THE BIRATIONAL GEOMETRY OF MODULI SPACES

LECTURE NOTES FOR THE CIMPA/TÜBİTAK/GSU SUMMER SCHOOL, ALGEBRAIC GEOMETRY AND NUMBER THEORY, 2–10 JUNE 2014

IZZET COSKUN

Abstract. The purpose of these lecture notes is to introduce the basics of the birational geometry of moduli spaces to students who have taken an introductory course in algebraic geometry. We concentrate on a few key ideas and examples. We define the cones of ample and effective divisors, compute them for a few examples such as the blowup of $\mathbb{P}^2$ at one or two points. Then we discuss the ample and effective cones of the Hilbert scheme of points on $\mathbb{P}^2$. Finally, in the last section, we give a guide to the literature on other moduli spaces. These are the notes for two lectures that I delivered at the CIMPA/TÜBITAK/GSU Summer School on Algebraic Geometry and Number Theory in Istanbul in 2014.

1. Introduction

The purpose of these two lectures is to introduce the fast developing field of birational geometry of moduli spaces to beginning students in algebraic geometry. Students who have taken an introductory course in algebraic geometry are the intended audience of these notes. Rather than developing the general theory, we will illustrate ideas via simple examples. We will concentrate on the Hilbert scheme of points on the plane and point the reader to the literature for the birational geometry of other moduli spaces such as the moduli space of curves or the Kontsevich moduli space of stable maps.

Let X be a smooth, projective variety over the complex numbers. We can ask the following basic questions about X.

(1) What are the embeddings of X into projective space?
(2) What are the rational maps from X into projective space?

Following the work of Kleiman, Kollár, Mori, Reid and others in the 1980s, it is customary to translate these problems into problems of convex geometry. The reader should consult [D], [KM] and [La] for detailed treatments of the subject.

Definition 1.1. A *divisor* on X is a finite linear combination $\sum_{i=1}^m a_i Y_i$, where $a_i \in \mathbb{Z}$ and Y_i are codimension one subvarieties of X. Let $\mathrm{Div}(X)$ denote the group of divisors on X. Two divisors D_1, D_2 are *numerically equivalent* ($D_1 \equiv D_2$) if they have the same intersection number $D_1 \cdot C = D_2 \cdot C$ for every curve $C \subset X$. Notice that we can extend the notion of numerical equivalence if we take $\mathbb{Q}$ or $\mathbb{R}$ as coefficients for divisors. The *Néron-Severi space* $\mathrm{NS}(X)$ of X is the $\mathbb{Q}$-vector space of $\mathbb{Q}$-divisors modulo numerical equivalence $\mathrm{Div}(X) \otimes \mathbb{Q}/\equiv$.

Given a divisor D on X, one associates the line bundle $\mathcal{O}_X(D)$ on X [Ha, II.6.13]. One can define the concepts for divisors or line bundles to get equivalent theories. We will use them interchangeably depending on convenience.

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As discussed in the lectures of Chris Peters and Olivier Debarre, to each line bundle we can associate a first Chern class in $H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})$. If two line bundles have the same Chern class, then the corresponding divisors are numerically equivalent. Consequently, $\mathrm{NS}(X)$ is finite dimensional since it is a quotient of a finite dimensional vector space. Its dimension is called the *Picard number* $\rho(X)$ of X.

We will now introduce convex cones in $\mathrm{NS}(X) \otimes \mathbb{R}$ that capture the birational geometry of X. In the next section, we will compute several simple examples. In the last two sections, we will discuss these cones for various moduli spaces.

Definition 1.2. A divisor $\sum_{i=1}^n a_i Y_i$ is called *effective* if $a_i \geq 0$ for $1 \leq i \leq n$. A line bundle is called *effective* if $H^0(X,L) \neq 0$. A line bundle L on X is called *globally generated* or *base-point-free* if for every $x \in X$, there exists a section $s \in H^0(X,L)$ such that $s(x) \neq 0$. A divisor D is called base-point-free if $\mathcal{O}_X(D)$ is base-point-free. More generally, a linear system V, i.e., a vector subspace $V \subset H^0(X,L)$, is called *base-point-free* if for every $x \in X$, there exists a section $s \in V$ such that $s(x) \neq 0$.

There is an equivalence between morphisms $f : X \to \mathbb{P}^n$ and base-point-free linear systems on X [Ha, II.7.1]. Projective space $\mathbb{P}^n$ is equipped with the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ whose global sections are linear homogeneous polynomials. Given any point $p \in \mathbb{P}^n$, we can find a linear homogeneous polynomial not vanishing at p. Consequently, $\mathcal{O}_{\mathbb{P}^n}(1)$ is a globally generated line bundle on $\mathbb{P}^n$. Given a morphism $f : X \to \mathbb{P}^n$, we obtain a line bundle $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ on X and a base-point-free linear system V by pulling back the sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ via f. Conversely, given a line bundle L on X and a base-point-free linear system $V \subset H^0(X,L)$, we obtain a morphism $f : X \to \mathbb{P}^n$. Choose a basis of V, $s_0, \dots, s_n$. Consider the map

$$f : X \to \mathbb{P}^n, \quad x \mapsto [s_0(x) : \cdots : s_n(x)].$$

Since V is base-point-free and $s_0, \dots, s_n$ is a basis of V, for each $x \in X$, there exists s_i such that $s_i(x) \neq 0$. Consequently, f is a well-defined morphism from X to $\mathbb{P}^n$.

Let L be a globally generated line bundle on X and let $V \subset H^0(X,L)$ be a base-point-free linear system. Then the morphisms obtained by the complete linear system, i.e., using the entire vector space $H^0(X,L)$, and the subseries V are related by a projection. Choose a basis $s_0, \dots, s_m$ for V and complete this basis to a basis of $H^0(X,L)$. Then the map defined by V is the projection of the map defined by $H^0(X,L)$ from the linear space defined by $x_{m+1} = \cdots = x_n = 0$. Hence, in order to understand morphisms from X to projective space, we can restrict ourselves to morphisms defined by complete linear systems.

Exercise 1.3. Show that $\mathcal{O}_{\mathbb{P}^2}(2)$ is a globally generated line bundle on $\mathbb{P}^2$. Show that the linear system V spanned by x^2, y^2, z^2, xy, xz is base-point-free. Show that the morphism defined by $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is the second Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ given by $[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : xz : yz]$. Show that the morphism defined by V is the projection of the second Veronese embedding from $[0 : 0 : 0 : 0 : 0 : 1]$.

Definition 1.4. A line bundle L is called *very ample* if $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ for an embedding $f : X \to \mathbb{P}^n$. A line bundle L is called *ample* if $L^{\otimes m}$ is very ample for some $m > 0$. A divisor D is *ample* if $\mathcal{O}_X(D)$ is ample.

There are cohomological, numerical and analytic characterizations of ample line bundles. First, Serre's Theorem gives a cohomological characterization of ampleness. In fact, Hartshorne uses this characterization as the definition of ampleness in [Ha, II.7].

Theorem 1.5 (Serre)**.** *A line bundle L is ample if and only if for every coherent sheaf $\mathcal{F}$ on X there exists an integer $m > 0$ such that $\mathcal{F} \otimes L^{\otimes n}$ is globally generated for $n \geq m$. Furthermore, there exists an integer m' such that $\mathcal{F} \otimes L^{\otimes n}$ has no higher cohomology for $n \geq m'$.*

Theorem 1.5 is a fundamental result in algebraic geometry whose proof can be found in [Ha, II.7.6] or [La, 1.2.6]. More importantly for our purposes, the Nakai-Moishezon criterion gives a numerical characterization of ampleness.

Theorem 1.6 (Nakai-Moishezon Criterion)**.** *A line bundle L on X is ample if and only if for every positive dimensional subvariety $Z \subseteq X$ the intersection number $L^{\dim Z} \cdot [Z] > 0$.*

The reader can find a proof of the Nakai-Moishezon criterion in [La, 1.2.23]. By the Nakai-Moishezon criterion, if $D_1 \equiv D_2$, then D_1 is ample if and only if D_2 is ample. Consequently, we can extend the notion of ampleness to $\mathbb{Q}$-divisors: a $\mathbb{Q}$-divisor D is ample if a positive multiple mD clearing all the denominators of the coefficients is ample. We can extend the definition to $\mathbb{R}$-divisors by requiring that an ample divisor is a positive $\mathbb{R}$-linear combination of ample divisors. Furthermore, since ampleness is a numerical condition, the notion makes sense for divisor classes in $\mathrm{NS}(X)$ or $\mathrm{NS}(X) \otimes \mathbb{R}$.

Exercise 1.7. Using Serre's characterization of ampleness, show that if L and M are ample line bundles on X, then $L \otimes M$ is also an ample line bundle on X.

