CASTELNUOVO-MUMFORD REGULARITY AND BRIDGELAND STABILITY OF POINTS IN THE PROJECTIVE PLANE

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Abstract. In this paper, we study the relation between Castelnuovo-Mumford regularity and Bridgeland stability for the Hilbert scheme of n points on $\mathbb{P}^2$. For the largest $\lfloor \frac{n}{2} \rfloor$ Bridgeland walls, we show that the general ideal sheaf destabilized along a smaller Bridgeland wall has smaller regularity than one destabilized along a larger Bridgeland wall. We give a detailed analysis of the case of monomial schemes and obtain a precise relation between the regularity and the Bridgeland stability for the case of Borel fixed ideals.

1. Introduction

In this paper, we consider the relation between the Castelnuovo-Mumford regularity and the Bridgeland stability of zero-dimensional subschemes of $\mathbb{P}^2$. Our study is motivated by the following result which relates geometric invariant theory (GIT) stability and Castelnuovo-Mumford regularity.

Theorem. [HH13, Corollary 4.5] *Let $C \subset \mathbb{P}^{3g-4}$ be a c-semistable bicanonical curve. Then $\mathcal{O}_C$ is 2-regular.*

Note that *c-semistability* of curves [HH13, Definition 2.6] is a purely geometric notion concerning singularities and subcurves, whereas Castelnuovo-Mumford regularity is an algebraic notion regarding the syzygies of ideal sheaves.

For points in $\mathbb{P}^2$, a similar but weaker statement holds. A set of n points in $\mathbb{P}^2$ is GIT semistable if and only if at most $2n/3$ of the points are collinear, in which case the regularity is at most $2n/3$. However, the regularities of semistable points cover a broad spectrum. Our goal in this paper is to use Bridgeland stability to obtain a closer relationship between stability and regularity.

There is a distinguished half-plane $H = \{(s,t) | s > 0, t \in \mathfrak{R}\}$ of Bridgeland stability conditions for $\mathbb{P}^2$. Let ξ be a Chern character. The half-plane H admits a wall-and-chamber decomposition, where in each chamber the set of Bridgeland semistable objects with Chern character ξ remains constant.

The Bridgeland walls where an ideal sheaf of points is destabilized consist of the vertical line $s = 0$ and a finite set of *nested* semicircular walls $\mathcal{W}_c$ centered along

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the s-axis at $s = -c - \frac{3}{2} < 0$ [ABCH13]. Since the semicircular Bridgeland walls are nested, we can order them by inclusion. If an ideal sheaf $\mathcal{I}_Z$ is destabilized along the wall $\mathcal{W}_c$, then $\mathcal{I}_Z$ is Bridgeland stable in the region bounded by $\mathcal{W}_c$ and $s = 0$. Let $\sigma \prec \sigma'$ if all σ'-semistable ideal sheaves with Chern character ξ are σ-semistable. Consequently, Bridgeland stability induces a stratification of $\mathbb{P}^{2[n]}$

$$\mathbb{P}^{2[n]} = \coprod_{\alpha} X^{\alpha},$$

where

$$X^{\alpha} = \{Z \in \mathbb{P}^{2[n]} \,|\, \mathcal{I}_Z \text{ is } \alpha\text{-semistable but } \beta\text{-unstable } \forall \alpha \prec \beta\}$$

and α runs over a Bridgeland stability condition in each chamber. We have $\overline{X^{\alpha}} = \bigcup_{\beta \preceq \alpha} X^{\beta}$ (see Section 2). By [ABCH13] and [LZ], this stratification coincides with the stratification of $\mathbb{P}^{2[n]}$ according to base loci of linear systems.

Similarly, there is a stratification induced by Castelnuovo-Mumford regularity:

$$\mathbb{P}^{2[n]} = \coprod_{r \in \mathbb{Z}} X^{\text{r-reg}},$$

where $X^{\text{r-reg}}$ is the collection of ideals whose Castelnuovo-Mumford regularity is r. The regularity, being a cohomological invariant [Eis95, Chapter 20], is upper-semicontinuous and we have $\overline{X^{\text{r-reg}}} = \coprod_{r' \geq r} X^{r'\text{-reg}}$.

