

# FANO MANIFOLDS OF INDEX $n - 1$ AND THE CONE CONJECTURE

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ABSTRACT. The Morrison-Kawamata cone conjecture predicts that the actions of the automorphism group on the effective nef cone and the pseudo-automorphism group on the effective movable cone of a klt Calabi-Yau pair  $(X, \Delta)$  have finite, rational polyhedral fundamental domains. Let  $Z$  be an  $n$ -dimensional Fano manifold of index  $n - 1$  such that  $-K_Z = (n - 1)H$  for an ample divisor  $H$ . Let  $\Gamma$  be the base locus of a general  $(n - 1)$ -dimensional linear system  $V \subset |H|$ . In this paper, we verify the Morrison-Kawamata cone conjecture for the blow-up of  $Z$  along  $\Gamma$ .

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## 1. INTRODUCTION

Let  $n \geq 3$  be a positive integer. Let  $Z$  be an  $n$ -dimensional Fano manifold of index  $n - 1$  over an algebraically closed field  $k$  of characteristic zero. Let  $H$  be an ample divisor that satisfies  $-K_Z = (n - 1)H$ . Let  $V \subset |H|$  be a general  $(n - 1)$ -dimensional linear system with base locus  $\Gamma$  and let  $X$  be the blow-up of  $Z$  along  $\Gamma$ . In this paper, we verify the Morrison-Kawamata cone conjecture for  $X$ .

The nef and movable cones are among the most important invariants of a projective variety  $Y$ . Recall that a divisor  $D$  is nef if  $D \cdot C \geq 0$  for every curve  $C$ . A divisor  $D$  is movable if its stable base locus has codimension at least two in  $Y$ . Nef and movable divisors form two convex cones in the Néron-Severi space  $N^1(Y)$  of  $Y$ , which control the contractions and the birational contractions of  $Y$ , respectively.

For Fano varieties, these cones are as simple as possible. The nef cone is rational polyhedral by the Cone Theorem of Mori-Kawamata-Kollár-Reid-Shokurov [14, Theorem 3.7], and the movable and effective cones are rational polyhedral by a theorem of Birkar-Cascini-Hacon-McKernan [1]. However, once we leave the world of Fano varieties, the cones may be very complicated. Even for simple examples of Calabi-Yau varieties such as  $K3$  surfaces, the nef cone can have uncountably many extremal rays.

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Nevertheless, one can still hope for a satisfying description of these cones in the Calabi–Yau case if automorphisms are taken into account. The Morrison–Kawamata cone conjecture [17, 10] predicts that for a Calabi–Yau variety, there are rational polyhedral fundamental domains both for the action of automorphisms on the effective nef cone and for the action of certain birational automorphisms on the effective movable cone. We now recall the statement of the Morrison–Kawamata cone conjecture that we will need in this paper. We refer the reader to Section 1 of [25] for a generalization of the conjecture to the relative setting and for history and examples.

**The Morrison–Kawamata Cone Conjecture.** Throughout this paper, a *rational polyhedral cone* in a real vector space with a  $\mathbb{Q}$ -structure means a closed convex cone with finitely many extremal rays, each spanned by a rational vector. For  $X$  a normal projective variety, let  $N^1(X)$  denote the Néron-Severi space of Cartier divisors on  $X$  modulo numerical equivalence. We denote by  $N^1(X)_{\mathbb{Z}}$  the free abelian group in  $N^1(X)$  consisting of numerical classes of Cartier divisors. We denote the nef cone and the closure of the movable cone of  $X$  by  $\overline{A(X)}$  and  $\overline{M(X)}$ , respectively. Let  $B^e(X)$  denote the cone generated by effective Cartier divisors. We denote by  $\overline{A(X)}^e$  and  $\overline{M(X)}^e$  the intersections  $\overline{A(X)} \cap B^e(X)$  and  $\overline{M(X)} \cap B^e(X)$ , and call them the *effective nef cone* and the *effective movable cone*, respectively.

Define a *pseudo-isomorphism* from  $X_1$  to  $X_2$  to be a birational map  $X_1 \dashrightarrow X_2$  which is an isomorphism in codimension one. A *small  $\mathbb{Q}$ -factorial modification* (SQM) of  $X$  is a pseudo-isomorphism from  $X$  to another  $\mathbb{Q}$ -factorial projective variety. An SQM  $\alpha : X \dashrightarrow X'$  gives an identification of the vector spaces  $N^1(X)$  and  $N^1(X')$  by pushforward and pullback. Since the pullback of an ample divisor is movable, this identification embeds  $\overline{A(X')}^e$  as a subcone of  $\overline{M(X)}^e$ . This embedding depends on the map  $\alpha : X \dashrightarrow X'$ , so we denote the image by  $\overline{A(X', \alpha)}^e$ . Precomposing the SQM  $\alpha$  with a pseudo-automorphism of  $X$  gives another SQM  $\beta : X \dashrightarrow X'$ , and hence another embedding of  $\overline{A(X')}^e$  in  $\overline{M(X)}^e$ . We can sum this up by saying: *pseudo-automorphisms of  $X$  permute the nef cones  $\overline{A(X', \alpha)}^e$  of SQMs of  $X$  inside  $\overline{M(X)}^e$ .*

For an  $\mathbb{R}$ -divisor  $\Delta$  on a normal  $\mathbb{Q}$ -factorial variety  $X$ , the pair  $(X, \Delta)$  is *klt* if, for all resolutions  $\pi : \tilde{X} \rightarrow X$  with a simple normal crossing  $\mathbb{R}$ -divisor  $\tilde{\Delta}$  such that  $K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta)$ , the coefficients of  $\tilde{\Delta}$  are less than 1. In particular, if  $X$  is smooth and  $D$  is a smooth divisor on  $X$ , then  $(X, rD)$  is klt for any  $r < 1$ . We say that  $(X, \Delta)$  is a *klt Calabi–Yau pair* if  $(X, \Delta)$  is a  $\mathbb{Q}$ -factorial klt pair with  $\Delta$  effective such that  $K_X + \Delta$  is numerically trivial.

We denote the groups of automorphisms or pseudo-automorphisms of  $X$  which preserve a divisor  $\Delta$  by  $\text{Aut}(X, \Delta)$  and  $\text{PsAut}(X, \Delta)$ , respectively. Note that the action of  $\text{Aut}(X, \Delta)$  and  $\text{PsAut}(X, \Delta)$  on  $N^1(X)$  is determined by the images of the representations  $\text{Aut}(X, \Delta) \rightarrow \text{GL}(N^1(X)_{\mathbb{Z}})$  and  $\text{PsAut}(X, \Delta) \rightarrow \text{GL}(N^1(X)_{\mathbb{Z}})$ . We denote the images of these representations by  $\text{Aut}^*(X, \Delta)$  and  $\text{PsAut}^*(X, \Delta)$ . We say that a finite rational polyhedral cone  $\Pi$  is a *fundamental domain* for the action of a group  $G$  on a cone  $C$  if  $C = G \cdot \Pi$  and the interiors of  $\Pi$  and  $g\Pi$  are

disjoint for  $1 \neq g \in G$ . With these conventions in place, we are ready to state the Morrison–Kawamata cone conjecture.

**Conjecture 1.1** (Morrison–Kawamata). *Let  $(X, \Delta)$  be a klt Calabi–Yau pair. Then:*

- (1) *The number of  $\text{Aut}(X, \Delta)$ -equivalence classes of faces of the effective nef cone  $\overline{A(X)}^e$  corresponding to birational contractions or fiber space structures is finite. Moreover, there exists a finite rational polyhedral cone  $\Pi$  which is a fundamental domain for the action of  $\text{Aut}^*(X, \Delta)$  on  $\overline{A(X)}^e$ .*
- (2) *The number of  $\text{PsAut}(X, \Delta)$ -equivalence classes of nef cones  $\overline{A(X', \alpha)}^e$  in the cone  $\overline{M(X)}^e$  corresponding to marked SQMs with marking  $\alpha : X' \dashrightarrow X$  is finite. Moreover, there exists a finite rational polyhedral cone  $\Pi'$  which is a fundamental domain for the action of  $\text{PsAut}^*(X, \Delta)$  on  $\overline{M(X)}^e$ .*

The conjecture has been proved for Calabi–Yau surfaces by Looijenga–Sterk and Namikawa [22, 18], for klt Calabi–Yau pairs of dimension 2 by Totaro [25], for Calabi–Yau fiber spaces of dimension 3 over a positive-dimensional base by Kawamata [10], and for abelian varieties by the second author [21]. For Calabi–Yau 3-folds there are significant results by Oguiso–Peters [19], Szendrői [23], Uehara [26], and Wilson [27], but the full conjecture remains open.

In this paper, we verify the Morrison–Kawamata conjecture for certain blowups of Fano manifolds of index  $n - 1$ . We now explain our examples more precisely. Fano manifolds of index  $n - 1$  are called *del Pezzo manifolds* and were classified by Fujita in [5] and [6]. If  $Z$  is a del Pezzo manifold of dimension  $n \geq 3$ , then  $Z$  is one of the following:

- (1)  $Z$  is a linear section of the Grassmannian  $Gr(2, 5)$  in its Plücker embedding.
- (2)  $Z = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}$  is an intersection of two quadric hypersurfaces in  $\mathbb{P}^{n+2}$ .
- (3)  $Z$  is a cubic hypersurface in  $\mathbb{P}^{n+1}$ .
- (4)  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ .
- (5)  $Z = F(1, 2; 3)$  is the flag variety parameterizing full flags in  $k^3$ .
- (6)  $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .
- (7)  $Z \rightarrow \mathbb{P}^n$  is a double cover of  $\mathbb{P}^n$  branched along a quartic hypersurface.
- (8)  $Z$  is the blow-up of  $\mathbb{P}^3$  at a point.
- (9)  $Z$  is a sextic hypersurface in the weighted projective space  $\mathbb{P}(3, 2, 1, \dots, 1)$ .

Let  $-K_Z = (n - 1)H$  and let  $V \subset |H|$  be a general linear system of dimension  $n - 1$ . Then  $V$  has  $H^n = r$  base points  $\Gamma = \{p_1, \dots, p_r\}$ . Let  $[r]$  denote the set of integers  $\{1, \dots, r\}$ . Let  $\pi : X \rightarrow Z$  be the blowup of  $Z$  along  $\Gamma$  and let  $E_i$  denote the exceptional divisor over  $p_i$ . In this paper, we verify Conjecture 1.1 for  $X$ .

Note that if we are in case (8), so  $Z$  is the blowup of  $\mathbb{P}^3$  at a point, then  $H = 2\mathcal{O}(1) - E$  is the linear system of quadrics through the point, and so  $X$  is the blowup of  $\mathbb{P}^3$  in the base locus of a general net of quadrics. The conjecture has already been verified in this case in [20]. Also, in case (9) the conjecture turns out to be somewhat trivial; we give the details of the proof in Section 6. For the rest of the paper, therefore, we will concentrate on cases (1)–(7). To simplify notation, we will refer to case (1) throughout the paper as  $Gr(2, 5)$ , but the reader should bear in mind that our argument applies equally well to a linear section of this variety of dimension at least 3.

Let us explain how the cone conjecture applies to our examples. Let  $X$  be a smooth variety such that  $-K_X$  is semi-ample. Choose a sufficiently large integer  $m$  so that the line bundle  $-mK_X$  is base-point-free. By Bertini's Theorem, a general divisor  $D$  in the linear system  $|-mK_X|$  is smooth. If we define  $\Delta$  to be the  $\mathbb{Q}$ -divisor  $\Delta = \frac{1}{m}D$ , then  $K_X + \Delta$  is numerically trivial, and  $(X, \Delta)$  is a klt pair. So in this case the cone conjecture makes predictions about the action of automorphisms or pseudo-automorphisms on the nef cone  $\overline{A(X)}^e$  or movable cone  $\overline{M(X)}^e$ .