Exercise 1.8. Using Serre's characterization of ampleness, show that if L is an ample line bundle and M is any line bundle, then $L^{\otimes m} \otimes M$ is ample for all $m >> 0$.

The two previous exercises show that the set of ample divisor classes in $\mathrm{NS}(X)$ forms an open, convex cone. We will call this cone the *ample cone* of X and denote it by $\mathrm{Amp}(X)$. It is one of the basic invariants of X and encodes the embeddings of X into projective space.

Definition 1.9. A line bundle is called *nef* if its degree on every curve $C \subset X$ is nonnegative. A divisor is called *nef* if $D \cdot C \geq 0$ for every curve $C \subset X$.

By definition, being nef is a numerical condition. Hence, we can extend the notion to $\mathbb{Q}$ or $\mathbb{R}$ divisors and it makes sense to consider nef divisor classes in $\mathrm{NS}(X) \otimes \mathbb{R}$. Any non-negative linear combination of nef divisors is again nef. Furthermore, for each curve C, the condition $C \cdot D \geq 0$ defines a closed half-space in the Néron-Severi space. As C varies over all curves in X, the intersection of all these half-spaces is still closed. Hence, the set of nef divisor classes in $\mathrm{NS}(X) \otimes \mathbb{R}$ forms a closed, convex cone $\mathrm{Nef}(X)$ called the *nef cone*. Since the degree of an ample line bundle on a curve is strictly positive, we have the containment $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$. Since $\mathrm{Amp}(X)$ is an open convex cone and $\mathrm{Nef}(X)$ is a closed convex cone, the containment is strict. The celebrated theorem of Kleiman clarifies the relation between these two cones.

Theorem 1.10 (Kleiman's Criterion)**.** *The nef cone is the closure of the ample cone. The ample cone is the interior of the nef cone.*

The reader can find a proof of Kleiman's Theorem in [La, 1.4.9]. The pullback of an ample line bundle under a birational morphism is nef but not ample. Hence, one can view nefness as a birational version of ampleness. The following lemma is useful in describing nef cones of varieties.

Lemma 1.11. *A base-point-free line bundle is nef.*

Proof. Let L be base-point-free, then L defines a morphism $f : X \to \mathbb{P}^n$. Let C be an irreducible curve on X. Pick a point $p \in C$. Pick a hyperplane $H \subset \mathbb{P}^n$ not containing $f(p)$. Hence $D = f^{-1}(H)$ is a section of L that does not contain p. A codimension one subvariety (such as D) has negative intersection with an irreducible curve C if and only if $C \subset D$. If $C \not\subset D$, then C intersects D in finitely many points and the intersection number is the number of intersection points counted with multiplicity (which are all positive). Hence, $D \cdot C \geq 0$. We conclude that L is nef. □

The following two exercises will explore the concepts we have introduced so far for curves. They are intended for students who have some familiarity with the theory of curves at the level of [Ha, Chapter 4].

Example 1.12. Let E be an elliptic curve. Let $L = \mathcal{O}_E(p)$ for a point $p \in E$. Then show that $L^{\otimes 2} = \mathcal{O}_E(2p)$ is globally generated but not very ample (hint: it defines a $2:1$ map to $\mathbb{P}^1$). Show that $L^{\otimes m}$ is very ample for $m \geq 3$.

Exercise 1.13. Let C be a smooth, projective curve of genus g. Let $\mathcal{O}_C(K_C)$ be the canonical bundle of C. Recall that $\mathcal{O}_C(K_C)$ is the line bundle of degree $2g-2$ dual to the tangent bundle T_C. Global sections of $\mathcal{O}_C(K_C)$ are holomorphic one-forms. Given a divisor D on C, the Riemann-Roch Theorem for curves calculates the Euler characteristic of $\mathcal{O}_C(D)$ in terms of the degree of D and the genus g of C:

$$h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)) = \deg(D) - g + 1.$$

Prove the following assertions.

(1) A line bundle L on C is base-point-free if and only if $h^0(C, L(-p)) = h^0(C, L) - 1$ for every $p \in C$.
(2) A line bundle L on C is very ample if and only if $h^0(C, L(-p-q)) = h^0(C, L) - 2$ for every $p, q \in C$ (possibly equal).
(3) A line bundle L on C is ample if and only if its degree is positive.
(4) A line bundle of degree $d \geq 2g + 1$ on C is very ample.
(5) Give an example of a line bundle of degree $2g$ on a curve of genus $g > 2$ which is very ample. Give an example of a line bundle of degree $2g$ on a curve of genus $g > 2$ which is not very ample. Notice that unlike ampleness, very ampleness is not a numerical condition.
(6) If $g \geq 3$, show that the canonical bundle is very ample if and only if C is not hyperelliptic.

Next, we introduce another set of cones that play an important role in birational geometry. We begin by introducing the effective cone.

Example 1.14. Being effective is not a numerical condition. Let E be an elliptic curve and let L be a nontrivial, degree zero line bundle. Then $\mathcal{O}_E$ and L are numerically equivalent since they both have degree zero. However, the only effective degree zero line bundle on a curve is the trivial bundle. Observe that we cannot fix this problem by taking multiples. If L is a non-torsion degree zero line bundle, then no multiple of L has a section. On the other hand, if L is an m-torsion line bundle and is not torsion of any lower order, then $L^{\otimes k}$ has a section if and only if m divides k.

Since being effective is not numerical, we need to exercise some care. The *effective cone* $\mathrm{Eff}(X)$ is the cone generated by the classes of all effective divisors. Since a non-negative linear combination of two effective divisors is again effective, $\mathrm{Eff}(X)$ is a convex cone. In general, it is neither open nor closed (see [La, 1.5.1] for an example of a two-dimensional effective cone that contains one of its extremal rays but not the other). The closure $\overline{\mathrm{Eff}}(X)$ of $\mathrm{Eff}(X)$ in $\mathrm{NS}(X)$ is called the *pseudo-effective* cone. Thanks to a recent theorem of Boucksom, Demailly, Paun and Peternell [BDPP], $\overline{\mathrm{Eff}}(X)$ has an intrinsic characterization as the cone dual to the cone of movable curve classes. An irreducible curve is *movable* if its deformations cover a Zariski dense subset of X. If C is a movable, irreducible curve and D is an effective divisor, then $C \cdot D \geq 0$ since some deformation of C is not contained in D. The theorem of Boucksom, Demailly, Paun and Peternell verifies the converse. The interior of $\mathrm{Eff}(X)$ also has an intrinsic characterization.

Definition 1.15. A line bundle L is called *big* if for some multiple $m > 0$, the dimension of the image of the rational map defined by $L^{\otimes m}$ is equal to the dimension of X.

A celebrated theorem of Kodaira shows that a divisor is big if and only if it is numerically equivalent to the sum of an ample divisor and an effective divisor [La, 2.2.6, 2.2.7]. In particular, being big is a numerical condition. The open convex cone generated by big divisor classes is called the *big cone* and will be denoted by $\mathrm{Big}(X)$. An ample divisor is clearly big, hence $\mathrm{Amp}(X) \subset$

$\mathrm{Big}(X)$ and a big divisor is clearly effective, so $\mathrm{Big}(X) \subset \mathrm{Eff}(X)$. In fact, $\mathrm{Big}(X)$ is the interior of $\overline{\mathrm{Eff}}(X)$ and $\overline{\mathrm{Eff}}(X)$ is the closure of $\mathrm{Big}(X)$.

In this section, we have introduced several cones in $\mathrm{NS}(X)$. We have the containments $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$ and $\mathrm{Amp}(X) \subset \mathrm{Big}(X)$ and $\mathrm{Nef}(X), \mathrm{Big}(X) \subset \overline{\mathrm{Eff}}(X)$.

Remark 1.16. When one studies moduli spaces or birational geometry, one inevitably encounters singular varieties. It is possible to extend the discussion in this section to mildly singular varieties. On a singular variety the notions of Weil divisors and line bundles diverge. To obtain an equivalence with line bundles, one has to restrict to Cartier divisors (see [Ha, II.6.14, II.6.15]). Let X be a normal, projective variety over the complex numbers. A Weil divisor D on X is called $\mathbb{Q}$-*Cartier*, if there exists an integer $m > 0$ such that mD is Cartier. A variety is called $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. Since much of the theory described in this section is asymptotic, it extends to $\mathbb{Q}$-factorial, normal varieties without much trouble (see [La] for a systematic treatment). Many interesting moduli spaces such as the moduli space of curves or moduli spaces of Gieseker semi-stable sheaves on surfaces are constructed as GIT quotients and are $\mathbb{Q}$-factorial projective varieties. Hence, the theory applies to them.

2. Birational Geometry

In this section, we will demonstrate techniques for computing $\mathrm{Amp}(X)$ and $\mathrm{Eff}(X)$ in several simple examples. We will illustrate the close connection between these cones and the birational geometry of X.

