THE MODULI SPACE OF CURVES

In this section, we will give a sketch of the construction of the moduli space $\overline{\mathcal{M}}_g$ of curves of genus g and the closely related moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed curves of genus g using two different approaches. Throughout this section we always assume that $2g-2+n>0$. In the first approach, we will embed curves in a projective space $\mathbb{P}^N$ using a sufficiently high power n of their dualizing sheaf. Then a locus in the Hilbert scheme parameterizes n-canonically embedded curves of genus g. The automorphism group $\mathbb{P}GL(N+1)$ acts on the Hilbert scheme. Using Mumford's Geometric Invariant Theory, one can take the quotient of the appropriate locus in the Hilbert scheme by the action of $\mathbb{P}GL(N+1)$ to construct $\overline{\mathcal{M}}_g$. In the second approach, one first constructs the moduli space of curves as a Deligne-Mumford stack. One then exhibits an ample line bundle on the coarse moduli scheme of this stack.

1. Basics about curves

We begin by collecting basic facts and definitions about stable curves. A curve singularity (C,p) is called a *node* if locally analytically the singularity is isomophic to the plane curve singularity $xy=0$. A curve C is called *at-worst-nodal* or more simply *nodal* if the only singularities of C are nodes.

The dualizing sheaf ω_C of an at-worst-nodal curve C is an invertible sheaf that has a simple description. Let $\nu : C^\nu \to C$ be the normalization of the curve C. Let $p_1,\ldots,p_\delta$ be the nodes of C and let $\{r_i,s_i\}=\nu^{-1}(p_i)$. Then ω_C associates to an open subset U of C, rational differentials η on $\nu^{-1}(U)$ having at worst simple poles at r_i,s_i lying over the points $p_i\in U$ such that

$$\mathrm{Res}_{r_i}(\eta)+\mathrm{Res}_{s_i}(\eta)=0$$

for every pair r_i,s_i. If C is a connected, nodal curve of arithmetic genus g, then ω_C has degree $2g-2$ and $h^0(C,\omega_C)=g$.

Definition 1.1. A *stable* curve C of genus g is a connected, complete, at-worst-nodal curve of arithmetic genus g such that ω_C is ample.

Exercise 1.2. Let C be a connected, complete, at-worst-nodal curve of arithmetic genus $g\geq 2$. Show that the following three conditions are equivalent.

(1) ω_C is ample.
(2) If C_i^ν is a genus zero component of the normalization of C, then C_i^ν has at least three points mapped to nodes of C by ν.
(3) The automorphism group of C is finite.

Definition 1.3. An *n-pointed stable* curve $(C, p_1, \dots, p_n)$ of genus g is a connected, complete, at-worst-nodal curve C of arithmetic genus g and n distinct, smooth points $p_1, \dots, p_n \in C$ such that $\omega_C(\sum_{i=1}^n p_i)$ is ample. The points p_i are called *marked points*.

An isomorphism between marked curves

$$(C, p_1, \dots, p_n) \text{ and } (C', p_1', \dots, p_n')$$

is an isomorphism $\phi : C \to C'$ such that $\phi(p_i) = p_i'$ for $1 \le i \le n$.

Exercise 1.4. Let $(C, p_1, \dots, p_n)$ be an n-pointed, connected, complete, at-worst-nodal curve of arithmetic genus g such that $2g-2+n > 0$. Show that the following three conditions are equivalent.
(1) $\omega_C(\sum_{i=1}^n p_i)$ is ample.
(2) If C_i^ν is a genus zero component of the normalization of C, then C_i^ν has at least three points mapped to nodes or marked points of C by ν.
(3) The automorphism group of $(C, p_1, \dots, p_n)$ is finite.

Now we are ready to define several closely related functors.

Definition 1.5. The functor $\mathcal{M}_g$ associates to every S-scheme X, isomorphism classes of families $\pi : C \to X$ such that π is flat of relative dimension one and for every closed point $x \in X$, $\pi^{-1}(x)$ is a smooth curve of genus g.

