqcup S_{i,j}$ by letting $S_{i,j} = Y_i \cap Z_j$, where the intersection is the scheme theoretic intersection of Y_i and Z_j. The flattening stratification is the simultaneous common refinement of all the stratifications of $\mathcal{F}(k)$ for $k \geq m$.

More precisely, let Y_e^k be the stratum of the flattening stratification of $\mathcal{F}(k)$ on which $\mathcal{F}(k)$ is locally free of rank e. Let $P_1, \ldots, P_r$ be the polynomials that occur as the Hilbert polynomials of $\mathcal{F}_s$. Then set

$$S_i = \bigcap_{k=m}^{\infty} Y^k_{P_i(k)}.$$

The intersection defining S_i stabilizes after finitely many k. First, the support of S_i is determined by taking the intersection of at most $n+1$ of the schemes $Y^k_{P_i(k)}$:

$$\bigcap_{k=m}^{\infty} \mathrm{Supp}(Y^k_{P_i(k)}) = \bigcap_{k=m}^{m+n} \mathrm{Supp}(Y^k_{P_i(k)}).$$

This follows by noting that the Hilbert polynomials P_j have degree at most n. If the values of P_j and P_l agree at $n+1$ points, then $P_j = P_l$. Then we have a descending chain of ideals with fixed support, so the scheme structures also stabilize after finitely many k. We thus obtain a well-defined stratification of S into locally closed subschemes on which $\mathcal{F}(k)$ are all simultaneously locally free. This gives us the flattening stratification. $\square$

The flattening stratification allows us to put a scheme structure on the image of our map to the Grassmannian. More precisely, consider the incidence correspondence

$$I \subset \mathbb{P}^n \times G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P))).$$

The incidence correspondence has two projections

$$\pi_1 : I \to \mathbb{P}^n$$

and

$$\pi_2 : I \to G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P))).$$

For the rest of this section we will abbreviate $G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P)))$ simply by G. $\pi_2^* T(-m_P)$ where T is the tautological bundle on G is an idea sheaf of $\mathcal{O}_{\mathbb{P}^n \times G}$. Let us denote the corresponding subscheme by Y. The flattening stratification of $\mathcal{O}_Y$ over G gives a subscheme H_P of G corresponding to the Hilbert polynomial P. (Note that this is the scheme structure that we put on the set we earlier obtained.) The claim is that H_P represents the Hilbert functor and the universal family is the restriction W of Y to the inverse image of H_P.

Suppose we have a subscheme $X \subset \mathbb{P}^n \times S$ mapping to S via f and flat over S (and suppose the Hilbert polynomial is P). We obtain an exact sequence

$$0 \to f_* I_X(m_P) \to f_* \mathcal{O}_{\mathbb{P}^n \times S}(m_P) \to f_* \mathcal{O}_X(m_P) \to 0.$$

By the universal property of the Grassmannian G, this induces a map $g : S \to G$. Since

$$f_* I_X(m) = g^* \pi_{2*} I_Y(m)$$

for m sufficiently large, we see that $(1 \times g)^* \mathcal{O}_Y$ is flat with Hilbert polynomial P, hence g factors through H_P by the definition of the flattening stratification. Moreover, X is simply $S \times_{H_P} W$. This concludes the construction of $Hilb_P(\mathbb{P}^n/S)$.

Exercise 1.12. Verify the details of the above construction.

So far we have constructed the Hilbert scheme as a quasi-projective subscheme of the Grassmannian. To prove that it is projective it suffices to check that it is proper. This is done by checking the valuative criterion of properness. This follows from the following proposition [Ha] III.9.8.

Proposition 1.13. *Let X be a regular, integral scheme of dimension one. Let $p \in X$ be a closed point. Let $Z \subset \mathbb{P}^n_{X-p}$ be a closed subscheme flat over $X - p$. Then there exists a unique closed subscheme $\overline{Z} \in \mathbb{P}^n_X$ flat over X, whose restriction to $\mathbb{P}^n_{X-p}$ is Z.*

Exercise 1.14. Deduce from the proposition that the Hilbert scheme we constructed is projective.