Example 2.1. In this example, we will compute the ample and effective cones for the blowup X of $\mathbb{P}^2$ at a point p. Recall that X is the graph of the projection of $\mathbb{P}^2$ from p. By choosing appropriate coordinates, we may assume that $p = [0:0:1]$. The projection is given by $[x:y:z] \mapsto [x:y]$ and is defined away from p. The graph has equations

$$X = \{([x:y:z],[u:v]) | xv = yu\} \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

The first projection $\pi_1 : X \to \mathbb{P}^2$ is a birational map. Over $\mathbb{P}^2 - p$, π_1 has a well-defined inverse given by $[x:y:z] \mapsto ([x:y:z],[x:y])$. The inverse map is not defined at p. If $x = y = 0$, then every $[u,v] \in \mathbb{P}^1$ satisfies the equation $xv = yu$. Hence, the inverse image of $\pi_1^{-1}(p)$ is a rational curve E called the *exceptional curve*. The second projection π_2 defines a morphism $\pi_2 : X \to \mathbb{P}^1$ and exhibits X as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, where the fiber over $[u:v]$ is the line in $\mathbb{P}^2$ through p with slope $[u:v]$.

In order to understand $\mathrm{Amp}(X)$ and $\mathrm{Eff}(X)$, we first have to describe $\mathrm{NS}(X)$. If Y is a subvariety of Z of codimension at least two, then $\mathrm{Div}(Z) \cong \mathrm{Div}(Z-Y)$. Hence, $\mathrm{Div}(\mathbb{P}^2) \cong \mathrm{Div}(\mathbb{P}^2 - p)$. On the other hand, $\mathbb{P}^2 - p$ is isomorphic to $X - E$, so $\mathrm{Div}(\mathbb{P}^2) \cong \mathrm{Div}(X - E)$. Let H denote the pullback of the class of a line by π_1. Using the exact sequence [Ha, II.6]

$$\mathbb{Z}E \to \mathrm{Pic}(X) \to \mathrm{Pic}(X-E) \to 0,$$

we conclude that H and E generate the Picard group of X. In fact, $\mathrm{Pic}(X) \cong \mathbb{Z}H \oplus \mathbb{Z}E$. We can compute the intersection pairing on X. By taking two distinct lines that avoid p, it is clear that $H^2 = 1$. On the other hand, by taking a line avoiding p, it is also clear that $H \cdot E = 0$. The hardest calculation is E^2. Take two separate lines that pass through p and consider their proper transforms. Setting $y = xv$, we see that the equation of the line $y - ax$ becomes $x(v-a)$. The total transform of the lines vanish once along the exceptional divisor (whose equation is $x = 0$) and the proper transforms intersect the exceptional divisor at $[1:a]$. The proper transforms of the lines have class $H - E$ and are disjoint. We see that $(H-E)^2 = 0$, and, using $H^2 = 1, H \cdot E = 0$, we conclude that $E^2 = -1$. Therefore, the intersection pairing is given by

$$H^2 = 1, H \cdot E = 0, E^2 = -1.$$

In particular, H and E are numerically independent. Consequently, the Néron-Severi space is the two-dimensional $\mathbb{Q}$ vector space spanned by H and E.

The ample cone in $\mathrm{NS}(X)$ is simple to describe. We already know that X admits two morphisms π_1 and π_2 to $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively. The divisor classes that define these morphisms are H and $H-E$, respectively. Hence, these divisor classes are base-point-free. By Lemma 1.11, they are also nef. On the other hand, neither of these classes are ample. The morphism defined by H contracts the exceptional curve E and has $H \cdot E = 0$. The intersection number $(H-E)^2 = 0$ shows that $H-E$ is not ample. We conclude that H and $H-E$ are the two extremal rays that bound $\mathrm{Nef}(X)$.

Since X is a surface, the effective cone is dual to the ample cone under the intersection pairing. We conclude that the boundary rays of the effective cone are spanned by E and $H-E$. Alternatively, we can exhibit irreducible curves whose deformations cover a Zariski open set in X and have intersection number zero with E and $H-E$. If C is an irreducible curve and D is an effective divisor, then, as we remarked earlier, $C \cdot D \geq 0$ unless $C \subset D$. If deformations of the curve cover a Zariski open set, then we can find C such that $C \not\subset D$. Hence, $C \cdot D \geq 0$ for every effective divisor. Having $C \cdot D = 0$, forces D to lie on the boundary of the cone. Note that both H (the class of a line not passing through p) and $H-E$ (the proper transform of a line passing through p) are classes of irreducible curves whose deformations cover Zariski open sets in X. Therefore, E and $H-E$ span the extremal rays of $\mathrm{Eff}(X)$ since $E \cdot H = 0$ and $(H-E)^2 = 0$.

In conclusion, we learn that $\mathrm{Nef}(X)$ is the closed cone spanned by H and $H-E$ and $\mathrm{Eff}(X)$ is the closed cone spanned by E and $H-E$. A large enough multiple of any divisor $D = aH + b(H-E)$ with $a, b > 0$ defines an embedding of X. The divisor H defines the blowdown map $\pi_1 : X \to \mathbb{P}^2$, whereas the divisor $H-E$ defines the second projection $\pi_2 : X \to \mathbb{P}^1$.

Example 2.2. To give a slightly more sophisticated example, let us consider the blowup X of $\mathbb{P}^2$ at two points p_1, p_2. As in the previous example, one may write the equations for X explicitly. By choosing appropriate coordinates, we may assume that $p_1 = [0:0:1]$ and $p_2 = [0:1:0]$. Then the equations of X can be written as

$$\{([x:y:z],[u:v],[s:t]) | xv = yu, xt = zs\} \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

The reader should check that the first projection $\pi_1 : X \to \mathbb{P}^2$ is a birational morphism whose inverse is defined everywhere but the two points p_1, p_2. The inverse images of p_i are rational curves E_i for $1 \leq i \leq 2$. The projections to the two other factors $\pi_2, \pi_3 : X \to \mathbb{P}^1$ define two morphisms to $\mathbb{P}^1$. As in the previous example, the Picard group is isomorphic to $\mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$. We have the intersection pairing

$$H^2 = 1, H \cdot E_i = E_1 \cdot E_2 = 0, E_i^2 = -1.$$

(Exercise: The reader should verify these two statements.) Consequently, these classes are numerically independent and the Néron-Severi space is the three dimensional $\mathbb{Q}$-vector space spanned by H, E_1, E_2.

There are three self-intersection -1 rational curves on X: the two exceptional curves E_1, E_2 and the proper transform of the line joining p_1 and p_2 with class $H - E_1 - E_2$. The dual cone to the cone generated by these three curves is the cone spanned by $H, H-E_1$ and $H-E_2$. The classes H, $H-E_1$ and $H-E_2$ are base-point-free since they define the three projection morphisms π_1, π_2 and π_3, respectively. We conclude that the cone they span is contained in $\mathrm{Nef}(X)$. However, since this cone is dual to a cone generated by three effective curves, the nef cone cannot be any larger. Therefore, $\mathrm{Nef}(X)$ is the closed cone spanned by $H, H-E_1$ and $H-E_2$. Dually, the effective cone is the closed cone spanned by E_1, E_2 and $H-E_1-E_2$.

If we take any divisor D in the interior of $\mathrm{Nef}(X)$, then D is ample and a sufficiently large multiple defines an embedding of X. The divisor H, by its definition is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ from $\mathbb{P}^2$ and it defines the blow down map $\pi_1 : X \to \mathbb{P}^2$. Similarly, the divisors $H-E_1$ and $H-E_2$ are the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ via the projections π_2 and π_3, respectively. Hence, they define the two projections to $\mathbb{P}^1$. If D is a positive linear combination of H and $H - E_1$ (respectively, H and $H - E_2$), then the corresponding morphisms are the blowdown of E_2 (respectively, E_1) (i.e., the blowup of $\mathbb{P}^2$ at p_1 and p_2, respectively) defined by the projection $\pi_{1,2}$ (respectively, $\pi_{1,3}$) to the first two factors (respectively, to the first and third factor). Finally, if we take a positive linear combination of $H - E_1$ and $H - E_2$, we obtain the projection onto $\mathbb{P}^1 \times \mathbb{P}^1$. We thus see the correspondence between the points of $\mathrm{Nef}(X)$ and the morphisms from X to other projective varieties very explicitly.

Exercise 2.3. Compute the ample and effective cones of the blowup of $\mathbb{P}^2$ at three non-collinear points p_1, p_2, p_3. (Hint: Show that the effective cone is spanned by the three exceptional divisors E_1, E_2, E_3 lying over p_1, p_2, p_3 and the proper transforms of the three lines joining p_i, p_j for $1 \leq i < j \leq 3$. The ample cone is dual to this cone under the intersection pairing.)

Exercise 2.4. Compute the ample and effective cones of the blowup of $\mathbb{P}^2$ at three collinear points p_1, p_2, p_3. Describe how the answer differs from the previous exercise.