The functor $\overline{\mathcal{M}}_g$ associates to every S-scheme X, isomorphism classes of families $\pi : C \to X$ such that π is flat of relative dimension one and for every closed point $x \in X$, $\pi^{-1}(x)$ is a stable curve of genus g.

The functor $\mathcal{M}_{g,n}$ associates to every S-scheme X, isomorphism classes of families $\pi : C \to X$ together with n-sections $s_1, \dots, s_n$ of π such that π is flat of relative dimension one and for every closed point $x \in X$, $(\pi^{-1}(x), s_1(x), \dots, s_n(x))$ is a smooth n-pointed curve of genus g.

Finally, the functor $\overline{\mathcal{M}}_{g,n}$ associates to every S-scheme X, isomorphism classes of families $\pi : C \to X$ together with n-sections $s_1, \dots, s_n$ of π such that π is flat of relative dimension one and for every closed point $x \in X$, $(\pi^{-1}(x), s_1(x), \dots, s_n(x))$ is a stable n-pointed curve of genus g. In each of these cases, the functor associates to a morphism $f : Y \to X$ of S-schemes, the pull-back family f^*C.

Unlike the functors we have studied so far, none of these functors are representable. The main obstruction to representing $\mathcal{M}_g$ is the existence of curves with non-trivial automorphisms. Let C be a curve with a non-trivial automorphism ϕ. Then we can construct an isotrivial family of curves which is not trivial. Recall that a family $\pi : \mathcal{C} \to X$ is *isotrivial* if the fibers of π are isomorphic to a fixed curve C. However, an isotrivial family does not have to be isomorphic to the trivial family $X \times C \to X$ given by projection to the first factor. If the functor were representable by a scheme M, then each family over X would correspond to a morphism $f_\pi : X \to M$ such that the family is the pull-back of the universal family over M by the morphism f_π. However, both for the isotrivial family and the trivial family the morphism induced from X to M has to be the constant morphism with image the point representing the isomorphism class of C. Therefore, $\mathcal{M}_g$ cannot be representable. The next example explicitly constructs an isotrivial family which is not isomorphic to a product.

Example 1.6. Fix a hyperelliptic curve C of genus g. Let τ denote the hyperelliptic involution of C. Let S be any variety with a fixed-point-free involution. For concreteness, we can take S to be a $K3$-surface with a fixed point free involution i such that S/i is an Enriques surface E. If you would like to write down explicit equations, let C be the normalization of the plane curve defined by the equation $y^2 = p(x)$, where $p(x)$ is a polynomial of degree $2g+2$ with no repeated roots. The hyperelliptic involution is given by $(x, y) \mapsto (x, -y)$. Let Q_1, Q_2, Q_3 be three general ternary quadratic forms. Let the $K3$-surface S be defined by the vanishing of the three polynomials $Q_i(x_0, x_1, x_2) + Q_i(x_3, x_4, x_5) = 0$ with the involution that exchanges the triple (x_0, x_1, x_2) with (x_3, x_4, x_5). Consider the quotient of $C \times S$ by the fixed-point free involution $\tau \times i$. The quotient is a non-trivial family over the Enriques surface E; however, every fiber is isomorphic to C. If $\mathcal{M}_g$ were finely represented by a scheme, then this family would correspond to a morphism from E to it. However, this morphism would have to be constant since the moduli of the fibers is constant. The trivial family would also give rise to the constant family. Hence, $\mathcal{M}_g$ cannot be finely represented.

If $\pi : C \to S$ is a stable curve of genus g over a scheme S, then C has a relative dualizing sheaf $\omega_{C/S}$ with the following properties

(1) The formation of $\omega_{C/S}$ commutes with base change.
(2) If $S = \mathit{Spec}\ k$ where k is an algebraically closed field and $\tilde{C}$ is the normalization of C, then $\omega_{C/S}$ may be identified with the sheaf of meromorphic differentials on $\tilde{C}$ that are allowed to have

simple poles only at the inverse image of the nodes subject to the condition that if the points x and y lie over the same node then the residues at these two points must sum to zero.