Exercise 1.15. For a projective scheme X/S construct $Hilb_P(X/S)$ as a locally closed subscheme of $Hilb_P(\mathbb{P}^n/S)$.

Exercise 1.16. Suppose X and Y are projective schemes over S. Assume X is flat over S. Let $Hom(X, Y)$ be the functor that associates to any S scheme T the set of morphisms

$$X \times_S T \to Y \times_S T.$$

Using our construction of the Hilbert scheme and noting that a morphism may be identified with its graph construct a scheme that represents the functor $Hom(X, Y)$.

1.4. **The local structure of the Hilbert scheme.** In this subsection, we determine the Zariski tangent space to the Hilbert scheme $Hilb_P(X)$ at a subscheme Y. The best reference for this section is [K]. Kollár also discusses at length obstruction groups. Unfortunately, we will not cover this very important topic here.

Theorem 1.17. *Let X be a projective scheme over a field k and $Y \subset X$ be a closed subscheme with Hilbert polynomial P, then the Zariski tangent space to $Hilb_P(X)$ at $[Y]$ is naturally isomorphic to $Hom_Y(I_Y/I_Y^2, \mathcal{O}_Y)$. In particular, if X and Y are both smooth projective varieties, then the tangent space to the Hilbert scheme is given by $H^0(Y, N_{Y/X})$ the global sections of the normal bundle of Y in X.*

Proof. Let $i : \mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ be the morphism induced by the natural ring of maps $k[\epsilon]/(\epsilon^2) \to k$. The Zariski tangent space to $Hilb_P(X)$ at Y is given by morphisms

$$\phi : \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \to Hilb_P(X)$$

such that the image of $\phi \circ i$ is $[Y]$. Since the Hilbert functor is representable, these morphisms are in one-to-one correspondence with flat families of schemes over $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ such that the fiber over the image of i is Y. We will now characterize such morphisms and show that they are parameterized by $\mathrm{Hom}_Y(I_Y/I_Y^2, \mathcal{O}_Y)$.

We may reduce the calculation to the affine case. Our construction will then globalize to yield the result. Let $X = \mathrm{Spec}(A)$ for a finitely generated k-algebra A and let $Y = \mathrm{Spec}(B)$ with $B = A/I$ for some ideal I of A. Let $\tilde{A} = A \otimes_k k[\epsilon]/(\epsilon^2)$. Let $\tilde{B} = \tilde{A}/\tilde{I}$, where $(\tilde{I} + \epsilon A)/\epsilon A = I$. Our task is to characterize $\tilde{B}$ that are flat over $k[\epsilon]/(\epsilon^2)$. Flatness can be checked by considering ideals. The only non-zero proper ideal of $k[\epsilon]/(\epsilon^2)$ is the ideal (ϵ). Hence $\tilde{B}$ is a flat $k[\epsilon]/(\epsilon^2)$-module if and only if $\epsilon \otimes \tilde{B} = \epsilon B$. This holds if and only if $\epsilon \tilde{a} \in \tilde{I}$ implies that $\tilde{a} = \epsilon \tilde{x} (\mathrm{mod}\ I)$. We conclude that $\tilde{B}$ is flat over $k[\epsilon]/(\epsilon^2)$ if and only if $\epsilon A \cap \tilde{I} = \epsilon I$. Given $\tilde{j} \in \tilde{I}$, then $\tilde{j} = i + \epsilon a$, where a is determined modulo I. Given $i \in I$, the residue class of a modulo I such that $i + \epsilon a \in \tilde{I}$ is uniquely determined. We see that $\tilde{I}$ is then determined by a homomorphism $\phi : I \to A/I$. Hence, we can identify the flat $\tilde{B}$ over $k[\epsilon]/(\epsilon^2)$ with $\mathrm{Hom}_A(I, B) = \mathrm{Hom}(I/I^2, B)$. Globalizing we see that the Zariski tangent space to the Hilbert scheme at Y is given by homomorphisms $\phi : I_Y/I_Y^2 \to \mathcal{O}_Y$. □

Exercise 1.18. Let $Z \subset \mathbb{P}^n$ be a complete intersection of hypersurfaces of degree $d_1, \ldots, d_r$. Describe the Zariski tangent space to the Hilbert scheme at the point Z and calculate its dimension.