We now turn to a higher dimensional example to illustrate some new features that arise when one leaves the realm of surfaces.

Example 2.5. Recall from Olivier Debarre's lectures that the Grassmannian $\mathbb{G}(1,3)$ parameterizing lines in $\mathbb{P}^3$ is a 4-dimensional smooth variety. The Plücker map embeds $\mathbb{G}(1,3)$ into $\mathbb{P}^5$ as a quadric hypersurface with equation

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0.$$

Flag varieties provide a natural generalization of Grassmannians. The flag variety $\mathbb{F}(k_1, k_2, \ldots, k_r; n)$ parameterizes linear partial flags

$$\mathbb{P}^{k_1} \subset \mathbb{P}^{k_2} \subset \cdots \subset \mathbb{P}^{k_r} \subset \mathbb{P}^n.$$

For example, $\mathbb{F}(0,1;3)$ parameterizes pointed lines in $\mathbb{P}^3$, $\mathbb{F}(0,1,2;3)$ parameterizes triples $p \subset l \subset \Pi$ of consisting of a point p, a line l and a plane Π in $\mathbb{P}^3$.

Fix a line $\Lambda \subset \mathbb{P}^3$ spanned by the two points e_1, e_2. The Schubert variety $\Sigma_1(\Lambda)$ parameterizing lines in $\mathbb{P}^3$ that intersect the fixed line Λ is defined by the vanishing of the Plücker coordinate $x_{34} = 0$. Hence, Σ_1 is the quadric cone in $\mathbb{P}^5$ defined by the equations $x_{34} = x_{13}x_{24} - x_{14}x_{23} = 0$ with singular point $x_{13} = x_{14} = x_{23} = x_{24} = x_{34} = 0$. Consider the following subvarieties of flag varieties closely related to $\Sigma_1(\Lambda)$.

$$X_1 := \{(p,l) | p \in l \cap \Lambda\} \subset \mathbb{F}(0,1;3)$$

$$X_2 := \{(l,\Pi) | l, \Lambda \subset \Pi\} \subset \mathbb{F}(1,2;3)$$

$$X_3 := \{(p,l,\Pi) | p \in l \cap \Lambda, l, \Lambda \subset \Pi\} \subset \mathbb{F}(0,1,2;3)$$

The variety X_1 parameterizes pointed lines (p,l) such that the point p is in the intersection of l with the fixed line Λ. In particular, l has to intersect Λ and, unless the line $l = \Lambda$, the point p is uniquely determined by l. The variety X_2 parameterizes pairs (l, Π) such that Π contains the span of l and Λ. Hence, l has to intersect Λ and Π is uniquely determined by l unless l equals Λ. Finally, X_3 parameterizes triples (p, l, Π), where p is in the intersection of l and Λ and Π contains the span of l and Λ. If $l \neq \Lambda$, then p and Π are uniquely determined. Observe that by projecting to l, X_1, X_2 and X_3 admit morphisms to $\Sigma_1(\Lambda)$. All three morphisms are birational. The exceptional locus in the first two cases are $\mathbb{P}^1$ (the pairs (p,Λ) with $p \subset \Lambda$ and the pairs (Λ, Π) with $\Lambda \subset \Pi$) and in the last case $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 2.6. By noticing that both X_1 and X_2 are $\mathbb{P}^2$ bundles over $\mathbb{P}^1$ show that they are smooth. Show that X_3 is the blowup of $\Sigma_1(\Lambda)$ at the singular point and it is smooth. Hence, X_1, X_2 and X_3 all provide resolutions of singularities of $\Sigma_1(\Lambda)$. Show that both X_1 and X_2 are small resolutions in the sense that the exceptional locus has codimension 2. The exceptional locus of the projection $\pi_3 : X_3 \to \Sigma_1(\Lambda)$ is a divisor.

Notice that X_1 admits a rational map to X_2 by sending (p, l) to (l, Π), where Π is the span of l and Λ. This map is only a rational map and is not a morphism (the map is not defined when $l = \Lambda$). These types of rational maps are called *flops*. We will not make an attempt to define or explain their importance here. The reader who would like a systematic introduction to higher dimensional birational geometry can start with the article [CCJ] and the books [D], [KM] and [Ko].

We can describe the nef and effective cones of X_3 as in the previous examples. The following exercise will help you work these out.

Exercise 2.7. Show that the three projections to p, l and Π define three morphisms from X_3 to $\mathbb{P}^1$, $\Sigma_1(\Lambda)$ and $\mathbb{P}^1$, respectively. Show that the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{G}(1,3)}(1)$ via these maps generate $\mathrm{Nef}(X_3)$. Describe the morphisms one obtains from non-negative linear combinations of the corresponding divisor classes. Take special note that the projection to $\Sigma_1(\Lambda)$ results in a singular variety. Hence, even when one wishes to study the birational geometry of smooth varieties, one naturally encounters singular varieties. See [KM], [Ko2], [R] for an in depth discussion of the singularities that occur in the minimal model program. For recent developments see [BCHM].

3. The Hilbert scheme of points on the plane

In this section, we will introduce the Hilbert scheme of points on $\mathbb{P}^2$ and discuss its cones of ample and effective divisors. The reader who wishes to learn more about the geometry of Hilbert schemes of points can start with [G], [Le] and [N]. For more information on the birational geometry of the Hilbert scheme of points on surfaces, the reader can start with [ABCH], [BM], [BM2], [BC], [CH] and [Hui].

Let X be a smooth, projective variety. Let $n > 1$ be an integer. The configuration space $\mathrm{Config}_n(X)$ of n points on X parameterizes n-unordered tuples of points on X. Unfortunately, $\mathrm{Config}_n(X)$ is not compact since distinct points on X can tend to each other. The symmetric product $X^{(n)}$ gives a natural compactification of $\mathrm{Config}_n(X)$. Recall that $X^{(n)}$ is the quotient of the product X^n by the symmetric group action $\mathfrak{S}_n$ permuting the factors. When $\dim(X) = 1$, $X^{(n)}$ is a smooth, projective variety and gives a nice compactification of $\mathrm{Config}_n(X)$. When $\dim(X) \geq 2$, $X^{(n)}$ is singular. In this section, we will discuss the Hilbert scheme of points on X introduced by Grothendieck, which is a desingularization of $X^{(n)}$ when $\dim(X) = 2$.

Exercise 3.1. Given n points $[u_1 : v_1], \dots, [u_n : v_n]$ on $\mathbb{P}^1$, show that the homogeneous polynomial $\prod_{i=1}^n (v_i x - u_i y)$ of degree n is well-defined up to a scalar multiple. Hence, n unordered tuples of points on $\mathbb{P}^1$ can be uniquely parameterized by the coefficients of the corresponding polynomial up to scaling. Deduce that $\mathbb{P}^{1(n)} \cong \mathbb{P}^n$.

Exercise 3.2. Let X be a smooth, projective curve of genus g. Then, $X^{(n)}$ admits a morphism ϕ_n to $\mathrm{Pic}^n(X)$ by sending $\sum_{i=1^n} p_i$ to $\mathcal{O}_X\left(\sum_{i=1}^n p_i\right)$. Show that when $n > 2g-2$, ϕ_n realizes $X^{(n)}$ as a projective bundle over $\mathrm{Pic}^n(X)$. The fiber over a point $\mathcal{O}_X\left(\sum_{i=1}^n p_i\right)$ is the linear system $|\mathcal{O}_X(\sum_{i=1}^n p_i)|$. Show that ϕ_n is surjective if $n \geq g$, but in the range $g \leq n \leq 2g-2$ the fiber dimension of ϕ_n jumps over line bundles that have higher cohomology.

The idea of Grothendieck is to take Exercise 3.1 as a starting point. Rather than considering the set of distinct points Z on X, we can consider polynomials that vanish on Z. We get an ideal I_Z with the property that $h^0(\mathcal{O}_Z(k)) = n$ for all $k \geq 0$. Grothendieck proposes to take the set of all ideal sheaves $I \subset \mathcal{O}_X$ such that $h^0(\mathcal{O}_X/I(k)) = n$ as a compactification of $\mathrm{Config}_n(X)$. In fact, much more generally, he considers the set $\mathrm{Hilb}_P(X)$ of schemes Z whose ideal sheaves $I_Z \subset \mathcal{O}_X$ have a fixed Hilbert polynomial P. He shows that this set has naturally the structure of a projective scheme (see [HMo] for an explanation of the word naturally).