This naturally raises the question of comparing the two stratifications. We will show that a general scheme destabilized at one of the $\lfloor \frac{n}{2} \rfloor$ largest Bridgeland walls has smaller regularity than the general scheme destabilized along the larger walls. Our main theorem will be proved in Section 5:

Theorem. *Let $\mathfrak{p}_i$ be the maximal ideal of the closed point $p_i \in \mathbb{P}^2$, $i = 1, \ldots, s$. Let Z be the subscheme given by $\cap_{i=1}^{s} \mathfrak{p}_i^{m_i}$ and let n be its length. Define*

$$h := \max \left\{ \sum_{j=1}^{t} m_{i_j} \;\middle|\; p_{i_1}, \ldots, p_{i_t} \text{ are colinear} \right\}.$$

If $n \leq 2h - 3$, then Z is destabilized at the wall $\mathcal{W}_{\mathrm{reg}(Z)-1}$. In particular, general points destabilized at $\mathcal{W}_{k+1}$ have higher regularity than those destabilized at $\mathcal{W}_k$, $\forall k \geq \frac{n}{2} - 1$.

For zero-dimensional subschemes cut out by monomials, we have a more precise connection between regularity and Bridgeland stability:

Proposition. *Let Z be a zero-dimensional monomial scheme in $\mathbb{P}^2$. If the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$, then*

$$\frac{3}{4}(\mathrm{reg}(\mathcal{I}_Z) - 1) \leq \mu(Z) \leq \mathrm{reg}(\mathcal{I}_Z) - 1.$$

(1) *The left equality holds if and only if $\mathrm{reg}(\mathcal{I}_Z) + 1 = 2m$ is even and $\mathcal{I}_Z = \langle x^m, y^m \rangle$*
(2) *The right equality holds if and only if $\mathcal{I}_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \rangle$ with $\max_{1 \leq i \leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.*

In particular, for Borel fixed ideals, the regularity and the Bridgeland stability completely determine each other:

Corollary. *Let* $Z \subset \mathbb{P}^2$ *be a zero-dimensional monomial scheme whose ideal is Borel-fixed (which holds if it is a generic initial ideal, for instance). Then the ideal sheaf* $\mathcal{I}_Z$ *is destabilized at the wall* $\mathcal{W}_{\mathrm{reg}(\mathcal{I}_Z)-1}$.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

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2. Preliminaries on Bridgeland stability conditions

We briefly review the basics of Bridgeland stability conditions on $\mathbb{P}^2$. We refer the reader to [ABCH13] and [CH14] for more details. Let $\mathcal{D}^b(\mathbb{P}^2)$ be the bounded derived category of coherent sheaves on $\mathbb{P}^2$, and $K(\mathbb{P}^2)$ be the K-group of $\mathcal{D}^b(\mathbb{P}^2)$.

Definition 2.1. A *Bridgeland stability condition* on $\mathbb{P}^2$ consists of a pair $(\mathcal{A}, \mathcal{Z})$, where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}^b(\mathbb{P}^2)$ and $\mathcal{Z} : K(\mathbb{P}^2) \to \mathbb{C}$ is a homomorphism (called the *central charge*) satisfying

- if $0 \neq E \in \mathcal{A}$, $\mathcal{Z}(E)$ lies in the semi-closed upper half-plane $\{re^{i\pi\theta} \mid r > 0, 0 < \theta \leq 1\}$.
- $(\mathcal{A}, \mathcal{Z})$ has the Harder-Narasimhan property, which will be defined below.

Definition 2.2. Writing $\mathcal{Z} = -d + ir$, the *slope* $\mu(E)$ of $0 \neq E \in \mathcal{A}$ is defined by $\mu(E) = d(E)/r(E)$ if $r(E) \neq 0$ and $\mu(E) = \infty$ otherwise.

Definition 2.3. An object $E \in \mathcal{A}$ is called *stable* (resp. *semistable*) if for every proper subobject $F \subset E$ in $\mathcal{A}$, $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$).

Definition 2.4. The pair $(\mathcal{A}, \mathcal{Z})$ has the *Harder-Narasimhan property* if any nonzero object $E \in \mathcal{A}$ admits a finite filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each *Harder-Narasimhan factor* $F_i = E_i/E_{i-1}$ is semistable and $\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n)$.