2. Examples of Hilbert schemes

In this section, we will give explicit examples of Hilbert schemes.

2.1. **Hilbert schemes of points.** Let X be a smooth projective variety. We begin by considering Hilbert schemes of zero dimensional subschemes of X. Let $P = n$ be a positive integer. $Hilb_n(X)$ can be a very complicated scheme even when X is $\mathbb{P}^n$.

Exercise 2.1. Show that $Hilb_1(X)$ is isomorphic to X.

The n-th symmetric product $X^{(n)}$ of X is the quotient of the product X^n under the action of the symmetric group $\mathfrak{S}_n$ interchanging the factors of X^n.

Exercise 2.2 (Zero dimensional subschemes on a smooth curve)**.** Show that if C is a smooth curve, then $Hilb_n(C)$ is isomorphic to $C^{(n)}$. In particular, conclude that $Hilb_n(\mathbb{P}^1) \cong \mathbb{P}^n$.

Exercise 2.3. Suppose C is an irreducible curve with a node. Describe $Hilb_2(C)$. Is $Hilb_2(C)$ isomorphic to $C^{(2)}$?

Example 2.4 (Zero dimensional subschemes on a smooth surface)**.** The Hilbert scheme of points on smooth surfaces provide beautiful examples of smooth projective varieties.

Theorem 2.5 (Fogarty)**.** *Let X be a smooth, projective surface over an algebraically closed field k. Then $Hilb_n(X)$ is a smooth, irreducible projective variety of dimension $2n$. The Hilbert scheme admits a Hilbert-Chow morphism*

$$hc : Hilb_n(X) \to X^{(n)}.$$

$Hilb_n(X)$ is a small resolution of the symmetric product $X^{(n)}$.

Example 2.6 (Iarrabino's Examples)**.** In the previous examples, we saw that the Hilbert scheme of points on smooth curves and surfaces behave in the expected manner. These examples are the exception rather than the rule. Let X be a smooth projective variety of dimension $d \geq 3$. Then $Hilb_n(X)$ contains an irreducible component whose general point parameterizes a scheme Z consisting of n-reduced distinct points.

Exercise 2.7. Show that the locus in $Hilb_n(X)$ parameterizing schemes with n-reduced distinct points is smooth and has dimension dn. Show that this locus is isomorphic to the complement of the diagonals in the symmetric product $X^{(n)}$. Deduce that the closure of this locus is an irreducible component of $Hilb_n(X)$. We will call this component the *expected component.*

Even $Hilb_n(\mathbb{P}^3)$ may have components other than the expected component. One way to obtain other components of $Hilb_n(\mathbb{P}^3)$ is to use

Iarrabino's construction. It is more convenient to work with $Hilb_n(\mathbb{A}^3)$. The schemes obtained by this construction are supported at one point. Let I be the ideal in $k[x,y,z]$ generated by $\mathfrak{m}^k$ and an r dimensional subspace W of homogeneous polynomials of degree $k-1$, where $\mathfrak{m}$ is the maximal ideal at the origin. Then the support of I is clearly the origin. The Hilbert polynomial of I is the dimension of the vector space $k[x,y,z]/I$. We can choose a basis for $k[x,y,z]/I$ by taking all monomial of degree less than $k-1$ and a basis complementary to W among polynomials of degree $k-1$. Hence the dimension of $k[x,y,z]/I$ is