Given a klt Calabi-Yau pair, the fundamental difficulty in proving the cone conjecture is finding the necessary automorphisms and pseudo-automorphisms needed to verify the conjecture. In Theorem 3.1, we will show that, in our examples, the nef cone is equal to the effective nef cone and is rational polyhedral. As a consequence, we will deduce part (1) of Conjecture 1.1. Part (2) of Conjecture 1.1 is considerably more difficult. By abuse of notation, denote the pullback of  $H$  by the blowup map  $\pi$  also by  $H$ . Then  $-\frac{1}{n-1}K_X = H - \sum_{i \in [r]} E_i$  defines a morphism  $f : X \rightarrow \mathbb{P}^{n-1}$  whose general fiber is an elliptic curve. The elliptic fibration comes with  $r$  sections given by the base-points. This elliptic fibration structure with  $r$  sections provides the pseudo-automorphisms that we need to verify the Morrison–Kawamata cone conjecture. The proof involves some beautiful explicit classical geometry.

The organization of the paper is as follows. In §2, we will describe the varieties  $X$  and the elliptic fibrations  $f : X \rightarrow \mathbb{P}^{n-1}$  in greater detail. In particular, we will show that the fibers of  $f$  that are reducible occur in codimension two. In §3, we will compute the nef cone of  $X$  and prove the first part of Conjecture 1.1 for our examples. In §4, we will collect basic facts concerning certain flops of  $X$  that will allow us to construct a fundamental domain for the action of the group of pseudo-automorphisms. In §5, we will study the action of the group of pseudo-automorphisms on the movable cone in detail to prove the second part of Conjecture 1.1. Finally, in §6, we will explain the case (5).

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## 2. THE ELLIPTIC FIBRATION

In this section, we study the geometry of the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . We discuss the locus of reducible fibers in detail and show that it has codimension two.

Let  $Z$  be a Fano manifold of dimension  $n$  and index  $n - 1$  such that  $-K_Z = (n - 1)H$  for a very ample divisor  $H$ . Consider the embedding of  $Z$  in  $\mathbb{P}^N$  by the complete linear system  $|H|$ . Let  $V$  be a general  $(n - 1)$ -dimensional linear system of hyperplanes, and consider the corresponding rational map  $\tilde{f} : Z \dashrightarrow V^* = \mathbb{P}^{n-1}$ . The base locus of  $V$  in  $\mathbb{P}^N$  is a linear space  $\Lambda$  of dimension  $\mathbb{P}^{N-n}$ . The base locus of  $V$  restricted to  $Z$  is a union of  $r$  points  $\Gamma = \{p_1, \dots, p_r\}$ , where  $r = H^n$ . Let

$$g : X = \text{Bl}_\Gamma Z \rightarrow Z$$

be the blowup of  $Z$  along  $\Gamma$ . Let  $E_i$  denote the exceptional divisor over  $p_i$ . We then get a morphism

$$f : X \rightarrow \mathbb{P}^{n-1}$$

given by the sections of

$$g^*H - E_1 - \cdots - E_r = -\frac{1}{n-1}K_X.$$

In particular,  $-K_X$  is semi-ample, so the cone conjecture applies to  $X$  as explained in §1.

By adjunction, the smooth fibers of  $f$  are curves with trivial canonical bundle, therefore,  $f$  is an elliptic fibration. The exceptional divisors  $E_1, \dots, E_r$  give sections of the fibration, hence restrict to  $k(\mathbb{P}^{n-1})$ -rational points on the generic fiber  $X_\nu$  of  $f$ .

We now analyze the fibers of  $f$  in some detail. We first note that  $f$  is flat. By [15, Theorem 23.1],  $f$  is flat if and only if the fibers of  $f$  are equi-dimensional. Each fiber of  $f$  is the proper transform in  $X$  of the intersection of  $Z$  with a linear space  $S \cong \mathbb{P}^{N-n+1}$  containing  $\Lambda$ . If  $\dim(S \cap Z) > 1$ , then  $\dim(\Lambda \cap Z) = \dim(S \cap Z) - 1 \geq 1$ . This contradicts that  $\Lambda \cap Z$  is the finite set of points  $\Gamma$ . We conclude that  $f$  is flat and every fiber of  $f$  has dimension one. We already observed that the general fiber of  $f$  is a smooth genus one curve. However,  $f$  may have reducible and singular fibers.

**Lemma 2.1.** *Let  $C$  be an irreducible component of the fiber of  $f$  of degree  $d < r$ . Then  $g(C)$  contains exactly  $d$  of the points  $\Gamma = \{p_1, \dots, p_r\}$ .*

*Proof.* The fibers of  $f$  are curves of degree  $r$ . Suppose that  $C$  contains  $d + 1$  points  $p_{i_1}, \dots, p_{i_{d+1}}$  of  $\Gamma$ . Then by Bezout's Theorem,  $C$  is contained in the linear span of the points  $p_{i_1}, \dots, p_{i_{d+1}}$ . Since the linear span of the points  $p_1, \dots, p_r$  intersects  $Z$  in finitely many points, we obtain a contradiction. We conclude that  $g(C)$  can contain at most  $d$  points. However, since the image of the fiber under  $g$  contains all the points  $p_1, \dots, p_r$  and the residual curve (which has degree  $r - d$ ) can only contain at most  $r - d$  points, we conclude that  $g(C)$  contains exactly  $d$  of the points.  $\square$

We now run through the list of possible  $Z$  and describe the geometry in each case.

**(1)  $Z$  is the Grassmannian  $Gr(2, 5)$ .** Let  $Gr(2, 5)$  denote the Grassmannian parameterizing two-dimensional subspaces of  $W \cong k^5$ . For  $3 \geq a \geq b \geq 0$ , let  $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ , where  $F_i$  is an  $i$ -dimensional subspace of  $W$ , denote the Schubert variety parameterizing two-dimensional subspaces of  $F_{n-b}$  that intersect  $F_{n-1-a}$  non-trivially. We denote the class of the Schubert variety by  $\sigma_{a,b}$ . When  $b = 0$ , we will abuse notation and denote the Schubert variety by  $\Sigma_a(F_{n-1-a})$ . The codimension of the Schubert variety with class  $\sigma_{a,b}$  is equal to  $a + b$ .

The Plücker map embeds  $i : Gr(2, 5) \hookrightarrow \mathbb{P}(\wedge^2 W) \cong \mathbb{P}^9$ . The pullback of the hyperplane class via the Plücker embedding is the Schubert divisor with class  $\sigma_1$ . The canonical class is given by  $K_{Gr(2,5)} = -5\sigma_1 = -5i^*(H)$ . The degree of  $Gr(2, 5)$  is given by  $\sigma_1^6 = 5$ , which can be seen by repeated applications of Pieri's rule [7, I.5]. The linear system  $V \subset |H|$  is the set of hyperplanes that contain a general  $\Lambda \cong$

$\mathbb{P}^3 \subset \mathbb{P}^9$ . Hence,  $X$  is the blowup of  $Gr(2, 5)$  at the five points  $\Gamma = \{p_1, \dots, p_5\} = Gr(2, 5) \cap \Lambda$ . The fibers of  $f : X \rightarrow \mathbb{P}^5$  are elliptic normal curves of degree 5 in  $\mathbb{P}^4$ . Hence, the reducible fibers contain an irreducible component of degree one or two.

A line in the Grassmannian  $Gr(2, 5)$  parameterizes a pencil  $L_t$  of two-dimensional subspaces that contain a fixed vector  $p$  and are contained in a fixed three-dimensional subspace  $S$  of  $W$ . In particular, there exists a line in  $Gr(2, 5)$  through every point. Let  $\Omega_i$  be the two-dimensional subspace corresponding to the point  $p_i \in Gr(2, 5)$ . The span of a line  $l$  through  $p_i$  and the remaining points of  $\Gamma$  is a  $\mathbb{P}^4$  that intersects  $Gr(2, 5)$  in a degree 5 reducible curve containing  $l$  as a component.

Conversely, suppose that  $f$  has a reducible fiber containing a line  $l$ . By Lemma 2.1,  $l$  has to contain one of the points of  $\Gamma$ . Assume without loss of generality that  $l$  contains  $p_1$ . The linear spaces parameterized by  $l$  intersect  $\Omega_1$  non-trivially. Consequently, every line passing through  $p_1$  is contained in the Schubert variety  $\Sigma_2(\Omega_1)$ . Furthermore, every point of  $\Sigma_2(\Omega_1)$  contains a line passing through  $p_1$ . The Schubert variety  $\Sigma_2(\Omega_1)$  is a cone over the Segre variety  $\mathbb{P}^1 \times \mathbb{P}^2$  [3]. Restricted to this locus, the map  $f$  is the projection from the cone point. We conclude that the fiber contains a line passing through  $p_1$  precisely over fibers contained in a Segre embedding of  $\mathbb{P}^1 \times \mathbb{P}^2$  in  $\mathbb{P}^5$ . In particular, this locus has codimension two.

We can view  $Gr(2, 5)$  also as  $\mathbb{G}(1, 4)$  parameterizing projective lines in  $\mathbb{P}^4$ . The lines parameterized by a conic in  $\mathbb{G}(1, 4)$  sweep out a quadric surface in  $\mathbb{P}^4$ . If the plane spanned by the conic is contained in  $\mathbb{G}(1, \mathbb{P}V)$ , then the quadric surface is singular or a double plane, depending on the cohomology class of the plane. If the cohomology class of the plane is  $\sigma_{3,1}$ , then the quadric surface is a cone. If the cohomology class of the plane is  $\sigma_{2,2}$ , then the quadric surface is a double plane. If the plane of the conic is not contained in the Grassmannian, then the quadric surface is smooth. In particular, if two points on the conic correspond to two skew-lines, then the quadric surface is smooth. Given two points  $p, q$  in the  $\mathbb{G}(1, \mathbb{P}V)$ , the corresponding lines  $L_p$  and  $L_q$  in  $\mathbb{P}^4$  span a  $\mathbb{P}^3$  or  $\mathbb{P}^2$ . Since given any three lines in  $\mathbb{P}^3$ , there exists a (possibly reducible) quadric surface containing them, we conclude that there is a conic through any two points of  $Gr(2, 5)$ .

Suppose that a fiber of  $f$  contains an irreducible conic  $C$  passing through  $p_1$  and  $p_2$ . Since the line joining  $p_1$  and  $p_2$  is not contained in  $Gr(2, 5)$ , we conclude that the plane spanned by  $C$  cannot be contained in the Grassmannian. Thus the lines parameterized by  $C$  sweep out a smooth quadric surface. The lines  $\mathbb{P}\Omega_1$  and  $\mathbb{P}\Omega_2$  corresponding to the points  $p_1$  and  $p_2$  span a  $\mathbb{P}^3$  in  $\mathbb{P}W$ . Since the span of a quadric surface is  $\mathbb{P}^3$ , any quadric surface swept out by the lines parameterized by an irreducible conic containing  $p_1$  and  $p_2$  must be contained in this  $\mathbb{P}^3$ . We conclude that the Zariski closure of the union of the conics passing through  $p_1$  and  $p_2$  is contained in the Schubert variety  $\Sigma_{1,1}(F_3 \subset F_4 = \overline{\Omega_1\Omega_2})$ . By the previous paragraph, for every point  $p$  in  $\Sigma_{1,1}(F_3 \subset F_4 = \overline{\Omega_1\Omega_2})$ , there exists a conic containing  $p, p_1$  and  $p_2$ . Hence, the Zariski closure of the union of conics in the fibers of  $f$  containing  $p_1$  and  $p_2$  is precisely the Schubert variety  $\Sigma_{1,1}(F_3 \subset F_4 = \overline{\Omega_1\Omega_2})$ . This Schubert variety is isomorphic to  $Gr(2, 4)$  and is embedded in  $\mathbb{P}^9$  by the Plücker embedding as a quadric fourfold [3]. Its image under  $f$  is  $\mathbb{P}^3$  obtained by projecting  $\Sigma_{1,1}(F_3 \subset F_4 = \overline{\Omega_1\Omega_2})$  from the line joining  $p_1$  and  $p_2$ . We conclude that over a  $\mathbb{P}^3$

in  $\mathbb{P}^5$ , the fiber of  $f$  contains a (possibly reducible) conic passing through  $p_1$  and  $p_2$ . In particular, fibers of  $f$  are irreducible in codimension two.