Let $\mu_{min}(E)$ (resp. $\mu_{max}(E)$) denote the minimum (resp. maximum) slope of a Harder-Narasimhan factor of a coherent sheaf E with respect to the Mumford slope. For $s \in \mathbb{R}$, let $\mathcal{Q}_s$ and $\mathcal{F}_s$ be the full subcategory of $\mathrm{Coh}(\mathbb{P}^2)$ defined by

- $Q \in \mathcal{Q}_s$ if Q is torsion or $\mu_{min}(Q) > s$.
- $F \in \mathcal{F}_s$ if F is torsion-free, and $\mu_{max}(F) \leq s$.

Each pair $(\mathcal{F}_s, \mathcal{Q}_s)$ is a torsion pair [Bri08, Lemma 6.1], and induces a t-structure via tilting on $D^b(\mathbb{P}^2)$ with heart [HRS96]

$$\mathcal{A}_s = \{E \in \mathcal{D}^b(\mathbb{P}^2) \mid H^{-1}(E) \in \mathcal{F}_s, H^0(E) \in \mathcal{Q}_s, \text{ and } H^i(E) = 0 \text{ otherwise}\}.$$

Let L be the class of a line in $\mathbb{P}^2$.

Theorem. [Bri08, AB13, BM11] *For each* $s \in \mathbb{R}$ *and* $t > 0$, *define*

$$\mathcal{Z}_{s,t}(E) = -\int_{\mathbb{P}^2} e^{-(s+it)L} \mathrm{ch}(E).$$

Then the pair $(\mathcal{A}_s, \mathcal{Z}_{s,t})$ *defines a Bridgeland stability condition on* $\mathcal{D}^b(\mathbb{P}^2)$.

We thus obtain an upper half-plane H of Bridgeland stability conditions.

Fix a class ξ in the numerical Grothendieck group. If ξ has positive rank, define the *slope* and the *discriminant* by

$$\mu(\xi) = \frac{\mathrm{ch}_1(\xi)}{\mathrm{rank}\,(\xi)} \quad \Delta = \frac{1}{2}\mu(\xi)^2 - \frac{\mathrm{ch}_2(\xi)}{\mathrm{rank}\,(\xi)}.$$

For an ideal sheaf $\mathcal{I}_Z$ of n points, we have $\mu = 0$ and $\Delta = n$.

There exists a locally finite set of walls in the (s,t)-half plane depending on ξ such that the set of σ-(semi)stable objects of class ξ does not change as the σ varies in a chamber [Bri08, BM11, BM14]. These walls are called *Bridgeland walls.* For $\mathbb{P}^2$, the Bridgeland walls where a Gieseker semistable sheaf is destabilized consist of line $s = \mu(\xi)$ and a finite number of nested semicircles with center $(c, 0)$ with $c < \mu$ [ABCH13]. The largest semicircular wall is called the *Gieseker wall* and the smallest semicircular wall is called the *collapsing wall.* If $\xi = (1, 0, n)$, the Chern character of the ideal sheaf of a zero-dimensional subscheme of $\mathbb{P}^2$ of length n, then the wall with center $(c, 0)$ has radius $\sqrt{c^2 - 2n}$. Throughout the paper $\mathcal{W}_\mu = \mathcal{W}_\mu^n$ will denote the wall centered at $(-\mu - \frac{3}{2}, 0)$. An ideal sheaf destabilized along $\mathcal{W}_\mu$ is Bridgeland stable for all Bridgeland stability conditions outside $\mathcal{W}_\mu$ and not semistable for any Bridgeland stability condition contained in $\mathcal{W}_\mu$. All Bridgeland walls for $n \leq 9$ were explicitly computed in [ABCH13].

3. Monomial schemes

A *monomial subscheme* of $\mathbb{P}^2$ is a subscheme whose ideal is generated by monomials. For these schemes, the relation between Castelnuovo-Mumford regularity and Bridgeland stability is clear because the regularity is easy to compute and the Bridgeland stability is explicitly described by [CH14]. To reveal the relation, we need to study the combinatorics.

Proposition 3.1. *Let Z be a zero-dimensional monomial scheme in $\mathbb{P}^2$. If the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$, then*

$$\frac{3}{4}(\mathrm{reg}(\mathcal{I}_Z) - 1) \leq \mu(Z) \leq \mathrm{reg}(\mathcal{I}_Z) - 1.$$

1. *The left equality holds if and only if $\mathrm{reg}(\mathcal{I}_Z) + 1 = 2m$ is even and $\mathcal{I}_Z = \langle x^m, y^m \rangle$.*
2. *The right equality holds if and only if $\mathcal{I}_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \rangle$ satisfies $\max_{1 \leq i \leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.*

A zero-dimensional monomial subscheme Z in $\mathbb{P}^2$, in a suitable affine coordinate system, has defining ideal I_Z generated by a set of monomials

$$x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \tag{†}$$

where $a_1 > \ldots > a_{r-1} > a_r = 0$ and $0 = b_1 < b_2 < \ldots < b_r$.