$$n = \sum_{i=0}^{k-2} \frac{(i+2)(i+1)}{2} + \frac{k(k+1)}{2} - r.$$

We thus get a point in $Hilb_n(\mathbb{A}^3)$. We can vary I by changing the subspace W and the supporting point. Hence, this locus has dimension at least $\dim G(r, \frac{k(k+1)}{2}) + 3 = r(\frac{k(k+1)}{2} - r) + 3$. When k is large enough, this dimension can become larger than $3n$, thus, producing irreducible components of $Hilb_n(\mathbb{A}^3)$ of dimension larger than that of the expected component. For example, let $k = 8$ and $r = 24$. Then we get $n = 96$. The dimension of the expected component is 288. Whereas, the locus we have constructed has dimension at least 291. We conclude that $Hilb_{96}(\mathbb{A}^3)$ is reducible.

Exercise 2.8. Let X be a smooth, projective variety of dimension d. Assume that $Hilb_n(X)$ has an irreducible component other than the expected component of dimension at least dn. Show that for $m \geq n$, $Hilb_m(X)$ is reducible. In particular, conclude that $Hilb_n(X)$ is reducible for any threefold X if $n \geq 96$.

Exercise 2.9. Use Iarrabino's construction to conclude that $Hilb_n(X)$ is reducible for any fourfold if $n \geq 21$, any five fold if $n \geq 11$ and any six fold if $n \geq 10$. Deduce that $Hilb_n(X)$ is reducible if $\dim(X) \geq 6$ and $n \geq 10$.

Example 2.10. The Hilbert scheme of points $Hilb_n(X)$ may also have components that have smaller dimension than the dimension of the expected component. Consider $Hilb_8(\mathbb{A}^4)$. The expected component in this case has dimension 32. However, this Hilbert scheme has another component of dimension 25. Consider an ideal I generated by $\mathfrak{m}^3$ and a 7-dimensional subspace W of quadratic polynomials. Then $\dim(k[x,y,z,w]/I) = 25$. The dimension of the locus of ideals of this form is $\dim(G(7,10)) + 4 = 25$. Of course, it could be that this locus is contained in a larger dimensional component, possibly the expected component. To see that this is not the case, it suffices to show that the

Zariski tangent space to $Hilb_8(\mathbb{A}^4)$ at a general point has dimension 25. It then follows that this locus forms an irreducible component.

Exercise 2.11. Let W be a general 7-dimensional subspace of quadratic polynomials. Show that the Zariski tangent space to $Hilb_8(\mathbb{A}^4)$ has dimension 25. It is probably best to do this exercise using a computer package such as Macaulay2. Use a random polynomial generator to generate W and then calculate the dimension of the Zariski tangent space to $Hilb_8(\mathbb{A}^4)$ at I to see that this dimension is 25.

Exercise 2.12. Show that $Hilb_n(X)$ is reducible for $n \geq 8$ and $\dim(X) \geq 4$.

Remark 2.13. It is known that $Hilb_n(\mathbb{A}^m)$ is irreducible if $n \leq 7$. Hence, when $m \geq 4$, $Hilb_n(\mathbb{A}^m)$ is reducible if and only if $m \geq 8$. The techniques of this subsection can be optimized to show that $Hilb_n(\mathbb{A}^3)$ is reducible if $n \geq 78$. At present we do not know whether $Hilb_n(\mathbb{A}^3)$ is irreducible or not when $9 < n < 78$. More importantly, currently there does not seem to be a good way of testing whether a given zero dimensional scheme is in the closure of the expected component.