(3) In particular, if C is a stable curve over a field k, then $H^1(C, \omega_{C/k}^{\otimes n}) = 0$ if $n \geq 2$ and $\omega_{C/k}^{\otimes n}$ is very ample for $n \geq 3$. When $n = 3$ we obtain a tri-canonical embedding of stable curves to $\mathbb{P}^{5g-6}$ with Hilbert polynomial $P(m) = (6m-1)(g-1)$.

To see the third property observe that every irreducible component E of a stable curve C either has arithmetic genus 2 or more, or has arithmetic genus one but meets the other components in at least one point, or has arithmetic genus 0 and meets the other components in at least three points. Since $\omega_{C/k} \otimes \mathcal{O}_E$ is isomorphic to $\omega_{E/k}(\sum_i Q_i)$ where Q_i are the points where E meets the rest of the curve. Since this sheaf has positive degree it is ample on each component E of C, hence it is ample. $\omega_{E/k}(\sum_i Q_i)$ has positive degree on each component, hence $\omega_{C/k}^{1-n} \otimes \mathcal{O}_E$ has no sections for any $n \geq 2$. By Serre duality, it follows that $H^1(C, \omega_{C/k}^{\otimes n}) = 0$. To show that when $n \geq 3$, $\omega_{C/k}^{\otimes n}$ is very ample, it suffices to check that $\omega_{C/k}^{\otimes n}$ separates points and tangents.

Exercise 1.7. Check that when $n \geq 3$, $\omega_{C/k}^{\otimes n}$ separates points and tangents.

2. The GIT construction of the moduli space

Good references for this section are [HM] Chapter 4, [Mum3], [FKM] and [Ne]. Explaining the GIT construction in detail would take us too far afield. Instead we will briefly sketch the main ideas and refer you to the literature.

2.1. **Basics about G.I.T..** An algebraic group G is a group together with the structure of an algebraic variety such that the multiplication and inverse maps are morphisms of varieties. An action of an algebraic group G on a variety X is a morphism $f : G \times X \to X$ such that $f(gg', x) = f(g, f(g', x))$ and $f(e, x) = x$, where e is the identity of the group. The stabilizer of a point $x \in X$ is the closed subgroup of G fixing x. The orbit of a point x under G is the image of f restricted to $G \times \{x\}$.

For our purposes we can always restrict attention to $SL(n)$, $GL(n)$ or $\mathbb{P}GL(n)$. An algebraic group which is isomorphic to a closed subgroup of $GL(n)$ is called a linear algebraic group. A group is called geometrically reductive if for every linear action of G on k^n and every non-zero invariant point $v \in k^n$, there exists an invariant homogeneous polynomial that does not vanish on v. The group is called linearly reductive if the homogeneous polynomial may be taken to have degree one. Finally a group is called reductive if the maximal connected normal solvable subgroup is isomorphic to a direct product of copies of k^*. In characteristic zero these concepts coincide. In characteristic $p > 0$ a threorem of Haboush guarantees that every reductive group is geometrically reductive.

The question is to obtain a quotient of a variety under the action of a reductive group.

Lemma 2.1. *Let G be a geometrically reductive group acting on an affine variety X. Let W_1 and W_2 be two disjoint invariant closed orbits. Then there exists an invariant polynomial $f \in A(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$.*

Proof. Pick any $h \in A(X)$ such that $h(W_1) = 0$ and $h(W_2) = 1$. Consider the subspace spanned by h^g for $g \in G$. This is a finite dimensional subspace. To see this consider the function $H(g, x) = h(gx)$ in $A(G \times X) \cong A(G) \otimes A(X)$. We can write $H(g, x)$ as a finite sum $\sum_i F_i \otimes H_i$ in $A(G) \otimes A(X)$ of the generators of $A(G)$ and $A(X)$. Hence the subspace spanned by h^g for $g \in G$ is contained in the subspace spanned by the H_i. Pick a basis for this subspace $h_1, \ldots, h_n$. We obtain a rational representation of G on this subspace, hence a linear action on k^n making the morphism $\pi : X \to k^n$ given by $\pi(x) = (h_1(x), \ldots, h_n(x))$ into a G-morphism. Since G is geometrically reductive there is an invariant polynomial f that has the value zero on $\pi(W_1)$ and the value 1 on $\pi(W_2)$. $f \circ \pi$ is the desired polynomial. □