It is convenient to represent monomial subschemes by their block diagrams. The *block diagram* D for Z consists of b_r left-aligned rows of consecutive boxes such that the ith row counting from the bottom has a_j boxes if $b_j < i \leq b_{j+1}$. The lower left corner represents the monomial 1. The box to the right of (resp. above) x^iy^j represent $x^{i+1}y^j$ (resp. x^iy^{j+1}). With this interpretation, the box diagram D records the monomials in $\mathbb{K}[x,y]$ which are not in I_Z. The next figure shows an example.

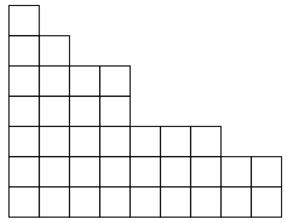

FIGURE 1. The block diagram for $\langle x^9, x^7y^2, x^4y^3, x^2y^5, xy^6, y^7\rangle$

We will always place the lower left corner of D at the origin and assume that the boxes in D are unit length.

Proof of Proposition 3.1. We briefly recapitulate the computation of $\mu(Z)$ in [CH14]. Index the rows of a box diagram D from bottom to top, and the columns from left to right. Let h_j (resp. ν_j) be the number of boxes in the jth row (resp. column). Let $r(D)$ and $c(D)$ be the number of rows and columns in D. Define the kth horizontal slope μ_k and the ith vertical slope μ_i' by

$$\mu_k = \frac{1}{k}\sum_{j=1}^{k}(h_j + j - 1) - 1, \quad \mu_i' = \frac{1}{i}\sum_{j=1}^{i}(\nu_j + j - 1) - 1.$$

Then the slope $\mu(Z)$ of Z is defined by

$$\mu(Z) = \max_{1\le k\le r(D), 1\le i\le c(D)}\{\mu_k, \mu_i'\}.$$

By [CH14, Theorem 1.6], the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$.

On the other hand, the regularity of $\mathcal{I}_Z$ can be computed from its minimal free resolution given by

$$0 \to \bigoplus_{i=1}^{r-1}\mathcal{O}(-a_i - b_{i+1}) \xrightarrow{M} \bigoplus_{i=1}^{r}\mathcal{O}(-a_i - b_i) \to \mathcal{I}_Z \to 0,$$

where M is the $r\times(r-1)$ matrix with entries

$$m_{i,i} = y^{b_{i+1}-b_i}, \quad m_{i+1,i} = -x^{a_i - a_{i+1}}, \quad \text{and} \quad m_{i,j} = 0 \ \text{ otherwise}.$$

Since $a_i + b_{i+1} - 1 \ge a_i + b_i$ for $i = 1, \ldots, r-1$ and $a_{r-1} + b_r - 1 \ge a_r + b_r$, the Castelnuovo-Mumford regularity $\operatorname{reg}(\mathcal{I}_Z)$ of $\mathcal{I}_Z$ is

$$\operatorname{reg}(\mathcal{I}_Z) = \max_{1\le i\le r-1}(a_i + b_{i+1} - 1).$$

If we place the block diagram D in the a-b plane with its lower left corner at the origin and set every box to be a unit square, then the points (a_i, b_{i+1}) are the vertices of D contained in the first quadrant. Hence, the block diagrams representing ideals with regularity l are precisely those which lie below and touch the line $a + b = l + 1$.

Fix the regularity to equal l. To maximize $\mu(Z)$ subject to $\operatorname{reg}(Z) = l$, we need to maximize μ_k and μ_i' under the condition that the box diagram lies below and touches the line $a + b = l + 1$. Since the box diagram of $I_Z = \langle x^l, x^{l-1}y, \ldots, y^l\rangle$ contains every positive integral lattice point under the line $a + b = l + 1$, it follows that Z gives the the maximum μ-value, which is $l - 1$. Note that $\mu_k = l - 1$ if and only if $h_1 = l, h_2 = l-1, \dots, h_k = l-(k-1)$. Hence, $\mu(Z) = l-1$ precisely when either $h_1 = l$ or $\nu_1 = l$. Equivalently, equality holds for $I_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \dots, y^{b_r}\rangle$ if I_Z satisfies $\max_{1\le i\le r-1}(a_i + b_{i+1} - 1) \le \max(a_1, b_r)$.