Theorem 3.3 (Grothendieck)**.** *There is a projective scheme* $\mathrm{Hilb}_P(X)$ *parameterizing ideal sheaves with Hilbert polynomial* P *in* X *such that* $\mathrm{Hilb}_P(X)$ *represents the Hilbert functor associating to each scheme* S *the families in* $S \times X$ *flat over* S *with Hilbert polynomial* P*.*

This is an important theorem to learn (see [M], [S]) and would make a nice reading project for any student who has studied [Ha, II and III]. For smooth curves, the Hilbert scheme and the symmetric product coincide. By Exercise 3.2, we understand the geometry of the Hilbert scheme of points on a smooth curve well, at least if the number of points is large compared to the genus. However, in higher dimensions the Hilbert scheme and the symmetric product are different. In this section, we are primarily interested in $\mathrm{Hilb}_n(\mathbb{P}^2)$, which we will abbreviate as $\mathbb{P}^{2[n]}$. To give the reader an idea of the points of $\mathbb{P}^{2[n]}$, we give some examples. Since we can always find a line that misses finitely many points, we will consider our examples in $\mathbb{C}^{2[n]}$ and write non-homogenous equations.

Example 3.4. Two distinct points $(0,0),(1,0)$ in $\mathbb{C}^2$ have ideal generated by $y, x(x-1)$ (exercise: prove this!). Then

$$\dim \mathbb{C}[x,y]/(y, x(x-1)) = 2$$

spanned by 1 and x. Next, consider the ideals $(y - ax, x^2)$. We have

$$\dim \mathbb{C}[x,y]/(y-ax, x^2) = 2$$

also spanned by 1 and x. Hence, these ideals also belong to the Hilbert scheme of two points. By varying $a \in \mathbb{C}$, we get a one-parameter family of such ideals. Unlike the symmetric product, which would only record the fact that there is a double point at the origin, the Hilbert scheme has a distinguished line $(y - ax)$ associated to each double point. For this reason, these length two schemes are typically denoted by a tangent vector with slope a.

Exercise 3.5. Show that $\mathbb{P}^{2[2]}$ is the blowup of $\mathbb{P}^{2(2)}$ along the diagonal. More generally, show that if X is a smooth variety, then the Hilbert scheme of two points $X^{[2]}$ is the blowup of the symmetric product $X^{(2)}$ along the diagonal.

Example 3.6. After a change of coordinates, we may assume that three non-collinear points are given by $(0,0),(1,0),(0,1)$. Their ideal is generated by the polynomials $I = (xy, x(x-1), y(y-1))$ (exercise: prove this). We see that $\mathbb{C}[x,y]/I$ is spanned by $1, x, y$, hence has dimension 3.

When the three points become collinear the equations change. Consider the points $(0,0),(0,1),(0,2)$. Their ideal is generated by $I = (x, y(y-1)(y-2))$. This time $\mathbb{C}[x,y]/I$ is spanned by $1, y, y^2$ and still has dimension 3.

When the three points collide, the possibilities become more interesting. First, we can have ideals of the form $I_1 = (x^3, y - ax - bx^2)$ for some $a, b \in \mathbb{C}$. Since $\mathbb{C}[x,y]/I_1$ is spanned by $1, x, x^2$, this is a point of $\mathbb{C}^{2[3]}$. More interestingly, consider the square of the maximal ideal $I_2 = (x^2, xy, y^2)$. Then $\mathbb{C}[x,y]/I_2$ is spanned by $1, x, y$ and hence belongs to $\mathbb{C}^{2[3]}$. The difference between I_1 and I_2 is that the scheme defined by I_1 is contained in a smooth curve (defined by $y - ax - bx^2$), whereas the scheme defined by I_2 is not contained in a smooth curve (prove this by showing that the Zariski tangent space is not one dimensional).

Example 3.7. A scheme supported at one point is called a *punctual scheme*. To understand points of $\mathbb{P}^{2[n]}$, it suffices to understand punctual schemes since any zero-dimensional scheme naturally decomposes into punctual schemes along its support. Let $I = (x^n, y - a_1 x - a_2 x^2 - \cdots - a_{n-1}x^{n-1})$. Then $\mathbb{C}[x,y]/I$ is spanned by $1, x, x^2, \cdots, x^{n-1}$, hence I is a point of $\mathbb{C}^{2[n]}$. These zero-dimensional schemes are called *curvilinear schemes* since they are contained in the smooth curve defined by $y - a_1 x - a_2 x^2 - \cdots - a_{n-1}x^{n-1}$. They form an $(n-1)$-dimensional smooth locus in the punctual Hilbert scheme of length n. A Theorem of Briançon (see [G] or [Le]) says that curvilinear schemes are dense in the punctual Hilbert scheme of a surface. Hence, the punctual Hilbert scheme of a surface is irreducible of dimension $n-1$. As a consequence, one can show that the locus of non-reduced schemes on a surface is an irreducible divisor. The next example shows that Briançon's Theorem fails when $\dim X > 2$.

Example 3.8 (Iarrabino's Example)**.** When the dimension of X is greater than 2, the Hilbert scheme of points is highly singular. In fact, it is not even a compactification of $\mathrm{Config}_n(X)$ as it might have many other components, some of different dimensions. Here we give an example due to Iarrabino that shows that $\mathbb{C}^{3[96]}$ is reducible.

$\mathrm{Config}_{96}(\mathbb{C}^3)$ is a 288-dimensional complex manifold and it is a Zariski open subset in $\mathbb{C}^{3[96]}$. Let $\mathfrak{m}$ denote the maximal ideal at the origin in $\mathbb{C}^3$ and let V be a 24-dimensional subspace of the space of homogeneous polynomials of degree 7 in 3 variables. Let $I = < V, \mathfrak{m}^8 >$ be the ideal generated by V and all homogeneous polynomials of degree 8. Recall that the dimension of the vector space of homogeneous polynomials of degree d in three variables is $\binom{d+2}{2}$. Then

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x,y,z]}{I} = \sum_{i=0}^{6} \binom{i+2}{2} + 12 = 96,$$

where 12 is the dimension of the space of degree 7 polynomials remaining after we quotient by V. Hence, ideals of this form belong to $\mathbb{C}^{3[96]}$. On the other hand, such ideals are determined by the choice of the vector space V, which are parameterized by the Grassmannian $G(24,36)$. The Grassmannian $G(24,36)$ has dimension $24 \times (36-24) = 288$. Finally, so far our ideals have been supported at the origin, but we can move the support to any other point in $\mathbb{C}^3$. We thus obtain a locus of $\mathbb{C}^{3[96]}$ of dimension at least $291 > 288$. This example shows that when $\dim(X) \geq 3$, there are schemes supported at one point that are not limits of smooth, distinct points. Hence, the locus of distinct points is not dense in the Hilbert scheme. It is an open problem to determine when a scheme is in the closure of the locus of distinct points.

In contrast to higher dimensions, by a theorem of Fogarty, the Hilbert scheme of a surface is as nice as possible.

Theorem 3.9 (Fogarty [F1])**.** *Let S be a smooth projective surface. The Hilbert scheme of points $S^{[n]}$ is a smooth, irreducible, projective variety of dimension $2n$. The configuration space* $\mathrm{Config}_n(S)$ *is a dense Zariski open subset of $S^{[n]}$.*

For the rest of this section, we will restrict to the case $S = \mathbb{P}^2$. The Hilbert schemes of points on surfaces play an important role in many branches of mathematics, including in algebraic geometry, topology, combinatorics, representation theory and mathematical physics. The reader who is interested in pursuing some of these topics can start with Haiman's work on the $n!$ conjecture [Hai] or Nakajima's work on the cohomology of the Hilbert scheme of points [N].

There is a morphism $h : \mathbb{P}^{2[n]} \to \mathbb{P}^{2(n)}$ called the *Hilbert-Chow morphism* that associates to a scheme Z the element of the symmetric product $\sum_{p \in \mathrm{Supp}(Z)} l_p(Z) p$ (see [Le]). We can understand the Néron-Severi space of $\mathbb{P}^{2[n]}$ in terms of the Hilbert-Chow morphism. The Hilbert-Chow morphism gives a *crepant resolution* of the symmetric product, i.e., h is a resolution such that $h^* K_{\mathbb{P}^{2(n)}} = K_{\mathbb{P}^{2[n]}}$. The exceptional locus of h is the irreducible divisor parameterizing non-reduced schemes. We will call this divisor B. The Picard group of $\mathbb{P}^{2(n)}$ is generated by a single element. We can pull it back via h. Geometrically, schemes whose support intersect a fixed line l in $\mathbb{P}^2$ give a section of this line bundle on $\mathbb{P}^{2[n]}$. We denote its class by H. Using h, it is easy to conclude that the Néron-Severi space of $\mathbb{P}^{2[n]}$ is the two-dimensional $\mathbb{Q}$-vector space spanned by H and B. In fact, Fogarty computed the Picard group over $\mathbb{Z}$.

Theorem 3.10 (Fogarty [F2])**.** *The Picard group of $\mathbb{P}^{2[n]}$ is isomorphic to $\mathbb{Z}H \oplus \mathbb{Z}\frac{B}{2}$.*

Unfortunately, the class $\frac{B}{2}$ is not effective, so it is harder to make sense of it geometrically. Since we are working over $\mathbb{Q}$, we can instead use the more geometric divisor B.