2.2. **Hilbert schemes of linear spaces and hypersurfaces.**

Example 2.14 (The Grassmannian)**.** Let

$$P_r(m) = \binom{m+r}{r}$$

be the Hilbert polynomial of a linear space $\mathbb{P}^r \subset \mathbb{P}^n$. By Bezout's Theorem, any subscheme of $\mathbb{P}^n$ with Hilbert polynomial P_r is a linear space in $\mathbb{P}^n$. By Exercise **??**, $Hilb_{P_r}(\mathbb{P}^n)$ is smooth. By the universal property of the Grassmannian $\mathbb{G}(r,n) = G(r+1, n+1)$, $Hilb_{P_r}(\mathbb{P}^n)$ admits a morphism

$$\phi : Hilb_{P_r}(\mathbb{P}^n) \to \mathbb{G}(r,n).$$

The morphism ϕ is a bijection on points, therefore, by Zariski's Main Theorem, an isomorphism. We conclude that $Hilb_{P_r}(\mathbb{P}^n) \cong \mathbb{G}(r,n)$.

Exercise 2.15 (The Hilbert scheme of hypersurfaces)**.** Let

$$P_{d,n}(m) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

Show that any subscheme of $\mathbb{P}^n$ with Hilbert polynomial $P_{d,n}$ is a hypersurface of degree d in $\mathbb{P}^n$. In particular, the ideal of such a scheme is generated by a single polynomial of degree d. Prove that the Hilbert scheme $Hilb_{P_{d,n}}(\mathbb{P}^n)$ is smooth and irreducible. Conclude that $Hilb_{P_{d,n}}(\mathbb{P}^n)$ is isomorphic to $\mathbb{P}^{\binom{n+d}{d}-1}$ parameterizing homogeneous polynomials of degree d in $n+1$ variables modulo scalars.

Example 2.16 (The orthogonal Grassmannian)**.** Let V be an n dimensional vector space. Let Q be a non-degenerate quadratic form on V. A subspace W of V is called *isotropic* with respect to Q if the restriction of Q vanishes on W. Let the *orthogonal Grassmannian* $OG(k,n)$ be the variety parameterizing k-dimensional Q-isotropic subspaces of V.

Exercise 2.17. Show that $OG(k,n)$ is a smooth, projective variety of dimension
$$\dim(OG(k,n)) = \frac{k(2n-3k-1)}{2}.$$
Show that every irreducible component of $OG(k,n)$ is homogeneous under the natural action of $SO(n)$.

Geometrically, $Q=0$ defines a smooth, quadric hypersurface in $\mathbb{P}V$. $OG(k,n)$ parameterizes linear spaces $\mathbb{P}^{k-1}$ contained in the hypersurface $Q=0$. Let $\mathbb{P}W$ be a linear space on $Q=0$. Consider the following short exact sequence of cohomology
$$0 \to N_{\mathbb{P}W/Q} \to N_{\mathbb{P}W/\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}V}(2)|_{\mathbb{P}W} \to 0.$$
Since $N_{\mathbb{P}W/\mathbb{P}V} \cong \mathcal{O}_{\mathbb{P}W}(1)^{\oplus n-k}$ and $h^1(N_{\mathbb{P}W/Q}) = 0$, by the long exact sequence of cohomology, we conclude that
$$h^0(\mathbb{P}W, N_{\mathbb{P}W/Q}) = \frac{k(2n-3k-1)}{2}.$$
It follows that the Hilbert scheme $Hilb_{P_{k-1}}(Q)$ is smooth, where P_{k-1} is the Hilbert polynomial of a linear space $\mathbb{P}^{k-1}$.

Exercise 2.18. Show that $Hilb_{P_{k-1}}(Q) \cong OG(k,n)$.

We now specialize to the case $n=2k$. $OG(k,2k)$ has two connected components. The cohomology group $H^{2k-2}(Q,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The two components of $OG(k,2k)$ are distinguished by the cohomology class of the linear spaces they parameterize. We conclude that the Hilbert scheme $Hilb_{P_{k-1}}(Q)$ is not connected for a smooth quadric hypersurface $Q \subset \mathbb{P}^{2k-1}$.

The previous example is in sharp contrast to Hilbert schemes of subspaces of $\mathbb{P}^n$. A celebrated theorem of Hartshorne asserts that the Hilbert schemes of subschemes in $\mathbb{P}^n$ are always connected.