To minimize $\mu(Z)$ subject to $\mathrm{reg}(Z) = l$, we use as few boxes as possible to minimize the slopes μ_k and μ'_i. A box diagram that touches the line $a+b = l+1$ at (a', b') contains the box diagram of the ideal $\langle x^{a'}, y^{b'}\rangle$. It follows that the ideal of Z should be of the form $\langle x^a, y^b\rangle$ with $a+b = l+1$. Then

$$\max_{1\le k\le r(D)}\{\mu_k\} = \mu_b = a + \frac{b-1}{2} - 1$$

and similarly

$$\max_{1\le i\le c(D)}\{\mu'_i\} = \mu'_a = b + \frac{a-1}{2} - 1$$

so that

$$\mu(Z) = \max\left(a + \frac{b-1}{2} - 1, b + \frac{a-1}{2} - 1\right)$$

Thus $\mu(Z)$ achieves the minimum when a and b are almost equal. If l is even, then $(a, b) = (\frac{l}{2}+1, \frac{l}{2})$ gives $\mu(Z) = \frac{3l}{4} - \frac{1}{2}$. If l is odd, then $(a, b) = (\frac{l+1}{2}, \frac{l+1}{2})$ gives $\mu(Z) = \frac{3l}{4} - \frac{3}{4}$. Furthermore, if $n > \frac{(l+1)^2}{4}$, then either the horizontal slope $\mu_{\frac{l+1}{2}}$ or the vertical slope $\mu'_{\frac{l+1}{2}}$ is strictly larger than $\frac{3l}{4} - \frac{3}{4}$. We conclude that $\frac{3l}{4} - \frac{3}{4} \le \mu(Z)$ with equality only if Z is the monomial ideal $\langle x^{\frac{l+1}{2}}, y^{\frac{l+1}{2}}\rangle$. □

Recall that an ideal I generated by monomials in x and y is *Borel fixed* if $x^i y^j \in I$ for some $j > 0$ implies $x^{i+1}y^{j-1} \in I$. Borel fixedness is one of the most important combinatorial properties in the study of monomial ideals. For instance, *generic initial ideals* with respect to a monomial order are Borel fixed. See [Eis95, Chapter 15] for a detailed discussion. We obtain the following corollary.

Corollary 3.2. *Let $Z \subset \mathbb{P}^2$ be a zero-dimensional monomial scheme whose ideal is Borel-fixed. Then the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mathrm{reg}(\mathcal{I}_Z)-1}$.*

Proof. A Borel-fixed ideal is of the form $\langle x^a, x^{a-1}y^{\lambda_{a-1}}, \dots, y^{\lambda_0}\rangle$ with $\lambda_0 > \dots > \lambda_{a-1} > 0$. Then $(i + \lambda_{i-1} - 1) \le \lambda_0 = \max(a, \lambda_0)$ for $i = 1, \dots, a$. The corollary follows from Proposition 3.1 (2). □

Every possible Betti diagram of a zero-dimensional scheme in $\mathbb{P}^2$ occurs as the Betti diagram of a monomial scheme [Eis05]. Let $\binom{k}{2} < n \le \binom{k+1}{2}$ and let Z be a scheme of length n. Then the regularity of Z can be any integer between k and n. Given $k \le l \le n$, take a box diagram D with n boxes and at most l rows such that $h_1 = l$ and $h_i \le l+1-i$ for $2 \le i \le l$. Since $n \le \binom{l+1}{2}$ such diagrams D exist. Moreover, $\mu(Z) = l-1 = \mathrm{reg}(\mathcal{I}_Z) - 1$, the maximum possible by Proposition 3.1.