Now that we have gathered this basic information about $\mathbb{P}^{2[n]}$, we can ask for the ample and effective cones of $\mathbb{P}^{2[n]}$. We already know that the Hilbert-Chow morphism h is a birational morphism from $\mathbb{P}^{2[n]}$ to $\mathbb{P}^{2(n)}$. However, h is not an isomorphism. It contracts the locus of non-reduced schemes. More concretely, fix $n-2$ distinct points $p_1, \dots, p_{n-2}$ different from the origin. Consider the curve induced in $\mathbb{P}^{2[n]}$ via the one-parameter family of schemes of length n given by the union of $p_1, \dots, p_{n-2}$ with schemes of length two supported at the origin. Under h this curve maps to a point. We conclude that H is a base-point-free divisor, which is not ample. Hence, it defines an extremal edge of the nef cone.

Finding the other extremal edge of the nef cone is harder. We begin by defining some rational maps to Grassmannians. Consider the standard exact sequence of sheaves

$$0 \to I_Z \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Z \to 0.$$

Twisting this sequence by $\mathcal{O}_{\mathbb{P}^2}(k)$, we get the exact sequence

$$0 \to I_Z(k) \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{O}_Z(k) \to 0.$$

The associated long exact sequence of cohomology yields the inclusion

$$H^0(\mathbb{P}^2, I_Z(k)) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)).$$

This is fancy notation for expressing the simple fact that homogeneous polynomials of degree k in three variables that vanish on the scheme Z is a subvector space of the vector space of all homogeneous polynomials of degree k in three variables. The latter vector space has dimension $N = \binom{k+2}{2}$. To require a polynomial to vanish at a point is one linear condition on the polynomials. If the conditions are independent, we would expect the vector space $H^0(\mathbb{P}^2, I_Z(k))$ to have dimension $N-n$. For a general set of points, these conditions will be independent and $N-n$ will be the dimension of the vector space. However, for special sets of points, the conditions may fail to be independent.

Example 3.11. Let p_1, p_2, p_3, p_4 be 4 distinct collinear points. Let Z be the zero-dimensional scheme consisting of their union. We would expect a scheme of length 4 to impose 4 conditions on polynomials of degree 2. However, any polynomial of degree 2 vanishing on Z, by Bezout's Theorem, must vanish on the line l they span. Hence, any degree two polynomial vanishing on Z is the product of the equation of l with any other linear form. Hence, there is a 3 dimensional space of polynomials of degree 2 vanishing on Z instead of the expected 2.

Exercise 3.12. Show that 4 points impose independent conditions on polynomials of degree 2 if and only if they are not collinear.

Exercise 3.13. Show that general points impose independent conditions on homogeneous polynomials of degree k (hint: choose the points inductively to reduce the dimension of the vector space by 1 at each stage. Suppose you have chosen m points such that the polynomials of degree k vanishing on them has dimension $N-m$. Pick such a polynomial f and let your $(m+1)$st point be any point not in the zero locus of f).

By sending a scheme Z to the vector space $H^0(\mathbb{P}^2, I_Z(k))$, we get a rational map

$$\phi_k : \mathbb{P}^{2[n]} \dashrightarrow G(N-k, N)$$

to the Grassmannian of $(N-k)$-dimensional subspaces of $H^0(\mathcal{O}_{\mathbb{P}^2}(k))$. In general, ϕ_k is only a rational map because some of the schemes may fail to impose independent conditions on polynomials of degree k, in which case, there isn't an $(N-k)$-dimensional subspace associated to them. For example, $\phi_2 : \mathbb{P}^{2[4]} \dashrightarrow G(2, 6)$ is not defined along the locus of collinear schemes of length 4.

When $k \geq n-1$, then ϕ_k is always a morphism. In other words, schemes of length n always impose independent conditions on polynomials of degree at least $n-1$. There are several ways of proving this fact. One may either use the theory of k-very ample line bundles ([LQZ] or [ABCH]) or one can use facts concerning resolutions of ideals of zero-dimensional schemes [BC]. Every zero-dimensional scheme in $\mathbb{P}^2$ has a minimal free resolution of the form

$$0 \to \oplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \oplus_{i=1}^{m+1} \mathcal{O}_{\mathbb{P}^2}(-a_i) \to I_Z \to 0,$$

where $n+1 \geq b_j$ for $1 \leq j \leq m$ and $n \geq a_i$ for $1 \leq i \leq m+1$ (see [E]). Twisting by $\mathcal{O}_{\mathbb{P}^2}(k)$ and taking cohomology, one can see that $h^1(\mathbb{P}^2, I_Z(k)) = h^2(\mathbb{P}^2, I_Z(k)) = 0$ for $k \geq n-1$. Since the Euler characteristic is constant, we conclude that $h^0(\mathbb{P}^2, I_Z(k))$ always has the expected dimension if $k \geq n-1$.

Consider the morphism $\phi_{n-1} : \mathbb{P}^{2[n]} \to G = G(\binom{n+1}{2} - n, \binom{n+1}{2})$. The pullback $\phi_{n-1}^* \mathcal{O}_G(1)$ is a base-point-free divisor. Hence, it is nef. On the other hand, ϕ_{n-1} is not an embedding. Every scheme of length n imposes independent conditions on polynomials of degree $n-1$; however, polynomials of degree $n-1$ do not suffice to cut out every scheme of length n. Suppose Z consists of n collinear points. Then any polynomial of degree $n-1$ vanishing on Z vanishes along the line containing Z. Hence, the vector space of polynomials of degree $n-1$ vanishing on Z is the vector space of polynomials of degree $n-1$ that are divisible by the equation of the line. If we take any other n points on the same line, this vector space does not change. Hence, $\phi_{n-1}^* \mathcal{O}_G(1)$ has degree zero on positive-dimensional subvarieties of $\mathbb{P}^{2[n]}$ and is not ample.

Exercise 3.14. Check that polynomials of degree n that are divisible by the equation of a fixed line form a vector space of dimension $\binom{n+1}{2} - n$.

In order to calculate the ample cone in our given basis, there remains to compute the class of $\phi_{n-1}^* \mathcal{O}_G(1)$. We can use test curves to compute this class. Fix $n-1$ general points Γ and a general line l disjoint from the points. Consider the curve A in $\mathbb{P}^{2[n]}$ obtained by taking the union of Γ with a point varying along l. Since none of these schemes are reduced, the resulting curve is disjoint from B. Its degree with respect to H is one. Finally, fix $\binom{n+1}{2} - n$ general points Ω and consider the linear spaces W of polynomials of degree $n-1$ that vanish at these points. Then subspaces of codimension n that intersect W give a section of $\mathcal{O}_G(1)$. There is a unique curve of degree $n-1$ containing $\Gamma \cup \Omega$. The line l intersects this curve in $n-1$ points. Consequently, we have the following intersection numbers

$$A \cdot H = 1, \ A \cdot B = 0, \ A \cdot \phi_{n-1}^* \mathcal{O}_G(1) = n-1.$$

Next, take a general pencil in $|\mathcal{O}_{\mathbb{P}^1}(n)|$ and consider the curve C induced in $\mathbb{P}^{2[n]}$. By the Riemann-Hurwitz formula, this pencil is ramified $2n-2$ times. The points in the pencil meet a general line once. Since the resulting map to G is constant it has degree zero on $\phi_{n-1}^* \mathcal{O}_G(1)$. We conclude that we have the following intersection numbers

$$C \cdot H = 1, \ C \cdot B = 2n-2, \ C \cdot \phi_{n-1}^* \mathcal{O}_G(1) = 0.$$

We conclude that the class of $\phi_{n-1}^* \mathcal{O}_G(1)$ is $(n-1)H - \frac{1}{2}B$. We have proved the following theorem.

Theorem 3.15 ([LQZ], see also [ABCH])**.** *The nef cone of $\mathbb{P}^{2[n]}$ is the closed cone spanned by H and $(n-1)H - \frac{1}{2}B$.*

Next, we describe the effective cone of $\mathbb{P}^{2[n]}$. The locus of nonreduced schemes B is the exceptional divisor of the Hilbert-Chow morphism. Consequently, it defines an extremal edge of the effective cone. The other extremal edge of the effective cone of $\mathbb{P}^{2[n]}$ is harder to compute and depends more subtly on the arithmetic properties of n. We will give some examples and refer the reader to the literature for the general answer.

Example 3.16 (n is a triangular number)**.** In $\mathbb{P}^{2[3]}$ the locus of collinear schemes $D_{\mathcal{O}(1)}$ forms a divisor. The class of $D_{\mathcal{O}(1)}$ can easily be computed as $H - \frac{1}{2}B$ using test families.