**(2)  $Z$  is the complete intersection of two quadric hypersurfaces in  $\mathbb{P}^{n+2}$ .** Let  $n \geq 3$ . If  $Z$  is a transverse intersection of two smooth quadric hypersurfaces  $Q_1$  and  $Q_2$  in  $\mathbb{P}^{n+2}$ , then, by adjunction,  $-K_Z = (n-1)H$ . The linear system  $V \subset |H|$  is the set of hyperplanes that contain a general plane  $\Lambda$  in  $\mathbb{P}^{n+2}$  and  $X$  is the blowup of  $Z$  at the four points  $\Gamma = \{p_1, \dots, p_4\} = Z \cap \Lambda$ . The fibers of  $f : X \rightarrow \mathbb{P}^{n-1}$  are quartic elliptic space curves. Hence, reducible fibers consist of either the union of a line and a cubic or two conics, where the cubic or either of the conics may be further reducible.

In a smooth quadric hypersurface  $Q$ , the lines that pass through a point  $p \in Q$  sweep out the codimension one quadric cone  $T_p Q \cap Q$ . Hence, the lines that pass through a point  $p_i$  on  $Z$  are contained in  $Y = T_{p_i} Q_1 \cap T_{p_i} Q_2 \cap Z$ . Conversely, since  $Y$  is a cone with vertex at  $p_i$ , every point of  $Y$  is contained in a line passing through  $p_i$ . Since  $Z$  is a transverse intersection of the two quadrics  $Q_1$  and  $Q_2$ , we conclude that  $Y$  is a codimension two subvariety of  $Z$ .

Similarly, conics that pass through two points  $p_1, p_2$  on  $Z$  sweep out a subvariety of  $Z$  of codimension at least two. This can be verified by a simple dimension count. In a homogeneous variety of the form  $Y = G/P$ , where  $G$  is a linear algebraic group and  $P$  is a parabolic subgroup, the space of rational curves in the class  $\beta$  is irreducible and has dimension  $-K_Y \cdot \beta + \dim(Y) - 3$  [11]. A quadric hypersurface in  $\mathbb{P}^{n+2}$  is homogeneous and contains a  $3n$ -dimensional family of conics. By Kleiman's Transversality Theorem [12], the locus of conics that are contained in the intersection of two quadrics has dimension  $3n - 5$ . Applying Kleiman's Transversality Theorem once more, we conclude that the space of conics containing two general fixed points has dimension  $n - 3$ . Therefore, these conics sweep out at most a codimension two subvariety of  $Z$ . The same argument bounds the dimension of reducible fibers in the remaining examples, but in each case we will give a much more explicit description of the reducible fibers.

**(3)  $Z$  is a smooth cubic hypersurface in  $\mathbb{P}^{n+1}$ .** Let  $n \geq 3$ . If  $Z$  is a smooth cubic hypersurface in  $\mathbb{P}^{n+1}$ , then  $-K_Z = (n-1)H$ , where  $H$  denotes the hyperplane class. The linear system  $V \subset |H|$  is the set of hyperplanes that contain a general line  $\Lambda$  in  $\mathbb{P}^{n+1}$  and  $X$  is the blowup of  $Z$  at the three points  $\Gamma = \{p_1, p_2, p_3\} = Z \cap \Lambda$ . The fibers of  $f$  are plane cubic curves. Hence, any reducible fiber is a union of a line and a conic, where the conic may also be reducible.

An *Eckardt point*  $p$  on a cubic hypersurface  $Z$  is a point where  $T_p Z \cap Z$  is a cone with vertex at  $p$ . A general cubic hypersurface does not contain any Eckardt points and a smooth cubic hypersurface can contain at most finitely many Eckardt points [4, Corollary 2.2]. Since  $\Lambda$  is a general line and  $Z$  has only finitely many Eckardt points, we may assume that the points  $\{p_1, p_2, p_3\} = Z \cap \Lambda$  are not contained in  $T_p Z \cap Z$  for any Eckardt point  $p \in Z$ .

Let  $q \in Z$  be a point which is not an Eckardt point. Then the space of lines in  $Z$  passing through  $q$  is a  $(2, 3)$  complete intersection in  $\mathbb{P}T_p Z$  [4, Lemma 2.1]. Hence, the space of lines passing through  $q$  is a variety of dimension  $n - 3$ . We conclude that lines that pass through  $q$  sweep out a subvariety of  $Z$  of dimension

$n - 2$ . Since every reducible fiber of  $f$  consists of a line passing through one of the points  $p_1, p_2$  or  $p_3$  and a residual conic, we conclude that the reducible fibers of  $f$  occur in codimension 2.

(4)  $Z$  is  $\mathbb{P}^2 \times \mathbb{P}^2$ . Let  $H_i$  denote the pullback of the hyperplane class via the projection  $\pi_i : Z \rightarrow \mathbb{P}^2$ . Then  $-K_Z = 3(H_1 + H_2)$ . The linear system  $V \subset |H_1 + H_2|$  is the set of hyperplanes containing a general  $\Lambda \cong \mathbb{P}^4$  in the Segre embedding of  $\mathbb{P}^2 \times \mathbb{P}^2$  in  $\mathbb{P}^8$ . Since  $(H_1 + H_2)^4 = 6$ ,  $X$  is the blowup of  $Z$  at the six points  $\Gamma = \{p_1, \dots, p_6\} = Z \cap \Lambda$ . The fibers of  $f : X \rightarrow \mathbb{P}^3$  are elliptic normal sextic curves in  $\mathbb{P}^5$  and have class  $3H_1^2H_2 + 3H_1H_2^2$ . Hence, a reducible fiber of  $f$  has an irreducible component of degree 1, 2 or 3. These curves can have various cohomology classes.

First, if a fiber of  $f$  contains a line  $l$ , then the cohomology class of  $l$  is either  $H_1^2H_2$  or  $H_1H_2^2$ . Curves in these classes are lines contained in a fiber of  $\pi_1$  or  $\pi_2$ . By Lemma 2.1,  $l$  contains one of the points  $p_i$ . Since both  $\pi_1$  and  $\pi_2$  have unique fibers containing  $p_i$ , we conclude that  $l$  has to be contained in one of the two  $\mathbb{P}^2$  containing  $p_i$ . Conversely, the span of any line passing through  $p_i$  and  $\Gamma$  is a  $\mathbb{P}^5$  containing  $\Lambda$  and gives a reducible fiber of  $f$ . We conclude that the lines in the fibers of  $f$  sweep out the planes that contain the points  $p_i$ . In particular, the locus of reducible fibers containing a line has codimension two in  $Z$ .

If the fiber contains an irreducible conic  $C$ , then  $C$  has class  $H_1^2H_2 + H_1H_2^2$ . Note that an irreducible curve of class  $2H_1^2H_2$  or  $2H_1H_2^2$  has to be contained in a fiber of  $\pi_1$  or  $\pi_2$ , respectively. If two of the points  $p_i, p_j$  were contained in the same fiber of one of the projections  $\pi_1$  or  $\pi_2$ , then the line joining the two points would be contained in  $Z$ . This would contradict the fact that  $Z \cap \Gamma$  is a finite set of points. Hence, an irreducible curve with class  $2H_1^2H_2$  or  $2H_1H_2^2$  cannot contain two of the points of  $\Gamma$ . By Lemma 2.1, we conclude that  $C$  has class  $H_1^2H_2 + H_1H_2^2$ . The projection of  $C$  to either factor is a line  $L_i$ . Hence,  $C$  is contained in  $L_1 \times L_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ . These lines are determined by the image of the projection of the two points  $p_1$  and  $p_2$  contained in  $C$ . Conversely, any conic contained in  $L_1 \times L_2$  and containing  $p_1$  and  $p_2$  lies in a fiber of  $f$ . We conclude that the conics in the fibers of  $f$  containing  $p_1$  and  $p_2$  sweep out the quadric surface  $L_1 \times L_2$ , hence form a codimension two subvariety of  $Z$ .

Finally, if the fiber decomposes into a union of two irreducible curves  $C_1 \cup C_2$  of degree three, then the curves must have classes  $2H_1^2H_2 + H_1H_2^2$  and  $H_1^2H_2 + 2H_1H_2^2$ , respectively. As in the previous paragraph, an irreducible cubic curve passing through three of the base points cannot have class  $3H_1^2H_2$  or  $3H_1H_2^2$ . We can choose five of the points  $p_1, \dots, p_5$  in  $\Gamma$  without any constraints. After reindexing, we may assume that  $C_1$  contains three of these points  $p_1, p_2, p_3$ . The projection of  $C_1$  by  $\pi_2$  is a line and the projection of  $C_1$  by  $\pi_1$  is a conic. Since the projection of the three points  $p_1, p_2, p_3$  by  $\pi_1$  are not collinear, there cannot be any reducible fibers consisting of the union of two cubics.

We conclude that the reducible fibers of  $f$  occur in codimension 2.

(5)  $Z$  is the flag variety  $F(1, 2; 3)$ . Let  $Z = F(1, 2; 3)$  be the flag variety parameterizing flags  $W_1 \subset W_2$  in  $k^3$ , where  $W_i$  is a subspace of dimension  $i$ . Alternatively, we can think of  $Z$  as the variety parameterizing pointed lines in  $\mathbb{P}^2$ . The Picard group of  $Z$  is generated by the two Schubert divisors. Let  $H_1$  denote the class of the Schubert divisor parameterizing pointed lines ( $p \in L$ ) such that  $L$  contains



a fixed point  $q$  on  $\mathbb{P}^2$ . Let  $H_2$  denote the class of the Schubert divisor parameterizing pointed lines ( $p \in L$ ) such that  $p$  is contained in a fixed line  $M$ . Then  $-K_Z = 2(H_1 + H_2)$ . The linear system  $V \subset |H_1 + H_2|$  is the set of hyperplanes containing a general  $\Lambda \cong \mathbb{P}^4$  in the Plücker embedding of  $F(1, 2; 3)$  in  $\mathbb{P}^7$ . Since  $(H_1 + H_2)^3 = 6$ ,  $X$  is the blowup of  $Z$  at the six points  $\Gamma = \{p_1, \dots, p_6\} = Z \cap \Lambda$ . The fibers of  $f : X \rightarrow \mathbb{P}^2$  are elliptic normal sextic curves in  $\mathbb{P}^5$  and have class  $H_1^2 + 2H_1H_2 + H_2^2 = 3H_1^2 + 3H_2^2$ . A reducible fiber of  $f$  has to contain an irreducible curve of degree one, two or three.

The flag variety  $F(1, 2; 3)$  admits two projections  $\pi_1 : F(1, 2; 3) \rightarrow Gr(1, 3) \cong \mathbb{P}^2$  and  $\pi_2 : F(1, 2; 3) \rightarrow Gr(2, 3) \cong (\mathbb{P}^2)^*$ . The reducible fibers can be easily described by considering their projections via these two maps.