Theorem 2.19 (Hartshorne's Theorem)**.** *Let P be the Hilbert polynomial of a subscheme of $\mathbb{P}^n$. Then $Hilb_P(\mathbb{P}^n)$ is connected.*

Example 2.20 (The Fano scheme of linear spaces)**.** Let X be a hypersurface of degree d in $\mathbb{P}^n$. Let P_r be the Hilbert polynomial of a linear space of dimension r. $Hilb_{P_r}(X)$ is classically called the *Fano scheme* of r-dimensional linear spaces.

Exercise 2.21. • Let X be a smooth cubic hypersurface in $\mathbb{P}^n$, $n \geq 4$. Show that the Fano scheme of lines $Hilb_{m+1}(X)$ is smooth and irreducible of dimension $2n-6$.

• Let X be a general hypersurface in $\mathbb{P}^n$ of degree $d \leq 2n-3$. Show that the Fano scheme of lines $Hilb_{m+1}(X)$ is smooth and of dimension $2n-3-d$.

• Let X be the Fermat quartic threefold defined by $Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4 = 0$. Show that every line on X is contained in one of the hyperplane sections $Z_i = \omega Z_j$ where ω is a fourth root of unity. Conclude that the Fano scheme of lines on X has 40 irreducible components. Calculate the class of the Fano scheme of lines on X in the Grassmannian $\mathbb{G}(1,4)$. Show that the automorphism group permutes these irreducible components. Conclude that each of these irreducible components are everywhere non-reduced. This example should convince you that the scheme structure on a Hilbert scheme can be very subtle.

• Find a smooth hypersurface X of degree $d > n$ for which

$$\dim(Hilb_{m+1}(X)) > 2n - 3 - d.$$

Problem 2.22. There are many open problems even for the Fano scheme of lines on hypersurfaces. For example, even the dimensions of these schemes are not always known. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $d \leq n$. Then a the Debarre-de Jong Conjecture asserts that the dimension of $Hilb_{2m+1}(X)$ is $2n-d-3$. The previous exercise asks you to verify this statement for a general hypersurface. However, the conjecture remains open for every smooth hypersurface. Prove the conjecture for $d \leq 5$. By results of Beheshti and Landsberg, Robles, the conjecture is known for $d = 6$. Currently it is open for $d > 6$.

Exercise 2.23. Let X be a general hypersurface of degree d. Determine the dimension of $Hilb_{P_r}(X)$.

2.3. **Some Hilbert schemes of curves.** We now turn to some examples of Hilbert schemes of curves in projective space.

Example 2.24 (The Hilbert scheme of conics in $\mathbb{P}^3$)**.** In this example, we analyze the Hilbert scheme $Hilb_{2m+1}(\mathbb{P}^3)$. The Hilbert polynomial of a smooth conic curve is $2m+1$. We need to determine what other subschemes of $\mathbb{P}^3$ have the same Hilbert polynomial. Towards this aim let's classify double line structures. Without loss of generality we may assume that the support of the line L is given by $x = y = 0$. The ideal of the double line must contain I_L^2. However, note that the Hilbert polynomial of the ideal $< x^2, xy, y^2 >$ is $3m + 1$. Therefore, the ideal must contain at least one more polynomial. This polynomial modulo I_L^2 maybe chosen of the form $xF(z, w) + yG(z, w)$. If the degrees of F and G are d, then the ideal $< x^2, xy, y^2, xF(z, w) + yG(z, w) >$ has Hilbert polynomial $2m + 1 + d$. We conclude that a double line with Hilbert polynomial $2m + 1$ is planar.

Exercise 2.25. Prove that any subscheme of $\mathbb{P}^3$ with Hilbert polynomial $2m + 1$ is a complete intersection of a linear polynomial H and a quadratic polynomial Q.

Exercise 2.26. Prove that the ideal of any double line of arithmetic genus $-d$ without embedded points has the form $< x^2, xy, y^2, xF(z, w)+ yG(z, w) >$ for some polynomials $F(z, w), G(z, w)$ of degree d.