Exercise 3.17. Let C be a smooth conic in $\mathbb{P}^2$ and let p_1, p_2, p_3 be three points on C. Take a pencil in the linear system $|\mathcal{O}_C(p_1+p_2+p_3)|$. Show that the induced curve in $\mathbb{P}^{2[3]}$ is disjoint from the locus of collinear points. Conclude that $D_{\mathcal{O}(1)}$ has class proportional to $H - \frac{1}{2}B$.

There is a smooth conic passing through any three non-collinear points. Hence, the curve described in the previous exercise is a moving curve which is disjoint from $D_{\mathcal{O}(1)}$. We conclude that $D_{\mathcal{O}(1)}$ spans the other extremal edge of the effective cone.

Similarly, in $\mathbb{P}^{2[6]}$ the locus of schemes that lie on a conic $D_{\mathcal{O}(2)}$ forms a divisor. As in the previous case, we can easily compute the class of this divisor.

Exercise 3.18. Fix a smooth cubic curve C and 6 points $p_1, \dots, p_6$ on C such that $\sum_{i=1}^6 p_i$ is not linearly equivalent to $\mathcal{O}_C(2)$. Consider the curve in $\mathbb{P}^{2[6]}$ induced by a general pencil in $|\mathcal{O}_C(\sum p_i)|$. Show that this curve is disjoint from $D_{\mathcal{O}(2)}$. Conclude that the class of $D_{\mathcal{O}(2)}$ is proportional to $2H - \frac{1}{2}B$.

Since the curve described in the previous exercise is a moving curve, we conclude that $D_{\mathcal{O}(2)}$ spans an extremal ray in $\mathrm{Eff}(\mathbb{P}^{2[6]})$.

More generally, when n is a triangular number of the form $n = \frac{k(k+1)}{2}$, the set of schemes that lie on a curve of degree $k-1$ forms an extremal effective divisor. The reader should compute its class (hint: $(k-1)H - \frac{1}{2}B$) and exhibit a moving curve disjoint from it (see [ABCH]).

Example 3.19 (n is one less or one more than a triangular number)**.** Similar constructions work when n is one less or one more than a triangular number. For example, when $n = 4$, consider the locus of schemes of length 4 that have a collinear subscheme of length 3. More generally, when $n = \frac{k(k+1)}{2} + 1$, an extremal ray of the effective cone is spanned by the divisor of schemes of length n that have a subscheme of length $n-1$ that is contained in a curve of degree $k-1$.

When $n = 2$, fix an auxiliary point p. Consider the divisor of schemes in $\mathbb{P}^{2[2]}$ that are collinear with p. More generally, when $n = \frac{k(k+1)}{2} - 1$, an extremal ray of the effective cone is spanned by the divisor of schemes that together with an auxiliary point p lie on a curve of degree $k-1$.

The first interesting case which cannot be reduced to the previous examples is $n = 12$. There is no longer an easily visible geometric condition on 12 points. Let us return to the previous examples. To say that three points are collinear can be rephrased as saying that the three points fail to impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^2}(1)$. Similarly, to say that six points lie on a conic can be rephrased as saying that the points fail to impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^2}(2)$. The idea is that for $n = 12$, we can look for a higher rank vector bundle such that the locus of 12 points that fail to impose independent conditions on the sections of that bundle is a divisor. Indeed, this idea works and generalizes to all n. Consider the bundle $T_{\mathbb{P}^2}(2)$. Using the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{O}_{\mathbb{P}^2}(3)^{\oplus 3} \to T_{\mathbb{P}^2}(2) \to 0,$$

we see that $h^0(\mathbb{P}^2, T_{\mathbb{P}^2}(2)) = 24$.

Given a rank r bundle E, asking for a section to vanish at a point is expected to impose r linear conditions on the space of sections. If we ask the sections to vanish at n points, assuming that the conditions are independent, we would expect to get a subvector space of codimension rn. Hence, we would expect the only section of $T_{\mathbb{P}^2}(2)$ that vanishes on 12 general points to be the zero section. This is indeed the case, but compared to the line bundle case more difficult to prove. We can consider the locus of 12 points that fail to impose independent conditions on sections of $T_{\mathbb{P}^2}(2)$. This is an effective divisor which spans an extremal ray of the effective cone of $\mathbb{P}^{2[12]}$. Its class is readily computable to be $7H - B$.

Whereas general points always impose independent conditions on sections of line bundles, general points may fail to impose independent conditions on sections of higher rank vector bundles. To see a simple example, consider the vector bundle $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ of rank 2 on $\mathbb{P}^2$. It has a three dimensional space of sections. If we ask that the sections vanish at a point, we would get a 2 dimensional space of sections rather than the expected 1 dimensional space of sections. The explanation in this case is easy. All the sections of the rank 2 vector bundle come from the $\mathcal{O}_{\mathbb{P}^2}(1)$ summand. Each point imposes one condition on the sections of $\mathcal{O}_{\mathbb{P}^2}(1)$. This example raises the higher rank interpolation problem.

Definition 3.20. A vector bundle E *has interpolation* for a sheaf F if $h^i(\mathbb{P}^2, E \otimes F) = 0$ for every i.

Problem 3.21 (Higher rank interpolation)**.** *Given a scheme Z, determine the invariants such that there exists a vector bundle E with those invariants satisfying interpolation for the ideal sheaf I_Z.*

If E satisfies interpolation for Z, then we can define a divisor on $\mathbb{P}^{2[n]}$ by considering

$$D_E := \{W \in \mathbb{P}^{2[n]} | h^1(E \otimes I_W) \neq 0\}.$$

Then D_E is an effective divisor that does not contain Z in its base locus. Using either the Grothendieck-Riemann-Roch formula or test curves, one can see that the class of D_E is $c_1(E)H - \frac{r(E)}{2}B$ [ABCH].

The interpolation problem in general is very hard. However, the higher rank interpolation problem has been solved when Z is a general zero dimensional scheme [Hui], when Z is a complete intersection [CH] and when Z is a monomial scheme [CH]. Explaining the solutions are beyond the scope of these lecture notes, so we refer the interested reader to the original literature. We simply state the result that is relevant for the effective cone of $\mathbb{P}^{2[n]}$. First, the Euler characteristic defines a pairing on the set of Chern characters. More precisely, $(\xi, \eta) = \chi(\xi^* \otimes \eta)$, where ξ^* is the Chern character of the dual bundle and χ is the Euler characteristic. Two Chern characters are orthogonal if $(\xi, \eta) = 0$. The *slope* of a vector bundle E is $\mu(E) = \frac{c_1(E)}{r}$, where $c_1(E)$ is the first Chern class of E and r is the rank of E.

Theorem 3.22 (Huizenga [Hui])**.** *Let Z be a general point in $\mathbb{P}^{2[n]}$. Then the minimal positive slope $\mu_{\min}$ for which there exists a vector bundle E with slope μ satisfying interpolation for Z is the minimal positive slope of a stable vector bundle orthogonal to I_Z. In particular, the extremal edge of the effective cone is spanned by the ray $\mu_{\min}H - \frac{1}{2}B$.*

Given n, it is easy to compute $\mu_{\min}$ in practice. This theorem has been generalized to describe the effective cone of any moduli space of Gieseker semi-stable sheaves on $\mathbb{P}^2$. See [CHW] for details.

One can also consider the birational models that one obtains from various divisors in the effective cone of $\mathbb{P}^{2[n]}$. For a complete description of the stable base locus decomposition for $n \leq 9$ and modular interpretations of the resulting models, see [ABCH]. For other rational surfaces such as $\mathbb{P}^1 \times \mathbb{P}^1$, del Pezzo surfaces or Hirzebruch surfaces, see [BC]. There is extensive literature for surfaces such as abelian surfaces, K3 surfaces or Enriques surfaces. We refer the reader to [BM] and [BM2] for further information and detailed references.

4. Other moduli spaces

There are many other moduli spaces whose birational geometry is studied very actively. In this last section, we will give an overview of the types of results known and some references to the literature. The literature is so vast that it would be futile to try to compile a complete set of references. Instead, we will guide the reader to a few papers that will help them enter the field. The reader can refer to [HMo], [C], [Far] and [CFM] for references and further details. For more information on algebraic curves, the reader should consult [Ha, IV] and [ACGH].

4.1. **The moduli space of curves.** Fix two nonnegative integers g, n such that $2g-2+n > 0$. Let $\mathcal{M}_{g,n}$ be the moduli space of curves parameterizing isomorphism classes of $(C, p_1, \dots, p_n)$, where C is a smooth genus g curve and $p_1, \dots, p_n$ are ordered, distinct marked point on C. This space has a modular compactification constructed by Deligne, Mumford and Knudsen.