A line in  $Z$  has cohomology class  $H_1^2$  or  $H_2^2$ . Geometrically, these classes parameterize pointed lines in  $\mathbb{P}^2$  where the line is a fixed line  $L$  or the point is a fixed point  $q$ , respectively. Hence, there is a unique line of each kind passing through a point  $p_i \in Z$  parameterizing the pointed line ( $q \in L$ ).

Conics in  $Z$  can have cohomology class  $H_1^2 + H_2^2$  or  $2H_i^2$ . The projection of a conic with class  $H_1^2 + H_2^2$  to both  $\mathbb{P}^2$  and  $(\mathbb{P}^2)^*$  is a line. Therefore, there is a unique conic with class  $H_1^2 + H_2^2$  containing two points  $(q_1, L_1)$  and  $(q_2, L_2)$ . It parameterizes pointed lines  $(q, L)$  such that  $q \in \overline{q_1, q_2}$  and  $L_1 \cap L_2 \in L$ . There cannot be any conics in the class  $2H_i^2$  that pass through two points  $p_1, p_2$  whose projections by  $\pi_i$  are distinct.

Finally, a cubic in  $Z$  can have class  $3H_i^2$  or  $2H_i^2 + H_j^2$ . Note that there cannot be cubics of this type passing through three general points. In the first case, the three points by the projection  $\pi_i$  has to coincide. In the second case, projection of the curve under  $\pi_i$  is a line. However, three general points are not collinear under this projection. Hence, there are no reducible fibers of this kind.

We conclude that  $f$  has finitely many reducible fibers.

**(6)  $Z$  is  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .** Let  $H_i$  denote the pullback of the hyperplane class by the  $i$ -th projection  $\pi_i : Z \rightarrow \mathbb{P}^1$ . Then  $-K_Z = 2(H_1 + H_2 + H_3)$ . The linear system  $V \subset |H_1 + H_2 + H_3|$  is the set of hyperplanes that contain a general  $\Lambda \cong \mathbb{P}^4$  in the Segre embedding of  $Z$  in  $\mathbb{P}^7$ . Since  $(H_1 + H_2 + H_3)^3 = 6$ ,  $X$  is the blowup of  $Z$  at the six points  $\Gamma = \{p_1, \dots, p_6\} = Z \cap \Lambda$ . The fibers of  $f : X \rightarrow \mathbb{P}^2$  are elliptic normal sextic curves in  $\mathbb{P}^5$  and have cohomology class  $2(H_1H_2 + H_1H_3 + H_2H_3)$ . A reducible fiber of  $f$  must have an irreducible curve of degree one, two or three.

The degree one component of a reducible fiber must have class  $H_iH_j$ . There is a unique line in  $Z$  in this class through every point. Curves of degree two may have cohomology class  $H_iH_j + H_iH_k$  or  $2H_iH_j$ . Curves in the latter class are necessarily reducible, so we can concentrate on curves of degree two in the class  $H_iH_j + H_iH_k$ . Since these curves have to pass through two points and these two points do not have the same  $i$ -coordinate, we conclude that there cannot be reducible fibers containing a curve of degree two. Finally, we consider irreducible cubics in the fiber. These can have cohomology class  $2H_iH_j + H_iH_k$  or  $H_1H_2 + H_1H_3 + H_2H_3$ . There cannot be irreducible curves with the first class and there is a unique curve with the second class passing through three points. We conclude that  $f$  has finitely many reducible fibers.

Finally, we deal with the following slightly different case:

**(7)  $Z \rightarrow \mathbb{P}^n$  is a double cover of  $\mathbb{P}^n$  branched along a smooth quartic  $Q$ .** This case is slightly different, since the line bundle  $H = -\frac{1}{n-1}K_Z$  is ample but not very ample:  $H$  is the pullback of  $\mathcal{O}_{\mathbb{P}^n}(1)$  under the double cover map. Nevertheless, we can treat this case in a very similar way to those above.

Let us denote the double cover by  $d : Z \rightarrow \mathbb{P}^n$ . The linear system  $V \subset |H|$  is the pullback to  $Z$  of the linear system of hyperplanes through a given general point  $p \in \mathbb{P}^n$ . On  $Z$  this linear system has base locus  $\{p_1, p_2\} = d^{-1}p$ , and  $X$  is the blowup of  $Z$  at these two points. The fibers of  $f$  are then preimages  $d^{-1}(L)$  of lines  $L$  in  $\mathbb{P}^n$  through  $p$ : the preimage of a line in  $\mathbb{P}^n$  is a double cover of  $\mathbb{P}^1$  branched over 4 points, hence an elliptic curve.

A fiber  $d^{-1}(L)$  of  $f$  is reducible if and only if  $L$  is a bitangent line of the quartic  $Q$  passing through  $p$ . Let us show that the locus of such lines has codimension 2 in  $\mathbb{P}^{n-1}$ .

To see that it has codimension at least 2, it suffices to exhibit a curve in  $\mathbb{P}^{n-1}$  (the space of lines in  $\mathbb{P}^n$  through  $p$ ) which is disjoint from the locus of bitangent lines. To do this, take a general plane section of the quartic, which is a smooth quartic plane curve. Such a curve has 28 bitangent lines (and these are exactly the bitangents of  $Q$  lying in the chosen plane). A general point  $p$  in the plane does not lie on any of these bitangent lines, so the pencil of lines in the plane through  $p$  does not intersect the locus of bitangents through  $p$ .

To see that the locus of bitangents through  $p$  has codimension equal to 2, it suffices to show that this locus intersects any plane  $\mathbb{P}^2 \subset \mathbb{P}^{n-1}$  in the space of lines through  $p$ . Such a plane corresponds to a subspace  $\mathbb{P}^3 \subset \mathbb{P}^n$ , and this intersects  $Q$  in a quartic surface in  $\mathbb{P}^3$ . The locus of bitangents to a quartic surface covers all of  $\mathbb{P}^3$ , so in particular there is such a bitangent through any point. This completes the proof that the locus of reducible fibers has codimension 2.

Finally, we note that any reducible fiber must be of the form  $C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are curves with  $H \cdot C_i = 1$  and  $p_i \in C_i$ .

We summarize the preceding discussion in the following statement:

**Proposition 2.2.** *Let  $Z$  be a Fano manifold of index  $n - 1$ , and  $V \subset |H|$  a general linear system of dimension  $n - 1$ . Let  $f : X \rightarrow \mathbb{P}^{n-1}$  be the elliptic fibration corresponding to  $V$ . Then  $f$  has reducible fibers in codimension 2.*

### 3. THE NEF CONE

In this section, we calculate the nef cone of the variety  $X$ , and thereby show that the first statement of Conjecture 1.1 is true for  $X$ .

We preserve the notation from the previous section. Recall that  $g : X \rightarrow Z$  is the blowup of the Fano manifold  $Z$  in  $\Gamma = \{p_1, \dots, p_r\}$  points. Let  $E_i$  denote the exceptional divisor over  $p_i$ . Let  $[r]$  denote the set  $\{1, 2, \dots, r\}$ . We will abuse notation and denote the pullbacks  $g^*H$  of divisor classes simply by  $H$ .

**Theorem 3.1.** *The nef cone of  $X$  has the following description.*

- (1) If  $Z$  is a linear section of  $Gr(2, 5)$ , then the nef cone of  $X$  is the cone spanned by the divisor classes  $\{H - \sum_{i \in I} E_i \mid I \subseteq [5]\}$ , where  $H$  denotes the hyperplane class of  $Gr(2, 5)$ .
- (2) If  $Z$  is the smooth complete intersection of two quadrics, then the nef cone of  $X$  is the cone spanned by  $\{H - \sum_{i \in I} E_i \mid I \subseteq [4]\}$ , where  $H$  denotes the hyperplane class of  $Z$ .
- (3) If  $Z$  is a smooth cubic hypersurface, then the nef cone of  $X$  is the cone spanned by  $\{H - \sum_{i \in I} E_i \mid I \subseteq [3]\}$ , where  $H$  denotes the hyperplane class of  $Z$ .
- (4) If  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ , then the nef cone of  $X$  is the cone spanned by the divisor classes  $H_1, H_2$ , and  $\{H_1 + H_2 - \sum_{i \in I} E_i \mid I \subseteq [6]\}$ , where  $H_i$  denotes the pullback of  $\mathcal{O}_{\mathbb{P}^2}(1)$  by the projection  $\pi_i : Z \rightarrow \mathbb{P}^2$ .
- (5) If  $Z = F(1, 2; 3)$ , then the nef cone of  $X$  is the cone spanned by the divisor classes  $H_1, H_2$ , and  $\{H_1 + H_2 - \sum_{i \in I} E_i \mid I \subseteq [6]\}$ , where  $H_i$  denotes the two Schubert divisors.
- (6) If  $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , then the nef cone of  $X$  is the cone spanned by the divisor classes  $H_1, H_2, H_3$ , and  $\{H_1 + H_2 + H_3 - \sum_{i \in I} E_i \mid I \subseteq [6]\}$ , where  $H_i$  denotes the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  by the projection  $\pi_i : Z \rightarrow \mathbb{P}^1$ .
- (7) If  $Z$  is a double cover of  $\mathbb{P}^n$  branched along a smooth quartic, then the nef cone of  $X$  is the cone spanned by the divisor classes  $H, H - E_1, H - E_2, H - E_1 - E_2$ .

*Proof.* The exceptional divisors  $E_i$  are isomorphic to  $\mathbb{P}^{n-1}$ . Let  $e_i$  denote the class of a line in  $E_i$ . Let  $l$  denote the class of a line in  $Z$ . In cases (1), (2) and (3), there is a family of lines in  $Z$  that contain exactly one of the points  $p_i$ . The proper transform of such a line in  $X$  is an effective curve with class  $l - e_i$ . The classes

$$\{l, e_1, \dots, e_r, l - e_1, \dots, l - e_r\}$$

span a subcone  $C$  of the cone of curves  $\overline{\text{Curv}(X)}$ . The dual of this cone  $\check{C}$  in  $N^1(X)$  contains the nef cone. The cone  $\check{C}$  also contains the divisors listed in the theorem.

Let  $D$  be a divisor contained in  $\check{C}$ . We may express  $D = aH - \sum_{i \in [r]} b_i E_i$ . Since  $D \cdot l = a$  and  $D \cdot e_i = b_i$ , we conclude that  $a, b_i \geq 0$ . After reordering the indices, we may assume that  $b_1 \geq b_2 \geq \dots \geq b_r$ . Similarly, pairing  $D$  with  $jl - e_1 - \dots - e_j$ , we conclude that for any divisor  $D \in \check{C}$ ,  $ja \geq \sum_{i=1}^j b_i$ . Hence, we can express  $D$  as

$$D = (a - \sum_{i=1}^r b_i)H + \sum_{j=1}^{r-1} (b_j - b_{j+1})(H - \sum_{i=1}^j E_i) + b_r(H - \sum_{i=1}^r E_i).$$

We conclude that the dual cone  $\check{C}$  is generated by the divisors listed in the theorem.