Exercise 2.27. Show that $Hilb_{2m+1}(\mathbb{P}^3)$ is smooth and irreducible of dimension 8. Let U be the tautological bundle of $(\mathbb{P}^3)^*$. Use the universal property of the Hilbert scheme to show that $\mathbb{P}(Sym^2U^*)$ admits a morphism to $Hilb_{2m+1}(\mathbb{P}^3)$. Using the previous exercise, show that this morphism is a bijection on points. Use Zariski's Main Theorem to conclude that

$$Hilb_{2m+1}(\mathbb{P}^3) \cong \mathbb{P}(Sym^2U^*).$$

We can use the Hilbert scheme of conics to solve enumerative questions about conics. Suppose now that the ground field is $\mathbb{C}$.

Question 2.28. How many conics in $\mathbb{P}^3$ intersect 8 general lines in $\mathbb{P}^3$?

As in the case of Schubert calculus, we can try to calculate this number as an intersection in the cohomology ring. The cohomology ring of a projective bundle over a smooth variety is easy to describe in terms of the Chern classes of the bundle and the cohomology ring of the variety.

Theorem 2.29. *Let E be a rank n vector bundle over a smooth, projective variety X. Suppose that the Chern polynomial of E is given by $\sum c_i(E)t^i$. Let S be the tautological line bundle of $\mathbb{P}E$. Let $\zeta = c_1(S^*)$. The cohomology of $\mathbb{P}E$ is isomorphic to*

$$H^*(\mathbb{P}E) \cong \frac{H^*(X)\ [\zeta]}{< \zeta^n + \zeta^{n-1}c_1(E) + \cdots + c_n(E) = 0 >}$$

Theorem **??** allows us to compute the cohomology ring of $Hilb_{2n+1}(\mathbb{P}^3)$. Recall that U^* on $\mathbb{P}^{3*}$ is a rank 3 vector bundle with Chern polynomial

$$c(U^*) = 1 + h + h^2 + h^3.$$

Using the splitting principle, we can assume that $c(U^*)$ splits into three linear factors

$$(1+x)(1+y)(1+z).$$

Then the Chern polynomial of $Sym^2(U^*)$ is given by

$$(1+2x)(1+2y)(1+2z)(1+x+y)(1+x+z)(1+y+z).$$

Multiplying this out and expressing it interms of the elementary symmetric polynomials in x, y, z, we see that

$$c(Sym^2(T^*)) = 1 + 4h + 10h^2 + 20h^3.$$

It follows that the cohomology ring of $Hilb_{2n+1}(\mathbb{P}^3)$ is given by

$$H^*(Hilb_{2n+1}(\mathbb{P}^3)) \cong \frac{\mathbb{Z}[h,\zeta]}{< h^4,\ \zeta^6 + 4h\zeta^5 + 10h^2\zeta^4 + 20h^3\zeta^3 >}$$

The class of the locus of conics interseting a line l is given by $2h + \zeta$. Since this locus is a divisor, its class can be checked by a calculation away from codimension at least 2. Consider the locus of planes in $\mathbb{P}^{3*}$ that do not contain the line l. Over this locus, there is a line bundle that associates to each point (H, Q) on $Hilb_{2n+1}(\mathbb{P}^3)$ the homogeneous quadratic polynomials modulo those that vanish at $H \cap l$. This line bundle is none other than the pull-back of $\mathcal{O}_{\mathbb{P}^{3*}}(2)$. The tautological bundle over $Hilb_{2n+1}(\mathbb{P}^3)$ maps to this bundle by evaluation. The locus where the evaluation vanishes is the locus of conics that intersect l. Hence, the class of the locus of conics that intersect l is the difference of the first chern classes of $\pi^*(c_1(\mathcal{O}_{\mathbb{P}^{3*}}(2))) - c_1(U) = 2h + \zeta$.