The main theorem for quotients of reductive group actions on affine varieties is the following:

Theorem 2.2. *Let G be a reductive group acting on an affine variety X. Then there exists a quotient affine variety Y and a G-invariant, surjective morphism $\phi : X \to Y$ such that*

(1) *For any open set $U \subset Y$, the ring homomorphism*
$$\phi^* : A(U) \to A(\phi^{-1}(U))$$
is an isomorphism of $A(U)$ with $A(\phi^{-1}(U))^G$.
(2) *If $W \subset X$ is a closed invariant subset, then $\phi(W)$ is closed in Y.*
(3) *If W_1 and W_2 are disjoint closed invariant sets, then their images under ϕ are disjoint.*

Proof. The main technical results are provided by a theorem of Haboush and a theorem of Nagata.

Theorem 2.3 (Haboush)**.** *Any reductive group G is geometrically reductive.*

Theorem 2.4 (Nagata)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra R. Then the ring of invariants R^G is finitely generated.*

In view of these theorems $A(X)^G$ is finitely generated. Hence we can let $Y = Spec\ A(X)^G$. The inclusion of $A(X)^G \to A(X)$ induces a morphism $\phi : X \to Y$. The claimed properties are easy to check for ϕ. □

Remark 2.5. The following are straightforward observations:

(1) For any open subset $U \subset Y$, (U, ϕ) is a categorical quotient of $\phi^{-1}(U)$ by G.
(2) The images of two points in X coincide if and only if the orbit closures of these two points intersect. Consequently, Y will be an orbit space if and only if the orbits of the G action on X are closed.

Remark 2.6. We will not prove Haboush's theorem here. The interested reader may consult the original paper [Hab]. Over the complex numbers reductive, geometrically reductive and linearly reductive coincide. This follows from the fact that any finite dimensional representation is decomposible to irreducible representations. Projection to the one-dimensional invariant subspace produces the desired invariant linear functional.

We now sketch the proof of Nagata's theorem. Since R is a finitely generated k-algebra, we can pick generators $f_1, dots, f_n$ that generate R. We can also assume that the subspace spanned by the f_i is G-invariant. (If not, we can replace it by a minimal G-invariant subspace, which is finite-dimensional by the argument in Lemma 2.1.) We thus obtain a linear G action on the subspace spanned by f_i by setting

$$f_i^g = \sum_j \alpha_{i,j}(g) f_j.$$

Let $S = k[X_1, \ldots, X_n]$. There is an action of G on S by setting

$$X_i^g = \sum_j \alpha_{i,j}(g) X_j.$$

There is a k-algebra homomorphism from S to R sending X_i to f_i that is compatible with the G actions. We are thus reduced to proving

Nagata's theorem in the case when G acts on S preserving degree, $Q \subset S$ is a G-invariant ideal with the induced action on $R = S/Q$. Under these assumptions we would like to see R^G is finitely generated.

Suppose not. Since S is Noetherian, there exists an ideal Q maximal among those that are G-invariant such that R^G where $R = S/Q$ is not finitely generated. Then if $J \neq 0$ is a G-invariant homogeneous ideal in R, then $(R/J)^G$ is finitely generated. Suppose first there is a homogeneous ideal Q with the desired properties.