We can also ask for the minimum possible $\mu(Z)$ given a scheme Z of length n and regularity l. If $0 < m \le \frac{l}{2}$ and $m(l+1-m) \le n < (m+1)(l-m)$, then the tallest rectangle with upper right vertex on the line $x+y = l+1$ is the $m \times (l-m+1)$ rectangle. Hence, $\mu(Z) \ge \mathrm{reg}(\mathcal{I}_Z) - \frac{1}{2} - \frac{m}{2}$. Equality occurs, for instance, when $n = m(l+1-m)$. In case, l is even (resp. odd) and $n > \frac{l}{2}(\frac{l}{2}+1)$ (resp. $n > (\frac{l+1}{2})^2$), then $\mu(Z) \ge \frac{3}{4}\mathrm{reg}(\mathcal{I}_Z) - \frac{1}{2}$ (resp. $\frac{3}{4}\mathrm{reg}(\mathcal{I}_Z) - \frac{3}{4}$). In particular, we conclude that

$$1 \le \mathrm{reg}(\mathcal{I}_Z) - \mu(Z) \le \frac{\sqrt{n}+1}{2}.$$

Equality is attained on the right hand side when $\text{reg}(\mathcal{I}_Z)$ is odd and $n = \frac{(\text{reg}(\mathcal{I}_Z)+1)^2}{4}$. We summarize this in the following proposition.

Proposition 3.3. *Let* Z *be a monomial scheme of length* n *and regularity* l. *If* $0 < m \leq \frac{l}{2}$ *and* $m(l+1-m) \leq n < (m+1)(l-m)$, *then*

$$1 \leq \text{reg}(\mathcal{I}_Z) - \mu(Z) \leq \frac{m}{2} + \frac{1}{2}.$$

In general,

$$1 \leq \text{reg}(\mathcal{I}_Z) - \mu(Z) \leq \frac{\sqrt{n}+1}{2}.$$

4. General points

In this section, we discuss the relation between Bridgeland stability and regularity for general points on $\mathbb{P}^2$.

Let $\binom{r}{2} < n \leq \binom{r+1}{2}$. Then, for a dense open set $U \in \mathbb{P}^{2[n]}$, the minimal free resolution of $\mathcal{I}_Z$ is the Gaeta resolution

$$0 \to \mathcal{O}^{\oplus a}(-r-1) \oplus \mathcal{O}^{\oplus \max(0,-b)}(-r) \to \mathcal{O}^{\oplus \max(0,b)}(-r) \oplus \mathcal{O}^{\oplus c}(-r+1) \to \mathcal{I}_Z \to 0,$$

where $a = n - \binom{r}{2} > 0$, $c = \binom{r+1}{2} - n \geq 0$ and $b = c - a + 1$ [Eis05]. The regularity of $\mathcal{I}_Z$ is r. Since regularity is upper-semicontinuous and $\mathbb{P}^{2[n]}$ is irreducible, there exists an open set U_1 containing U such that $\text{reg}(\mathcal{I}_Z) = r$ for $Z \in U_1$.

On the other hand, there exists an open dense set $U_2 \in \mathbb{P}^{2[n]}$ such that for $Z \in U_2$ the ideal sheaf $\mathcal{I}_Z$ is destabilized at the collapsing wall $\mathcal{W}_{\mu_n}$ with center $(-\mu_n - \frac{3}{2}, 0)$. By a general point of $\mathbb{P}^{2[n]}$, we will mean a point $Z \in U_1 \cap U_2$. For such Z, there exists a precise relation between the regularity k and the Bridgeland slope μ_n. Huizenga computed μ_n for all n [Hui, Theorem 7.2]. The slope μ_n is the smallest positive slope of a stable vector bundle on the parabola $\mu^2 + 3\mu + 2 - 2n = 2\Delta$, where μ is the slope and Δ is the discriminant. The computation of μ_n, while easy for any given n, depends on a fractal curve. Consequently, it is hard to write a closed formula.

Luckily, there are good bounds for μ_n. Let

$$\mathcal{S} = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \dots \right\} \cup \left\{ \alpha > \phi^{-1} = \frac{\sqrt{5}-1}{2} \right\}$$

consisting of consecutive ratios of Fibonacci numbers and numbers larger than the inverse of the golden ratio. Let $n = \binom{k}{2} + s$ with $0 \leq s < k$. By [ABCH13, Theorem 4.5], we have

$$\mu_n = \begin{cases} k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \in \mathcal{S} \\ k - 1 - \frac{k-s}{k+1} & \text{if } 1 - \frac{s+1}{k+1} \in \mathcal{S}. \end{cases}$$