Similarly, in cases (4) and (5), there are families of lines with class  $l_1$  and  $l_2$  dual to  $H_1$  and  $H_2$ , respectively, in  $Z$  passing through exactly one of the points  $p_i$ . The proper transform of these lines in  $X$  are effective curves with class  $l_i - e_j$ . The classes

$$\{l_1, l_2, e_1, \dots, e_r, l_1 - e_1, \dots, l_1 - e_r, l_2 - e_1, \dots, l_2 - e_r\}$$

span a subcone  $C$  of the cone of curves  $\overline{\text{Curv}}(X)$ . The dual of this cone  $\check{C}$  in  $N^1(X)$  contains the nef cone and the cone generated by the divisors listed in the theorem. In fact, by an identical argument, the divisors listed in the theorem generate the dual cone  $\check{C}$ . Given a divisor  $D = a_1H_1 + a_2H_2 - \sum_{i=1}^r b_iE_i$  with  $b_1 \geq \dots \geq b_r$  in the dual cone  $\check{C}$ , by pairing with curves with class  $jl_i - e_1 - \dots - e_j$ , we conclude that  $ja_i \geq \sum_{i=1}^j b_i$ . Hence, we can express  $D$  as

$$\begin{aligned} D &= (a_1 - b_1)H_1 + (a_2 - b_1)H_2 + \sum_{j=1}^{r-1} (b_j - b_{j+1})(H_1 + H_2 - \sum_{i=1}^j E_i) \\ &\quad + b_r(H_1 + H_2 - \sum_{i=1}^r E_i). \end{aligned}$$

In case (6), there are families of lines with classes  $l_1, l_2$  and  $l_3$  dual to  $H_1, H_2$  and  $H_3$ , respectively, in  $Z$  passing through exactly one of the points  $p_i$ . The proper transform of these lines in  $X$  are effective curves with classes  $l_1 - e_i, l_2 - e_i$ , and  $l_3 - e_i$ . We thus obtain a subcone  $C$  of the cone of curves  $\overline{\text{Curv}}(X)$  generated by curve classes

$$\{l_1, l_2, l_3, e_1, \dots, e_6, l_1 - e_1, \dots, l_1 - e_6, l_2 - e_1, \dots, l_2 - e_6, l_3 - e_1, \dots, l_3 - e_6\}.$$

As in the previous two paragraphs, the divisors listed in the theorem generate the dual cone  $\check{C}$ .

Finally in case (7), the components of reducible fibers give curves on  $X$  with classes  $l - e_1, l - e_2$ , where the curve class  $l$  satisfies  $H \cdot l = 1$ . So we obtain a subcone  $C$  of the cone of curves generated by  $\{l - e_1, l - e_2, e_1, e_2\}$ . Again, the divisors listed generate the dual cone  $\check{C}$ .

To conclude the proof it remains to show that the divisors listed in the theorem are nef. In cases (1)-(3) and (7), the divisor  $H$ , in cases (4)-(5), the divisors  $H_1$  and  $H_2$  and in case (6), the divisors  $H_1, H_2, H_3$  are semi-ample, in particular nef, being the pullbacks of ample divisors by morphisms. The divisor

$$H - \sum_{i=1}^r E_i = -\frac{1}{n-1}K_X$$

is base-point-free since this is the divisor that defines the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . All the other divisors listed in parts (1)-(6) of the theorem are of the form  $D = -\frac{1}{n-1}K_X + E_{i_1} + \dots + E_{i_j}$ . If  $C$  is an irreducible curve such that  $C \cdot D < 0$ , then  $C$  is necessarily contained in one of the exceptional divisors  $E_i$ . However, since the exceptional divisor  $E_i$  is a projective space, any effective curve class on  $E_i$  is proportional to  $e_i$ . Since  $e_i \cdot D \geq 0$ , we obtain a contradiction. It follows that the cone  $\check{C}$  is the nef cone. Furthermore, by the Base-Point-Free Theorem [14, Theorem 3.3], all of these divisors are semi-ample. Both  $D$  and  $D - \frac{1}{n-1}K_X$  are nef, and moreover  $D - \frac{1}{n-1}K_X = -\frac{2}{n-1}K_X + E_{i_1} + \dots + E_{i_j}$  has top self-intersection number

$$2^n H^n - 2^n \sum_{i \notin \{i_1, \dots, i_j\}} E_i^n - \sum_{i \in \{i_1, \dots, i_j\}} E_i^n = 2^n r - 2^n (r - j) - j = (2^n - 1)j > 0$$

so is big. Hence, by the Base-Point-Free Theorem,  $D$  is semi-ample.  $\square$

**Corollary 3.2.** *The first statement of Conjecture 1.1 holds for  $X$ .*

*Proof.* By Theorem 3.1,  $\overline{A(X)}^e$  equals  $\overline{A(X)}$ , a rational polyhedral cone. The automorphism group  $\text{Aut}(X)$  acts on  $N^1(X)$  as a subgroup of  $\text{GL}(N^1(X)_{\mathbb{Z}})$  and moreover preserves  $\overline{A(X)}$ . This implies that the image of  $\text{Aut}(X)$  in  $\text{GL}(N^1(X)_{\mathbb{Z}})$  must be finite, since any automorphism  $g$  must permute the primitive integral vectors on the (finitely many) extremal rays of  $\overline{A(X)}$ , and this permutation determines the image of  $g$  in  $\text{GL}(N^1(X)_{\mathbb{Z}})$ .

Therefore, we have a finite group of integral linear transformations acting on the rational polyhedral cone  $\overline{A(X)}$ . It is then straightforward to produce a rational polyhedral fundamental domain  $\Pi$  for the action.  $\square$

*Remark 3.3.* In fact, one can show that in all but two of our examples the automorphism group  $\text{Aut}(X)$  is trivial, so that  $\overline{A(X)}^e$  is the fundamental domain. The exceptions are the following

- The case  $Z = Gr(2, 5)$ : here there are automorphisms inducing any permutation of the 5 base-points of the linear system  $V$ , so we get a group of automorphisms isomorphic to the symmetric group  $S_5$ .
- The case  $Z \rightarrow \mathbb{P}^n$  is a double cover branched over a quartic: here we have the involution switching the two sheets of the covering. This restricts to the hyperelliptic involution on any smooth fiber of  $X \rightarrow \mathbb{P}^{n-1}$ .

#### 4. FLOPS OF $X$

In this section we record some facts we need about flops. We show that the flops we need for the proof of Conjecture 1.1 do in fact exist, and prove a lemma about classes of certain effective curves on flops. Finally, we show that in our examples, flops preserve the property of having rational polyhedral nef cone.

**Lemma 4.1.** *Let  $Y$  be a projective variety, and let  $g : Y \dashrightarrow Y'$  be the flop of an extremal ray  $R \subset \overline{\text{Curv}(Y)} \cap K^{\perp}$ . Let  $\Gamma$  be a curve which intersects exactly one curve  $C$  with  $[C] \in R$ , and suppose the intersection is transverse and consists of  $k$  points. Then the proper transform  $\Gamma'$  of  $\Gamma$  on  $Y'$  has numerical class  $[\Gamma] + k[C]$ .*

*Proof.* Let  $U \subset Y$  and  $U' \subset Y'$  be the two maximal open subsets that are isomorphic under the flop  $g$ . Choose a very ample divisor  $A$  which intersects  $C$  and  $\Gamma$  transversely and is disjoint from  $C \cap \Gamma$ . Let  $A'$  denote the strict transform of  $A$  in  $Y'$ . Since  $g$  is an isomorphism on  $U$ , we get  $A \cdot \Gamma$  transverse intersection points of  $A'$  and  $\Gamma'$  on  $U'$ . Let  $C'$  be the cocenter of the flop. Next, we must consider intersection points of  $A'$  and  $\Gamma'$  along  $C'$ . The divisor  $A'$  has multiplicity  $A \cdot C$  at every point of  $C'$ , and  $\Gamma'$  meets  $C'$  transversely in  $k$  points. So for a general choice of  $A$ , we obtain a further contribution of  $k(A \cdot C)$  to the intersection number  $A' \cdot \Gamma'$ . We conclude that for a general very ample divisor  $A$ , we have

$$A' \cdot \Gamma' = A \cdot \Gamma + k(A \cdot C).$$

Since this is true for all general very ample divisors  $A$ , and these span  $N^1(Y)$ , we conclude that

$$[\Gamma'] = [\Gamma] + k[C].$$

□

We will use this lemma to calculate the numerical classes of fiber components on flops of our varieties. These classes will be important in later sections, since they give bounds on the nef cones of these flops. We will see in the next section that in each case it suffices to consider a short list of flops. The numerical classes of fiber components on these flops are recorded below. In each case,  $F$  denotes the class of a fiber of the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ .

- (1)  $Z = Gr(2, 5)$ . In this case, we need to flop the locus of lines, conics, and cubics through one of the base-points  $p_1$ . This creates numerical classes of fiber components of the following kinds:

$$F + (l - e_1), \quad F + (2l - e_1 - e_i), \quad F + (3l - e_1 - e_i - e_j).$$

- (2)  $Z$  is an intersection of quadric hypersurfaces in  $\mathbb{P}^{n+2}$ . In this case, we need to flop the locus of lines and conics through a given point  $p_1$ ; it is not necessary to flop cubics. As in the previous case, this creates classes of the following kinds:

$$F + (l - e_1), \quad F + (2l - e_1 - e_i).$$

- (3)  $Z$  is a cubic hypersurface in  $\mathbb{P}^{n+1}$ . In this case, we will not need to perform any flops on  $X$ .
- (4)  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ . Here it will be necessary to perform longer sequences of flops: it may be that we need to flop a component of a fiber, then on the new space flop the proper transform of the other component, and so on. One can apply Lemma 4.1 repeatedly to show that the result is always a class of the form  $mF + C$ , where  $C$  is the original class we flopped, and  $m$  is a positive integer. In our cases the relevant classes on these flops are

$$\begin{aligned} mF + (l_1 - e_1) \quad (m = 1, 2) \\ mF + (l_1 + l_2 - e_i - e_j) \quad (m = 1, 2, 3) \\ mF + (2l_1 + l_2 - e_1 - e_j - e_k) \quad (m = 1, \dots, 4) \end{aligned}$$

- (5)  $Z = F(1, 2; 3)$ ,  $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . In these cases, we will avoid explicit calculations by using facts specific to threefolds.

**Lemma 4.2.** *Let  $X$  be one of our examples. Let  $R$  be an isolated extremal ray of  $\overline{Curv}(X)$  which lies in  $K_X^\perp$ . Then the flop of  $R$  exists. More generally, if  $X'$  is an SQM of  $X$  obtained by a sequence of flops of classes in  $K^\perp$ , and  $R$  is an isolated extremal ray of  $\overline{Curv}(X')$  which lies in  $K_{X'}^\perp$ , then the flop of  $R$  exists.*

*Proof.* For any  $X'$  as in the statement, the anti-canonical divisor  $-K_{X'}$  is semi-ample, so there exists a  $\mathbb{Q}$ -divisor  $\Delta$  such that  $(X', \Delta)$  is a klt Calabi–Yau pair. Given a ray  $R$  as in the statement of the lemma, it suffices to find an effective divisor  $D$  such that  $D \cdot R < 0$ . For then,  $(X, \Delta + \epsilon D)$  is a klt pair for sufficiently

small  $\epsilon$ , and  $R$  is a  $(K_X + \Delta + \epsilon D)$ -negative extremal ray. By the Cone Theorem, the contraction of  $R$  exists. Therefore, by Birkar–Cascini–Hacon–McKernan [1, Corollary 1.4.1], the flop of  $R$  exists.