Exercise 2.30. Verify that the locus of conics intersecting a line l has class $2h + \zeta$ by the method of undetermined coefficients. Let A be the class of a pencil of conics contained in a fixed plane. Let B be the class of conics obtained by intersecting a fixed quadric surface by a pencil of planes. Show that $A \cdot h = 0, A \cdot \zeta = 1$ and $B \cdot h = 1, B \cdot \zeta = 0$. In other words, A, B give a dual basis to h, ζ. Use A, B to calculate the class of conics intersecting the line l.

Finally, to find the number of conics that intersect 8 general lines, we compute $(2h + \zeta)^8$ using the presentation of the cohomology ring. Note that $\zeta^5 h^3 = 1$ since h^3 corresponds to a point on $\mathbb{P}^{3*}$ and ζ restricted to a fiber of $\pi : \mathbb{P}Sym^2 U^* \to \mathbb{P}^{3*}$ is the class of a hyperplane. Multiplying the relation $\zeta^6 + 4h\zeta^5 + 10h^2\zeta^4 + 20h^3\zeta^3 = 0$ by h^2 and solving, we see that $\zeta^6 h^2 = -4$ Continuing to solve inductively, we conclude that $\zeta^7 h = 6, \zeta^8 = -4$. Finally, an easy calculation shows that $(2h+\zeta)^8 = 92$.

We can invoke Kleiman's Transversality Theorem to deduce that there are 92 smooth conics intersecting 8 general lines in $\mathbb{P}^3$. $Hilb_{2m+1}(\mathbb{P}^3)$ is not a homogeneous variety. However, $\mathbb{P}GL(4)$ has a dense open orbit on $Hilb_{2m+1}(\mathbb{P}^3)$ because any two non-singular conics in $\mathbb{P}^3$ are projectively equivalent. Since any conic intersecting 8 general lines is non-singular, by Kleiman's Transversality Theorem applied to the locus of non-singular conics in $Hilb_{2m+1}(\mathbb{P}^3)$, we conclude that the intersections of the cycles are transverse.

Exercise 2.31. Show that given 8 general lines the only conics intersecting all 8 are smooth conics.

Exercise 2.32. Calculate the number of conics that intersect $8-2i$ general lines and contain i general points for $0 \leq i \leq 3$.

Exercise 2.33. Calculate the class of the locus of conics that are tangent to a plane in $\mathbb{P}^3$. Find the number of conics that are tangent to a general plane and intersect 7 general lines.

Exercise 2.34. Generalize the previous discussion to conics in $\mathbb{P}^n$. Show that $Hilb_{2m+1}(\mathbb{P}^n)$ is isomorphic to $\mathbb{P}Sym^2(U^*)$, where U is the tautological bundle of $\mathbb{G}(2,n)$.

Exercise 2.35. Calculate the cohomology ring of $Hilb_{2m+1}(\mathbb{P}^4)$. Determine the number of conics that intersect $11-2i-3j$ general planes, i general lines and j general points in $\mathbb{P}^4$.

Exercise 2.36. Let $Q \subset \mathbb{P}^5$ be a smooth quadric hypersurface. Show that $Hilb_{2m+1}(Q)$ is the blow-up of the Grassmannian $G(3,6)$ along the orthogonal Grassmannian $OG(3,6)$

$$Hilb_{2m+1}(Q) \cong \mathrm{Bl}_{OG(3,6)} G(3,6).$$

In particular, conclude that the rank of the Picard group of $Hilb_{2m+1}(Q)$ is three. Show that the Picard group is generated by $\mathcal{O}_{G(3,6)}(1)$ and the two exceptional divisors E_1, E_2 of the blow-up lying over the two connected components of $OG(3,6)$.

More generally, let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Show that

$$Hilb_{2m+1}(Q) \cong \mathrm{Bl}_{OG(3,n+1)} G(3,n+1).$$

Conclude that when $n > 5$, the Picard group of $Hilb_{2m+1}(Q)$ has rank two.

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