Furthermore, by [ABCH13, Lemma 4.1, Corollary 4.8], the inequalities

$$\mu_{n-1} \leq \mu_n \leq \begin{cases} k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \geq \frac{1}{2} \\ k - 1 - \frac{k-s}{k+1} & \text{if } \frac{s}{k-1} \leq \frac{1}{2} \end{cases}$$

hold. When k is odd and $s = \frac{k-1}{2}$, then $\frac{s}{k-1} = \frac{1}{2} \in \mathcal{S}$ and $\mu_n = k - \frac{3}{2}$. When k is even and $n = \binom{k}{2} + \frac{k}{2} + 1$, then the positive root x_p of $\frac{1}{2}(\mu^2 + 3\mu + 2) - n = \frac{1}{2}$ satisfies $x_p > k - \frac{3}{2}$. By [Hui, Theorem 7.2], we conclude that $\mu_n > k - \frac{3}{2}$. Combining these inequalities we deduce the following proposition.

Proposition 4.1. *Let Z be a general point of $\mathbb{P}^{2[n]}$. Let $\mathcal{W}_{\mu_n}$ be the collapsing wall.*

(1) *If $n = \binom{k}{2}$, then $\mu_n = \mathrm{reg}(\mathcal{I}_Z) - 1$.*
(2) *If $n = \binom{k}{2} + s$ with $\frac{1}{2} \geq \frac{s}{k-1} > 0$, then*
$$\mathrm{reg}(\mathcal{I}_Z) - 1 - \frac{\max(k-s, \lceil \phi^{-1}(k+1) \rceil)}{k+1} \leq \mu_n \leq \mathrm{reg}(\mathcal{I}_Z) - 1 - \frac{k-s}{k+1}$$
and the right inequality is an equality if $1 - \frac{s+1}{k+1} \in \mathcal{S}$.
(3) *If $n = \binom{k}{2} + s$ with $\frac{s}{k-1} \geq \frac{1}{2}$, then*
$$\mathrm{reg}(\mathcal{I}_Z) - \frac{3}{2} \leq \mu_n \leq \mathrm{reg}(\mathcal{I}_Z) - 2 + \frac{s}{k-1}$$
and the right inequality is an equality if $\frac{s}{k-1} \in \mathcal{S}$.

In particular, $\mathrm{reg}(\mathcal{I}_Z) - 2 < \mu_n \leq \mathrm{reg}(\mathcal{I}_Z) - 1$ for a general Z.
We point out that the sets $U_1 - U_2$ and $U_2 - U_1$ are both nonempty in general.

Example 4.2. The minimum regularity for a scheme Z of length 7 is 4 and $\mu_7 = \frac{12}{5}$ [Hui, Table 1]. Consider the monomial scheme generated with defining ideal $\langle x^4, xy, y^4 \rangle$. The regularity of this scheme is 4 but it is destabilized along the wall $\mathcal{W}_3$. Hence, this monomial scheme is a point of U_1 which is not in U_2.

Example 4.3. The minimum regularity for a scheme Z of length 9 is 4. For a complete intersection scheme of type $(3,3)$, the minimal resolution is
$$0 \to \mathcal{O}(-6) \to \mathcal{O}(-3) \oplus \mathcal{O}(-3) \to \mathcal{I}_Z \to 0.$$
Hence, the regularity is 5. On the other hand, the general scheme and a complete intersection scheme both have $\mu = 3$ [ABCH13], [CH14, Theorem 5.1]. Hence, the complete intersection scheme is in U_2 but not in U_1.

5. Outer walls of the Bridgeland manifold

In general, it is hard to test whether a specific ideal sheaf $\mathcal{I}_Z$ is destabilized along a given wall $\mathcal{W}_\mu$. However, for the largest $\lfloor \frac{n}{2} \rfloor$ semicircular Bridgeland walls, one can give a concrete characterization of the ideal sheaves destabilized along the wall. This characterization allows us to compute the regularity.

Let Y^n_μ denote the locally closed subset of $\mathbb{P}^{2[n]}$ parameterizing subschemes Z destabilized along $\mathcal{W}_\mu$. By the one-to-one correspondence between the Bridgeland walls and Mori walls [ABCH13], we may rephrase [ABCH13, Proposition 4.16] as follows.

Proposition 5.1. *Let $n \leq k(k+3)/2$. Let $\mathcal{W}_k$ be the wall with center $x = -k - \frac{3}{2}$.*

(a) *If $n \leq 2k+1$, then Y^n_k parameterizes Z that have a linear subscheme of length $k+2$ but no linear subscheme of length greater than $k+2$;*
(b) *If $n = 2k+2$, then Y^n_k parameterizes Z that are contained in a conic or have a linear subscheme of length $k+2$ but does not have a linear subscheme of length greater than $k+2$.*

Fatabbi's theorem [Fat94] allows us to say more about the regularity of the schemes destabilized along $\mathcal{W}_k$.