To show that the necessary effective divisor  $D$  exists, we use the theorem of Boucksom–Demailly–Păun–Peternell [2, Theorem 0.2], which asserts that the cone of pseudoeffective divisors  $\overline{\text{Eff}}(X')$  is dual to the cone of moving curves on  $X'$ . Now suppose  $R$  is an extremal ray in  $K^\perp$  spanned by a curve  $C$ . Any curve  $C'$  numerically a multiple of  $C$  must be contained in a fiber of the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ , and moreover cannot be a whole fiber (since the class  $F$  does not span an extremal ray). Since  $X$  has reducible fibers in codimension 2, the same is true of any flop  $X'$ . Hence, in particular, the curve  $C$  cannot be moving. Since the ray  $R$  spanned by  $C$  is an isolated extremal ray, the closure of the cone of moving curves does not contain  $R$ . By the theorem of Boucksom–Demailly–Păun–Peternell, there is an effective divisor  $D$  with  $D \cdot C < 0$ , as required.  $\square$

## 5. THE MOVABLE CONE

In this section, we prove part (2) of Conjecture 1.1 for our examples. We first study the action of the pseudo-automorphisms  $\text{PsAut}(X/\mathbb{P}^{n-1})$  preserving the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$  on the relative effective movable cone  $\overline{M}(X/\mathbb{P}^{n-1})^e$ . We find a rational polyhedral fundamental domain for this action. We further show that the union of the nef cones of finitely many SQMs lie over this fundamental domain. Since these nef cones are rational polyhedral cones, we obtain a rational polyhedral fundamental domain for the action of pseudo-automorphisms on  $\overline{M}(X)^e$ . We use a combination of general results from minimal model theory and explicit analysis of our examples.

As explained in the introduction, the effective nef cones of all the SQMs of a variety  $X$  are contained in  $\overline{M}(X)^e$ . The next lemma shows that in our examples the converse is also true: the effective movable cone is the union of the effective nef cones of the SQMs of  $X$ . Given a contraction morphism  $f : X \rightarrow Y$ , an effective divisor  $D$  on  $X$  is called *f-vertical* if  $f_*(D) \neq Y$ .

**Lemma 5.1.** *Let  $X$  be a smooth projective variety such that some power of  $-K_X$  defines a contraction morphism  $f : X \rightarrow Y$  with  $\dim Y = \dim X - 1$ . Suppose that every  $f$ -vertical divisor on  $X$  is a multiple of  $-K_X$ . Then*

$$\overline{M}(X)^e = \bigcup \overline{M}(X')^e,$$

where the union on the right is over all SQMs  $X' \dashrightarrow X$ .

*Proof.* We must show that if  $D$  is any effective  $\mathbb{R}$ -divisor on  $X$  whose numerical class lies in  $\overline{M}(X)$ , then that class belongs to the nef cone of some SQM of  $X$ . Since, by assumption,  $-K_X$  is already semi-ample on  $X$ , we may assume that  $D$  is not a multiple of  $-K_X$ . Since there are no  $f$ -vertical divisors other than multiples of  $-K_X$ , we have  $D \cdot F > 0$ , where  $F$  is the generic fiber of  $f : X \rightarrow Y$ . Therefore,  $D$  is  $f$ -big. Hence, by [14, Lemma 3.23], the  $\mathbb{R}$ -divisor  $-mK_X + D$  is big for sufficiently large values of  $m$ . Dividing by  $m$ , we see that  $-K_X + \epsilon D$  is big for sufficiently small positive values of  $\epsilon$ .

Now choose a positive integer  $r$  such that  $-rK_X$  is base-point-free, and let  $\Delta$  be a smooth divisor in the linear system  $|-rK_X|$ . Setting  $\Theta = \frac{1}{r}\Delta$ , the pair  $(X, \Theta)$  is klt, hence so is  $(X, \Theta + \epsilon D)$  for sufficiently small positive numbers  $\epsilon$ . Moreover,

$$\Theta + \epsilon D \equiv -K_X + \epsilon D$$

is big for sufficiently small  $\epsilon$  by the previous paragraph, and

$$(1) \quad K_X + \Theta + \epsilon D \equiv \epsilon D$$

is effective. By Birkar–Cascini–Hacon–McKernan [1], the pair  $(X, \Theta + \epsilon D)$  has a minimal model  $X'$ . By Equation (1), the proper transform of  $D$  on  $X'$  is nef. Finally, since  $D \in \overline{M}(X)$ , there cannot be a  $D$ -negative divisorial contraction, so the birational map  $X \dashrightarrow X'$  is an SQM.  $\square$

**Corollary 5.2.** *The conclusion of Lemma 5.1 holds for each of our examples  $X$ .*

*Proof.* By construction each of our examples has an elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . We need to show that there are no  $f$ -vertical divisors other than multiples of  $-K_X$ . Each fibration  $f : X \rightarrow \mathbb{P}^{n-1}$  is equidimensional, so any irreducible vertical divisor  $D$  must be a component of a subset  $f^{-1}(Y)$  where  $Y \subset \mathbb{P}^{n-1}$  has codimension 1. By Proposition 2.2, the fiber of  $f$  over any codimension-one point is irreducible, so we must have  $D = f^{-1}(Y)$ , in other words some multiple of  $D$  is pulled back from a divisor on  $\mathbb{P}^{n-1}$ . The image of  $\text{Pic}(\mathbb{P}^{n-1}) \hookrightarrow \text{Pic}(X)$  is generated by some (rational) multiple of  $-K_X$ , so  $D = -qK_X$  for some rational number  $q$ , completing the proof.  $\square$

The next lemma exploits the combinatorial structure of the chamber decomposition of the movable cone to reduce the problem of finding a fundamental domain to a more “local” one. The upshot is that we do not need to understand all SQMs of a given variety  $X$ , but only those which are “close” to  $X$ .

**Lemma 5.3.** *Let  $X$  be a variety such that*

$$\overline{M}(X)^e = \bigcup \overline{A}(X')^e,$$

where the union on the right is over all SQMs  $X' \dashrightarrow X$ . Suppose that there is a collection of SQMs  $\{X_i\}_{i \in I}$  with the following property: for each SQM  $X_\alpha$  of  $X$  such that  $\overline{A}(X_\alpha)$  shares a codimension-one face with one of the cones  $\overline{A}(X_i)$ , there exists an SQM pseudo-automorphism  $\varphi$  of  $X$  such that  $\varphi_*(\overline{A}(X_\alpha)) \subset U$ , where  $U$  denotes the union  $\bigcup_{i \in I} \overline{A}(X_i)$ . Then  $\text{PsAut}^*(X) \cdot U = \overline{M}(X)^e$ .

*Proof.* Form a graph with a vertex for the nef cone of each SQM and an edge connecting  $\overline{A}(X_\alpha)$  and  $\overline{A}(X_\beta)$  if there is a log flip  $X_\alpha \dashrightarrow X_\beta$ . By Lemma 5.1, each SQM  $X_\alpha \dashrightarrow X$  consists of a finite sequence of log flips. Therefore, any two vertices of the graph are at a finite distance in the graph metric.

We will prove the lemma by induction on the distance  $d$  of a given nef cone  $\overline{A}(X_\alpha)$  from  $U$ . For  $d = 1$ , the assumption of the lemma gives  $\overline{A}(X_\alpha) \subset \text{PsAut}^*(X) \cdot U$ . Now suppose that all nef cones at a distance no more than  $d$  are contained in  $\text{PsAut}(X) \cdot U$ , and suppose  $\overline{A}(X_\alpha)$  is a cone at distance  $d + 1$ . Then there is a neighboring nef



cone  $\overline{A(X_\beta)}$  at distance  $d$  from  $U$ . By the induction hypothesis, there is a pseudo-automorphism  $\varphi$  such that  $\varphi_*(\overline{A(X_\beta)}) \subset U$ . Now pseudo-automorphisms act on the graph of nef cones via automorphisms, so  $\varphi_*(\overline{A(X_\alpha)})$  must either be contained in  $U$  or be a neighbor of a nef cone in  $U$ . In the second case, by assumption there is another pseudo-automorphism  $\psi$  such that  $\psi_*\varphi_*(\overline{A(X_\alpha)}) \subset U$ , so in either case we have  $\overline{A(X_\alpha)} \subset \text{PsAut}^*(X) \cdot U$ . By induction,  $\text{PsAut}(X) \cdot U$  contains every nef cone  $\overline{A(X_\alpha)}$ , therefore equals  $\overline{M(X)}^e$ .  $\square$

The next lemma describes the relative effective movable cone of the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . We recall that a Cartier divisor  $D$  is called *f-movable* (respectively, *f-effective*) if  $\text{codim}(\text{Supp}(\text{Coker}(f^*f_*\mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D)))) \geq 2$  (respectively,  $f_*\mathcal{O}_X(D) \neq 0$ ).

**Proposition 5.4.** *The relative effective movable cone of  $f : X \rightarrow \mathbb{P}^{n-1}$  is*

$$\overline{M(X/\mathbb{P}^{n-1})}^e = B := \{x \in N^1(X/\mathbb{P}^{n-1}) : x \cdot F > 0\} \cup \{0\},$$

where  $F$  denotes the class of a fiber of  $f$ .

*Proof.* The inclusion  $\overline{M(X/\mathbb{P}^{n-1})}^e \subseteq B$  is easy to see. An  $f$ -effective divisor  $D$  has nonnegative degree on the generic fiber of  $f$ , and has degree 0 only if it is vertical. Since the only vertical divisors are multiples of  $-\frac{1}{n-1}K_X$ , a vertical divisor is zero in  $N^1(X/\mathbb{P}^{n-1})$ .

To see the reverse inclusion, we use a standard argument involving Grauert's theorem on semicontinuity of cohomology [8, Corollary 12.9]. Suppose  $D$  is a divisor on  $X$  such that  $D \cdot F > 0$ . Then the restriction  $D|_{X_y}$  to any (fixed) irreducible fiber  $X_y$  is ample. Replacing  $D$  by a positive multiple if necessary, we have that  $D|_{X_y}$  is base-point-free, in particular, effective. Grauert's theorem implies that a section of  $D|_{X_y}$  is the restriction of a section  $s \in \mathcal{O}_X(D)(f^{-1}(U))$  for some open set  $U \subset \mathbb{P}^{n-1}$ . Hence,  $f_*(\mathcal{O}_X(D)) \neq 0$ ; that is,  $D$  is  $f$ -effective. To see that  $D$  is  $f$ -movable, replacing  $D$  by a positive multiple, we can assume that the restriction  $D|_{X_y}$  to every irreducible fiber  $X_y$  is base-point-free. Again by Grauert's theorem, a section of  $D|_{X_y}$  is the restriction of a section  $s \in \mathcal{O}_X(D)(f^{-1}(U))$  for some open set  $U \subset \mathbb{P}^{n-1}$ . Since  $D|_{X_y}$  is base-point-free, this shows that the support of the sheaf  $\text{Coker}(f^*f_*\mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D))$  does not contain any point of any irreducible fiber of  $f$ . We saw in Section 2 that  $f$  has reducible fibers in codimension 2, so  $D$  is  $f$ -movable. We have shown that every integral point of the cone  $B$  belongs to  $\overline{M(X/\mathbf{P})}^e$ . Since  $B$  is spanned by its integral points, this completes the proof of the reverse inclusion.  $\square$

Now we will study the action of pseudo-automorphisms on the movable cone in more detail. The action on  $N^1(X)$  is somewhat complicated, but fortunately we do not need to understand it in detail. It will suffice to understand the action on the space  $N^1(X/\mathbb{P}^{n-1})$ . The generic fiber  $X_\eta$  of the fibration  $f : X \rightarrow \mathbb{P}^{n-1}$  is an elliptic curve. Given two rational points  $p, q \in X_\eta$ , their difference  $p - q \in \text{Pic}^0(X_\eta)$  defines a translation on  $X_\eta$ , which extends to a pseudo-automorphism of  $X$ . The group of translations is called the *Mordell-Weil group* and is denoted by  $\text{MW}(f)$ . The hyperelliptic involution on  $X_\eta$  also defines a pseudo-automorphism of

$X$ . In fact, the group of relative pseudo-automorphisms  $\text{PsAut}(X/\mathbb{P}^{n-1})$  is the  $\mathbb{Z}_2$ -extension of  $\text{MW}(f)$  generated by the hyperelliptic involution. We will now describe a fundamental domain for the action of  $\text{PsAut}(X/\mathbb{P}^{n-1})$  on  $N^1(X/\mathbb{P}^{n-1})$ .