I claim that $(R/J)^G$ is integral over $R^G/(J \cap R^G)$. Suppose $f \in (R/J)^G$. Pick $h \in R$ such that the image of h in R/J is f. We would like to find $h_0 \in R^G$ such that $(h)^t - h_0$ for some positive integer t is in R^G. Look at the finite-dimensional, G-invariant subsapce M generated by h^g. [Unfortunately, there is potential for confusion between h^g and $(h)^t$. The first denotes the g-translate of h, the second denotes the t-th power of h. To distinguish between these two, we will put parentheses around h in the latter case.] Since J is invariant, $h^g - h$ is in J for every g. We conclude that $M \cap J$ has codimension 1 in M. We can write every element in M uniquely as $ah + h'$ where $a \in k$ and $h' \in M \cap J$. Sending $ah + h'$ to a defines a G-invariant linear functional l on M.

There is an action of G also on M^*. If we let $h, j_2, \ldots, j_n$ be a basis of M where $j_i \in M \cap J$, we can identify M^* with k^r in terms of the dual basis. The linear functional l corresponds to the vector $(1, 0, \ldots, 0)$. Since G is geometrically reductive, there exists an invariant homogeneous polynomial $F \in k[X_1, \ldots, X_n]$ of degree $t \geq 1$ such that the coefficient of X_1^t does not vanish. Consider the morphism $k[X_1, \ldots, X_n]$ sending X_1 to h and X_i to j_i for $i > 1$.If h_0 is the image of F, $h^t - h_0$ belongs to J. We conclude that $(R/J)^G$ is integral over $R^G/(J \cap R^G)$.

If A is a finitely generated k-algebra which is integral over a subalgebra B, then B is finitely generated. Hence in our case, $R^G/(J \cap R^G)$ is finitely generated. In fact, $(R/J)^G$ is a finite $R^G/(J \cap R^G)$-module.

Choose a non-zero homogeneous element f of R^G of degree at least one. If f is not a zero-divisor, $fR \cap R^G = fR^G$. Since R^G/fR^G is finitely generated, $(R^G/fR^G)_+$ is finitely generated as an ideal. Hence R^G_+ is finitely generated as an ideal in R^G. Hence R^G is a finitely generated k-algebra.

Exercise 2.7. Modify the last paragraph of the proof in case f is a zero-divisor. Hint: Consider the homogeneous ideal I of elements of

R that annihilate f. Since $R^G/(fR \cap R^G)$ and $R^G/I \cap R^G$ are both finitely generated, there is a finitely generated subalgebra of R^G that surjects onto both these algebras

In order to handle the non-homogeneous case, we may assume that R^G is a domain. By the homogeneous case S^G is finitely generated. R^G is integral over $S^G/Q \cap S^G$. It suffices to show that the field of fractions of R^G is a finitely generated extension of k. Let T be the set of non-zero divisors of R. Form the ring of fractions of R with respect to T. Let m be the maximal ideal. The field of fractions of R^G may be identified with a subfield of $T^{-1}R/m$. Since $T^{-1}R/m$ is the field of fractions of the finitely generated k-algebra $R/m \cap R$, this follows.

Example 2.8. Everyone's favorite example is the action of $GL(n)$ on the space of $n \times n$ matrices M_n by conjugation. The space of matrices is isomorphic to affine space $\mathbb{A}^{n^2}$. Hence, the coordinate ring is $k[a_{i,j}]$, $1 \leq i, j \leq n$. Any conjugacy class has a representative in Jordan canonical form which is unique upto a permutation of the Jordan blocks. Since the set of eigenvalues of a matrix is invariant under conjugation, we see that the elementary symmetric polynomials of the eigenvalues, i.e. the coefficients of the characteristic polynomial, are invariant under the action. Conversely, suppose that a polynomial is invariant under conjugation. If the eigenvalues are distinct, we can diagonalize the matrix by connjugation. Hence the polynomial must be a symmetric function of the eigenvalues. If the eigenvalues are repeated, the diagonal matrix is in the closure of the orbits with non-trivial Jordan blocks. We conclude that any invariant polynomial is a symmetric polynomial of the eigenvalues. Since the elementary symmetric polynomials generate the ring of symmetric polynomials, we conclude that the ring of invariant functions is generated by the coefficients of the characteristic polynomial.

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