Definition 4.1. An n-pointed genus g marked curve $(C, p_1, \dots, p_n)$ is *stable* if C is a reduced, connected, at-worst-nodal curve of arithmetic genus g, $p_1, \dots, p_n$ are distinct, ordered, smooth points of C and the marked curve has finitely many automorphisms.

Let us explain the terms of this definition. We are assuming that the only singularities of the curve are nodes, that is locally analytically each singularity looks like $xy = 0$ on the plane. To say that the marked curve has finitely many automorphisms means that there are finitely many automorphisms $f : C \to C$ such that $f(p_i) = p_i$ for $1 \leq i \leq n$. This condition can be reduced to a very explicit combinatorial condition which is easy to check in practice. A curve of arithmetic genus $g \geq 2$ has finitely many automorphisms ([ACGH]). A smooth curve E of genus one is a complex torus and has a one-dimensional family of automorphisms. If we require that the automorphism fix a point on E, then there will be finitely many automorphisms (in fact, except for the square and hexagonal lattices, the automorphism group will be $\mathbb{Z}/2\mathbb{Z}$). The automorphism group of $\mathbb{P}^1$ is $\mathbb{P}SL_2(\mathbb{C})$. Any automorphism that fixes 3 points is necessarily the identity. Consequently, a nodal marked curve with $2g - 2 + n > 0$ has finitely many automorphisms if and only if in the normalization of the curve every genus zero component has at least 3 points mapping either to a node or to a marked point.

Alternatively, one can rephrase the stability condition by requiring that $\omega_C(\sum_{i=1}^n p_i)$ is ample, where ω_C is the dualizing sheaf of C. This condition better generalizes to higher dimensions.

The Deligne-Mumford-Knudsen moduli space $\overline{\mathcal{M}}_{g,n}$ parameterizes isomorphism classes of n-pointed genus g stable curves (see [HMo] and [DM]). It is one of the most important and well-studied objects in mathematics and plays a central role in algebraic geometry, topology, hyperbolic geometry, complex analysis and mathematical physics. Despite being well-studied, many questions about the birational geometry of $\overline{\mathcal{M}}_{g,n}$ are still open. For example, thanks to the work of Eisenbud, Harris and Mumford [HMu], [H] and [EH], the canonical class of $\overline{\mathcal{M}}_g$ lies in $\mathrm{Big}(\overline{\mathcal{M}}_g)$ if $g \geq 24$. A natural question is to describe the image of the rational map obtained by the multiples of the canonical divisor. This is known as the *canonical model* of $\overline{\mathcal{M}}_g$. Describing it is the subject of the Hassett-Keel program. The main goal of the program is to construct the canonical model of $\overline{\mathcal{M}}_g$ by considering the birational transformations that take place as one considers the divisors $K_{\overline{\mathcal{M}}_g} + a\delta$ and decreases a from 1 to 0, where δ is the total boundary divisor. More generally, many of the questions we have discussed in the previous section for the Hilbert scheme of points on $\mathbb{P}^2$ are open for $\overline{\mathcal{M}}_{g,n}$. The following kinds of results are known:

(1) The ample and effective cones of $\overline{\mathcal{M}}_{g,n}$ are known for small values of g and n. For example, Keel and McKernan [KM] have determined the ample cones of $\overline{\mathcal{M}}_{0,n}$ for $3 \leq n \leq 7$, Gibney and Farkas have determined the ample cones of $\overline{\mathcal{M}}_g$ for $g \leq 24$ [FG]. The effective cone of $\overline{\mathcal{M}}_{0,n}$ is known for $n \leq 6$ thanks to the work of Keel, Vermeire [V], Hassett and Tschinkel [HT]. The effective cone of $\overline{\mathcal{M}}_2$ is easy. Rulla has determined the effective cones of $\overline{\mathcal{M}}_{2,1}$, $\overline{\mathcal{M}}_3$ [R1].

(2) Many of the birational models of $\overline{\mathcal{M}}_{g,n}$ are known for very small values of g and n. The Hassett-Keel program has been carried out for $g = 2, 3$ and partially carried out for $g = 4, 5$ by many mathematicians including Hyeon and Lee [HyL], Hassett [Has2], Fedorchuk, Casalaina-Martin, Jensen and Laza [CaJL]. A few steps of the Hassett-Keel program has been carried out in arbitrary genus (see, for example, [HH1], [HH2] and [AFSV]).

(3) Certain slices of the ample cone are known. For example, the intersection of the ample cone with the $\lambda - \delta$ plane is the cone generated by λ and $11\lambda - \delta$ [CoH], [GKM]. The moduli

space of curves has a stratification by topological type. Strata are indexed by dual graphs. There is a conjecture due to Fulton predicting that the ample cone of $\overline{\mathcal{M}}_{g,n}$ is dual to the cone of curves spanned by the components of the one-dimensional strata in the topological stratification (see [GKM] and [FG]).

(4) Many examples of effective divisors have been constructed. Eisenbud, Harris, Mumford constructed effective divisors to prove that the canonical class of $\overline{\mathcal{M}}_g$ is big [HMu], [H], [EH]. Logan extended these results to $\overline{\mathcal{M}}_{g,n}$ [Lo]. Farkas generalized these results using syzygy theoretic constructions [Far]. Castravet and Tevelev [CT] have constructed divisors on $\overline{\mathcal{M}}_{0,n}$ based on combinatorial objects known as hypertrees. Recently, people have found new constructions giving divisors not spanned by Castravet-Tevelev divisors (see [O] for example). Except when g and n are very small, the complete description of the effective cone is not known. However, it is known that these cones can be very complicated. For example, the effective cones of $\overline{\mathcal{M}}_{1,n}$ are not finitely generated as soon as $n \geq 3$ [CC2].

(5) The cones may become simpler if we consider unordered points. The effective cones of $\overline{\mathcal{M}}_{0,n}/\mathfrak{S}_n$ and $\overline{\mathcal{M}}_{1,n}/\mathfrak{S}_n$ are generated by boundary divisors [KM] [CC2]. If $G \subset \mathfrak{S}_n$, the effective cones of $\overline{\mathcal{M}}_{g,n}/G$ are not known. However, it is known that if G has at least three orbits, the effective cone of $\overline{\mathcal{M}}_{1,n}/G$ is not finite, polyhedral [CC2].

4.2. **The Kontsevich moduli space.** Let X be a smooth projective variety. Fix the class of a curve $\beta \in H_2(X, \mathbb{Z})$. Then the Kontsevich moduli space of genus zero stable curves $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parameterizes isomorphism classes of $(C, p_1, \dots, p_n, f)$, where $(C, p_1, \dots, p_n)$ is an n-pointed genus g stable curve, $f : C \to X$ is a morphism such that $f_*[C] = \beta$ and the datum has a finite automorphism group. An automorphism of the datum is a map $h : C \to C$ such that $h(p_i) = p_i$ for $1 \leq i \leq n$ and $f = f \circ h$. For detailed introductions to the geometry of Kontsevich moduli spaces, we refer the reader to [FP] and [C].

When $g > 0$, these spaces typically have many components of different dimensions even when the target X is a simple variety such as $\mathbb{P}^2$.

Exercise 4.2. Show that $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)$ has three components. Two of these components have dimension 9 and one has dimension 10.

In general, we know very little about these spaces. However, when X is a homogeneous variety such as $\mathbb{P}^n$, a Grassmannian $G(k, n)$ or a flag variety $F(k_1, \dots, k_r; n)$, then the Kontsevich moduli space of genus zero maps is an irreducible, normal, $\mathbb{Q}$-factorial variety with at-worst finite quotient singularities [FP]. These spaces play a crucial role in Gromov-Witten theory. Although many questions about their birational geometry remain open, the following types of theorems are known.

(1) The effective cones of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ are known for $r \geq d$ [CHS2]. Similarly, the effective cones of $\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ are known for $n \geq k + d$ [CS].

(2) The ample cones of $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^r, d)$ are known for small values of m and d [CHS]. In general, knowing the ample cone of $\overline{\mathcal{M}}_{0,m}(F(k_1, \dots, k_r; n), \beta)$ is reduced to knowing the ample cone of the Deligne-Mumford moduli space of genus 0 curves with marked points [CHS].

(3) The birational geometry of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ has been studied for very small values of d such as $2, 3$ [Ch], [CC1] and [CC3]. Some aspects of the birational geometry of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ have been studied in general [CC1].

4.3. **Other moduli spaces.** There are many variations on the moduli spaces described so far. We will not attempt to discuss these variations. The space of stable quotients (see [Coo]), the Quot scheme (see [J]), spaces of complete quadrics (see [Hue]), moduli spaces of abelian varieties (see [CFM]) and moduli spaces of polarized K3 surfaces (see [GHS]) are some of the moduli spaces algebraic geometers encounter frequently. Many mathematicians have been actively studying the cones of effective and ample divisors on these moduli spaces and are working on describing the birational models of these moduli spaces. I hope these notes will motivate you to explore some of these questions.

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