Observe that

$$\begin{aligned} N^1(X/\mathbb{P}^{n-1}) &\cong N^1(X)/\langle -K_X \rangle \\ &= (\text{Pic}(X)/\ker \rho) \otimes \mathbb{R} \\ &= \text{Pic}(X_\eta) \otimes \mathbb{R}, \end{aligned}$$

where  $\rho$  denotes the natural restriction homomorphism  $\text{Pic}(X) \rightarrow \text{Pic}(X_\eta)$ . An element  $x \in \text{MW}(f)$  acts on  $N^1(X/\mathbb{P}^{n-1})$  by

$$\varphi_x : y \mapsto y + (y \cdot F)x.$$

The involution  $\iota$  acts by

$$\iota : y \mapsto 2(y \cdot F)E_1 - y.$$

Using this information, we obtain the following.

**Proposition 5.5.** *Let  $X$  be one of our examples. Then  $N^1(X/\mathbb{P}^{n-1})$  has a basis consisting of sections of the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . If  $S_1, \dots, S_k$  is a basis of rational sections (where  $k = \dim N^1(X/\mathbb{P}^{n-1})$ ) then a fundamental domain for the action of  $\text{PsAut}(X/\mathbb{P}^{n-1})$  on  $\overline{M}(X/\mathbb{P}^{n-1})^e$  is the following:*

$$V = \mathbb{R}_+ \cdot \left\{ S_1 + \sum_i a_i (S_i - S_1) : 0 \leq a_i \leq 1, \sum_i a_i \leq \frac{k-1}{2} \right\}.$$

*Proof.* First, by Theorem 3.1, we see that the kernel of  $N^1(X) \rightarrow N^1(X/\mathbb{P}^{n-1})$  is spanned by  $-\frac{1}{n-1}K_X$  in each example. So  $\dim N^1(X/\mathbb{P}^{n-1}) = \dim N^1(X) - 1$ . Now  $N^1(X)$  has a basis consisting of the exceptional divisors  $\{E_i\}$  together with a basis of  $N^1(Z)$ , where  $Z$  is the underlying Fano manifold. The exceptional divisors are always sections of the elliptic fibration, and their classes in  $N^1(X/\mathbb{P}^{n-1})$  remain linearly independent. So to complete the proof of the first claim, in each case, we must identify  $\rho(Z) - 1$  additional rational sections of the fibration whose classes in  $N^1(X/\mathbb{P}^{n-1})$  are linearly independent of the classes  $E_i$ . If  $\rho(Z) = 1$ , there is nothing to do. In the other cases, the rational sections we need are as follows:

- (1)  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ : here an additional rational section is given by the preimage of the line in (say) the first copy of  $\mathbb{P}^2$  through the points  $\pi_1(p_1)$  and  $\pi_1(p_2)$ . This divisor has class  $H_1 - E_1 - E_2$ .
- (2)  $Z = F(1, 2; 3)$ : here an additional rational section is given by the divisor of all pointed lines which pass through the intersection point of the pointed lines corresponding to  $p_1$  and  $p_2$ . This divisor has class  $H_1 - E_1 - E_2$ .
- (3)  $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ : here the two additional rational sections can be chosen to be the pullbacks of the projections  $\pi_1(p_1)$  and  $\pi_1(p_2)$ . These have classes  $H_1 - E_1$  and  $H_1 - E_2$ .

This completes the proof of the first statement. To prove the second statement, note first that by Proposition 5.4 the cone  $\overline{M}(X/\mathbb{P}^{n-1})^e$  is the span of the affine

hyperplane

$$W = \{x \in N^1(X/\mathbb{P}^{n-1}) \mid x \cdot F = 1\}.$$

Furthermore, the action of  $\text{PsAut}(X/\mathbb{P}^{n-1})$  preserves  $W$ . So it suffices to prove that the region

$$V_1 := \left\{ S_1 + \sum_i a_i(S_i - S_1) : 0 \leq a_i \leq 1, \sum_i a_i \leq \frac{k-1}{2} \right\}$$

is a fundamental domain for the action of  $\text{PsAut}(X/\mathbb{P}^{n-1})$  on the hyperplane  $W$ .

For definiteness, let us fix  $S_1$  as the base-point of our group of translations. Then we have a basis of  $\text{MW}(f)$  given by the differences  $\{S_i - S_1\}$ . By the formula for the action of  $\text{MW}(f)$ , these differences acts on  $W$  by translations. We can write an arbitrary element of  $W$  in the form

$$S_1 + \sum_{i=2}^k a_i(S_i - S_1)$$

for some real numbers  $a_i$ , and then applying appropriate elements of  $\text{MW}(f)$  we can transform this to a class with  $0 \leq a_i \leq 1$  for all  $i$ . Moreover, it is clear that two classes with  $0 < a_i < 1$  for all  $i$  are not in the same orbit of  $\text{MW}(f)$ . So the ‘‘cubical’’ region

$$C = \left\{ S_1 + \sum_{i=2}^k a_i(S_i - S_1) \mid 0 \leq a_i \leq 1 \right\}$$

is a fundamental domain for the action of  $\text{MW}(f)$  on  $W$ . Finally, to obtain a fundamental domain for the whole group  $\text{PsAut}(X/\mathbb{P}^{n-1})$ , we must consider the action of the involution  $\iota$ . The formula for the action of  $\iota$  shows that  $\iota$  acts on  $W$  as follows:

$$S_1 + \sum_{i=2}^k a_i(S_i - S_1) \mapsto (1 + \sum_{i=2}^k a_i)S_1 - \sum_{i=2}^k a_i S_i.$$

If we then apply translation by the element  $\sigma = \sum_i(S_i - S_1) \in \text{MW}(f)$ , we obtain  $S_1 + \sum_i(1 - a_i)S_i$ . Therefore, the composition  $t_\sigma \circ \iota$  maps the region  $C$  to itself via the formula

$$t_\sigma \circ \iota : S_1 + \sum_{i=2}^k a_i(S_i - S_1) \mapsto S_1 + \sum_{i=2}^k (1 - a_i)(S_i - S_1).$$

Since all the coefficients of points in  $C$  satisfy  $0 \leq a_i \leq 1$ , at least one of the quantities  $\sum_{i=2}^k a_i$  and  $\sum_{i=2}^k (1 - a_i)$  does not exceed  $\frac{k-1}{2}$ . This shows that  $V_1$  is indeed a fundamental domain of the action on the hyperplane  $W$ .  $\square$

**Proposition 5.6.** *For each  $X$  in our list and each SQM  $X'$  of  $X$  obtained by a sequence of flops of fiber components, the nef cone  $\overline{A(X')}$  is rational polyhedral. Moreover, it is spanned by semi-ample divisors; in particular,  $\overline{A(X')}^e = \overline{A(X)}$ .*

*Proof.* We will prove the dual statement that the closed cone of curves  $\overline{\text{Curv}}(X')$  is rational polyhedral. Let  $R$  be an extremal ray of  $\overline{\text{Curv}}(X')$  which lies in  $K^\perp$  and is different from the ray spanned by  $F$ . Then there is an effective divisor  $D'$  on  $X'$  whose base locus is a union of fiber components and such that  $D' \cdot R < 0$ . To see this, start with an ample divisor  $A$  on  $X$  such that  $A \cdot F = 1$ . Applying pseudo-automorphisms to  $A$  produces effective divisors of the form

$$(2) \quad D = A + \sum_i a_i (S_i - S_1) - \frac{m}{n-1} K_X,$$

where the  $a_i$  are any chosen integers,  $m$  is some positive integer, and the  $S_i$  are a basis of sections of  $f$ . Now one can check that since  $R$  is not the ray spanned by the class  $F$ , we have  $(S_i - S_1) \cdot R \neq 0$  for some  $i$ . Moreover  $-K_X \cdot R = 0$ , and so we can choose the integers  $a_i$  suitably so that

$$D \cdot R = A \cdot R + \sum_i a_i (S_i - S_1) \cdot R < 0.$$

Note that since  $D$  is the proper transform of the ample divisor  $A$  under a pseudo-automorphism  $X \dashrightarrow X$  over  $\mathbb{P}^{n-1}$ , the base locus of  $D$  consists only of fiber components. Finally, let  $D'$  be the proper transform of  $D$  on  $X'$ : since  $X'$  is obtained from  $X$  by flopping fiber components, the base locus of  $D'$  is still a union of fiber components.

To prove the proposition, now suppose that  $\overline{\text{Curv}}(X')$  is not rational polyhedral. By the Cone Theorem, there must be an infinite sequence  $\{C_i\}$  of irreducible curves on  $X'$  such that the corresponding rays  $R_i$  are extremal rays of  $\overline{\text{Curv}}(X')$  and converge to a ray in  $K^\perp$ . Since there are only finitely many classes of curves that lie in the fibers of  $f$ , we may assume that the curves  $C_i$  are not fiber components. Then  $D \cdot C_i \geq 0$  for all divisors  $D$  of the form given by Equation (2), and hence  $D \cdot R \geq 0$  for all such divisors  $D$ . Then, by the previous paragraph,  $R$  must be the ray spanned by the class  $F$  of a fiber. By Lemma 4.1, the class  $F$  is always in the interior of the top-dimensional face  $\overline{\text{Curv}}(X') \cap K^\perp$ , so it is impossible that  $\lim_i R_i = \mathbb{R}_+ F$ . This is a contradiction, so  $\overline{\text{Curv}}(X')$  and hence  $A(X')$  are rational polyhedral cones.

Finally, to prove that  $A(X')$  is spanned by semi-ample divisors, we apply the Base-point-free Theorem [14, Theorem 3.3]. If  $X'$  is obtained from  $X$  by a sequence of flops, then  $-K_{X'}$  is base-point-free, in particular nef. We claim that if  $D$  is any nef Cartier divisor on  $X'$ , then  $D$  is semi-ample. If  $D$  is a multiple of  $-K_{X'}$  there is nothing to prove, so assume it is not. The class  $aD - K_{X'}$  is nef for any  $a \geq 0$ , so by [14, Proposition 2.61]  $aD - K_{X'}$  is big if and only its top self-intersection number is strictly positive. Any iterated self-intersection of  $-K_{X'}$  is an effective cycle, so by the Nakai–Moishezon criterion, for any  $a > 0$  we have

$$(aD - K)^n \geq aD \cdot (-K)^{n-1} = a'D \cdot F$$

(where  $a' = a \cdot (n-1)^{n-1}$ ). Since  $D$  is nef, if it were the case that  $D \cdot F = 0$ , then we would have  $D \cdot C = 0$  for every fiber component  $C$  on  $X'$ . By Theorem 3.1 and Lemma 4.1 fiber components on any flop span a codimension-one subspace  $V \subset N_1$ , and the dual space  $V^\perp \subset N^1$  is spanned by  $-K$ . Since by assumption

$D$  is not a multiple of  $-K$ , we conclude that  $D \cdot F > 0$ . This shows that for any  $a > 0$  the class  $aD - K_{X'}$  is nef and big, so by the Base-point-free Theorem  $D$  is semi-ample.  $\square$

**Corollary 5.7.** *If  $X$  is one of our examples and  $X'$  is an SQM of  $X$ , then any  $K$ -trivial extremal ray  $R$  of  $\overline{\text{Curv}}(X')$  is isolated, and the flop of  $R$  exists.*

*Proof.* By the previous proposition,  $\overline{\text{Curv}}(X')$  is rational polyhedral, so any extremal ray  $R$  is isolated. The flop of  $R$  then exists by 4.2.  $\square$

**Proposition 5.8.** *For each  $X$  in our list, there is a finite collection  $\{X_i\}_{i \in I}$  of SQMs of  $X$  such that the cone  $V$  is contained in the image of the union*

$$\bigcup_{i \in I} \overline{A(X_i)}^e$$

*under the projection map  $N^1(X) \rightarrow N^1(X/\mathbb{P}^n)$ .*

*Proof.* The first step of the proof is to prove that the region  $V$  is covered by the union of a finite collection of relative effective nef cones  $\overline{A(X_i/\mathbb{P}^{n-1})}^e$ , where the  $X_i$  are SQMs of  $X$  obtained by flopping classes of fibral curves. (Note that these cones are automatically rational polyhedral, since there are only finitely many classes of fiber components on any SQM, and the relative effective nef cone is the same as the relative nef cone by Proposition 5.4.) We will do this case-by-case below.