Proposition 5.2. *(Fat points) Let $\mathfrak{p}_i$ be the maximal ideals of distinct closed points $p_i \in \mathbb{P}^2$, $i = 1, \ldots, s$. Let Z be the subscheme given by $\cap_{i=1}^{s} \mathfrak{p}_i^{m_i}$ and suppose that Z is of length n. Define*

$$h := \max \left\{ \sum_{j=1}^{t} m_{i_j} \;\middle|\; p_{i_1}, \ldots, p_{i_t} \text{ are collinear} \right\}.$$

If $n \leq 2h-3$, then Z is destabilized at the wall $\mathcal{W}_{\mathrm{reg}(Z)-1}$. In particular, a general member of Y^n_{k+1} has a higher regularity than a general member of Y^n_k, $\forall k \geq \frac{n}{2} - 1$.

Proof. The assumption $n \leq 2h - 3$ allow us to apply [Fat94, Theorem 3.3] and conclude that the regularity of Z equals h. We shall prove that Z has no linear subschemes of length $h+1$. Let L be a linear subcheme of Z supported on $p_{i_1}, \ldots, p_{i_t}$. Let f be a linear form vanishing on $p_{i_1}, \ldots, p_{i_t}$. Then $\mathfrak{p}_{i_j} = \langle f, g_{i_j} \rangle$ for some linear form g_{i_j} and f and $\mathfrak{p}_{i_j}^{m_{i_j}}$, $j = 1, \ldots, t$ are contained in the ideal I_L of L.

For the length of L to be as large as possible, we take the smallest possible ideal that contains $f + \sum_{j=1}^{t} \mathfrak{p}_{i_j}$. Since $\mathfrak{p}_{i_j}^{m_{i_j}} = \langle f^{m_{i_j}}, f^{m_{i_j}-1} g_{i_j}, \ldots, g_{i_j}^{m_{i_j}} \rangle$, any ideal containing $f + \sum_{j=1}^{t} \mathfrak{p}_{i_j}$ must also contain $g_{i_j}^{m_{i_j}}$. It follows that $\langle f, g_{i_1}^{m_{i_1}} \rangle \cap \ldots \cap \langle f, g_{i_t}^{m_{i_t}} \rangle$ defines a linear subscheme of Z of maximal length $\sum_{j=1}^{t} m_{i_j}$ supported on the cycle $\sum_{j=1}^{t} m_{i_j} p_{i_j}$. Since the regularity h is the maximum that the degree $\sum_{j=1}^{t} m_{i_j}$ can achieve, it is the maximum length of a linear subscheme of Z. Now, since $n \leq 2(h-2) + 1$ by assumption, we may apply Proposition 5.1 and obtain the first assertion.

General points Z of Y^n_k, $k \geq \frac{n}{2} - 1$, have no multiplicities i.e. $m_i = 1$, $\forall i$; have $k+2$ collinear points; and the rest are in general position. This corresponds to the case $h = k+2 \geq \frac{n}{2} + 1 > \left[\frac{n}{2}\right]$, so Fatabbi's theorem applies and $\mathrm{reg}(\mathcal{I}_Z) = h = k+2$. ☐

In general, the relation between regularity and the Bridgeland slope is not monotonic. Let Z_1 and Z_2 be two schemes of length n destabilized along $\mathcal{W}_{\mu(Z_1)}$ and $\mathcal{W}_{\mu(Z_2)}$, respectively. It may happen that while $\mathrm{reg}(Z_1) > \mathrm{reg}(Z_2)$, we have $\mu(Z_1) < \mu(Z_2)$.

Example 5.3. Let Z_1 and Z_2 be the monomial scheme defined by $\langle x^4, y^4 \rangle$ and $\langle x^6, x^5 y, x^4 y^2, x y^3, y^4 \rangle$, respectively. Both are of length 16, and by the arguments of Section 3, we see that $\mathrm{reg}(\mathcal{I}_{Z_1}) = 7, \mathrm{reg}(\mathcal{I}_{Z_2}) = 6$ and $\mu(Z_1) = \frac{9}{2}, \mu(Z_2) = 5$.

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