Given this, we complete the proof as follows. If  $D$  is a divisor which maps into the interior of the cone  $\overline{A(X_i/\mathbb{P}^{n-1})}^e$ , in other words a relatively ample divisor, then by [14, Lemma] the divisor  $D - \frac{m}{n-1}K_X$  is ample on  $X_i$  for  $m$  sufficiently large. Since each  $X_i$  is obtained by flopping fiber components, the class  $-\frac{1}{n-1}K_X$  belongs to  $\overline{A(X_i)}^e$ , so this proves that  $\overline{A(X_i)}^e$  surjects onto the interior of  $\overline{A(X_i/\mathbb{P}^{n-1})}^e$ . But now since both these cones are rational polyhedral, in fact the image of  $\overline{A(X_i)}^e$  must be the whole cone  $\overline{A(X_i/\mathbb{P}^{n-1})}^e$ .

So in each case we must identify a finite set of SQMs  $X_i$  obtained by flopping fiber components such that  $V$  is covered by the union of the cones  $\overline{A(X_i/\mathbb{P}^n)}$ .

- (1)  $Z = Gr(2, 5)$ . Here  $k = 5$ , so the final condition in the definition of  $V_1$  is  $\sum_i a_i \leq 2$ . Let  $x$  be a point in  $V_1$ : since  $a_i \geq 0$  for  $i = 2, 3, 4, 5$  we have  $x \cdot (l - l_i) \geq 0$  for these values of  $i$ . So in order to make the class  $x$  relatively nef we may need to flop classes of the form  $l - l_1, 2l - l_1 - l_j$  where  $\sum_{i \neq j} a_i > 1$  and  $3l - l_1 - l_j - l_k$  where  $\sum_{i \neq j, k} a_i > 1$ . Note that since  $a_i \leq 1$  we never need to flop the class of a quartic.

To check that this sequence of flops makes  $x$  relatively nef, we need to check its degree on the classes of the new fiber components created. As explained in Section 4 these new classes are of the form  $F + (l - l_1), F + (2l - l_1 - l_j), F + (3l - l_1 - l_j - l_k)$ . But now

$$x \cdot (F + (l - l_1)) = 2 - \sum_i a_i \geq 0$$

by the description of  $V_1$ , and so

$$x \cdot (F + (2l - l_1 - l_j)) = 2 - \sum_i a_i + a_j \geq 0,$$

$$x \cdot (F + (3l - l_1 - l_j - l_k)) = 2 - \sum_i a_i + a_j + a_k \geq 0$$

also. Therefore  $V_1$  is covered by the relative nef cones of the finitely many SQMs obtained by flopping some sets of curves with classes  $l - l_1, 2l - l_1 - l_j, 3l - l_1 - l_j - l_k$ .

- (2)  $Z = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}$ : Here  $k = 4$  so we have the condition  $\sum_i a_i \leq \frac{3}{2}$ . In this case, to make a class  $x \in V_1$  relatively nef we may have to flop curves of the form  $l - l_1$  and  $2l - l_1 - l_j$  where  $\sum_{i \neq j} a_i > 1$ . Since  $a_i \leq 1$  we never need to flop the class of a cubic.

Again we must check such a sequence of flops makes  $x$  relatively nef. In this case the new fiber components have classes  $F + (l - l_1)$  or  $F + (2l - l_1 - l_j)$ , and the same calculation as above shows that these numbers are nonnegative.

- (3)  $Z$  is a cubic hypersurface in  $\mathbb{P}^{n+1}$ . Here  $k = 3$  so we have the condition  $\sum_i a_i \leq 1$ : that is,  $V_1$  is the unit simplex. By Theorem 3.1 the projection  $\overline{A(X)}^e \rightarrow V$  is surjective.
- (4)  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ : Here  $k = 7$  so we have the condition  $\sum_i a_i \leq 3$ . In this case, the sequence of flops we need to perform may be more complicated: we may have to flop two components of the same fiber in sequence. In more detail, one checks that divisors of the form

$$x = S_1 + \sum_{i=2}^7 a_i (S_i - S_1)$$

with  $0 \leq a_i \leq 1$  and  $\sum_i a_i \leq 3$  have intersection numbers  $x \cdot C \geq -4$  with fiber components  $C$ . Since  $x \cdot F = 1$  by definition of the region  $V_1$ , one has  $x \cdot (4F + C) \geq 0$  for all  $C$  which are components of fibers. By the discussion in Section 4, this shows that after an appropriate sequence of flops the class  $x$  becomes relatively nef.

- (5, 6)  $Z = F(1, 2; 3)$  or  $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . These varieties have dimension 3, so Kawamata's proof of the relative version of the cone conjecture [10] applied to  $f : X \rightarrow \mathbb{P}^2$  shows that in both cases there are finitely many flops  $\{X_i\}$  such that  $V$  is contained in the union of the cones  $\overline{A(X_i/\mathbb{P}^2)}$ . For the purposes of the cone conjecture, it is not necessary to identify these flops explicitly.
- (7)  $Z \rightarrow \mathbb{P}^n$  is a double cover branched on a quartic. Here  $k = 2$  so we have the condition  $\sum_i a_i \leq \frac{1}{2}$ . That is,  $V_1$  is the interval  $[0, \frac{1}{2}]$  in the affine line  $W$ . Again we have that  $\overline{A(X)}^e$  surjects onto  $V$ . (In this case, the image of  $\overline{A(X)}^e$  is strictly larger than  $V$ , due to the extra automorphism of  $X$  described at the end of Section 3.)

□

We can now complete the proof of the cone conjecture.

**Theorem 5.9.** *The second statement of Conjecture 1.1 holds for  $X$ .*

*Proof.* The idea of the proof is to show that given any SQM  $X_\alpha$  of  $X$ , there is a pseudo-automorphism  $\varphi$  such that  $\varphi_*\overline{A(X_j)} = \overline{A(X_\alpha)}$ , where  $X_j$  is one of the finitely many SQMs  $\{X_i\}$  in Proposition 5.8. By Lemma 5.3, it suffices to prove this when  $\overline{A(X_\alpha)}$  shares a codimension-one face with one of the cones  $\overline{A(X_i)}$ . We first assume the following claim:

**Claim 5.10.** *Every codimension-one face of a cone  $\overline{A(X_i)}$  which intersects the interior of the movable cone is dual to the class of a fiber component.*

The claim implies that if  $\overline{A(X_\alpha)}$  shares a codimension-one face with  $\overline{A(X_i)}$ , then  $X_\alpha$  is obtained from  $X_i$  by flopping the class of a fiber component. Since the relative nef cones  $\overline{A(X_i/\mathbb{P}^{n-1})}$  cover the fundamental domain  $V$ , there exists some  $j$  and some pseudo-automorphism  $\varphi$  such that  $\varphi_*\overline{A(X_j/\mathbb{P}^{n-1})} = \overline{A(X_\alpha/\mathbb{P}^{n-1})}$ . Choosing an ample class  $A_\alpha$  on  $X_\alpha$ , the previous Proposition implies there is an ample class  $A_j$  on  $X_j$  such that

$$\varphi_*(A_j) = A_\alpha + \frac{m}{n-1}K_X$$

for some integer  $m$ .

If  $m \geq 0$ , then we can add  $-\frac{m}{n-1}K_X$  to both sides of this equation. Since pseudo-automorphisms preserve  $\frac{1}{n-1}K_X$ , this gives

$$\varphi_*(A_j - \frac{m}{n-1}K_X) = A_\alpha.$$

Moreover, since  $\frac{1}{n-1}K_X$  is semi-ample on  $X_j$ , the class  $A_j - \frac{m}{n-1}K_X$  belongs to  $\overline{A(X_j)}$ , and we conclude that  $\varphi_*(\overline{A(X_j)}) = \overline{A(X_\alpha)}$ .

If  $m \leq 0$ , then  $A_\alpha + \frac{m}{n-1}K_X$  belongs to  $\overline{A(X_\alpha)}$  provided we know that  $-\frac{1}{n-1}K_X$  is nef on  $X_\alpha$ . By Claim 5.10,  $X_\alpha$  is obtained from some  $X_i$  by flopping the class of some fiber component, so  $-\frac{1}{n-1}K_X$  remains nef (indeed base-point-free) on  $X_\alpha$ . So again in this case we have  $\varphi_*(\overline{A(X_j)}) = \overline{A(X_\alpha)}$  as required.

There remains to verify Claim 5.10. First, assume that  $\dim X = 3$ . By [14, Theorem 6.15], the flop of a smooth threefold is again smooth. Hence, in this case, all  $X_i$  occurring in Proposition 5.8 are smooth threefolds. By Mori's theorem [16], there are no small  $K$ -negative extremal contractions on smooth threefolds, so a codimension-one face separating two nef cones must be dual to the class of a fiber component.

Next, let us verify Claim 5.10 for the case  $Z = \mathbb{P}^2 \times \mathbb{P}^2$ . We wish to prove that there are no small  $K$ -negative extremal contractions of any of the flops  $X_i$ . First, we note that each  $X_i$  is smooth. To see this, note that any sequence of flops of  $X$  restricts to a sequence of flops of any smooth threefold  $Y \in |H|$  inside  $X$ . As explained in the previous paragraph, any sequence of flops applied to  $Y$  yields a smooth threefold  $Y_i$ . If now the resulting fourfold  $X_i$  were singular, there would be a smooth threefold  $Y_i \subset X_i$  which is the support of a Cartier divisor, and which intersects the singular locus of  $X_i$ . This is impossible, so we conclude that  $X_i$  must be smooth.

Now let us show that there are no small  $K$ -negative extremal contractions of  $X_i$ . By the Cone Theorem [14], any  $K$ -negative extremal ray of the cone of curves  $\overline{\text{Curv}}(X_i)$  is spanned by a rational curve  $C$  with  $0 < -K_X \cdot C \leq \dim X + 1 = 5$ . Since  $-K_X = 3H$ , the only possibility is that  $H \cdot C = 1$ . By basic deformation theory [13, I.2.17], at the point corresponding to the inclusion of the curve  $C$ , the space of maps  $\mathbb{P}^1 \rightarrow X$  modulo isomorphisms of the map has dimension at least  $-K_X \cdot C + \dim X - 3 = 4$ . Now suppose that the deformations of  $C$  swept out only a surface  $S$  inside  $X_i$ . Then the space of curves through 2 general points of  $S$  must have dimension at least 2. By Bend and Break [13, II.5], there must be a deformation of  $C$  which is reducible. But  $C$  is a generator of an extremal ray of the cone curves  $\overline{\text{Curv}}(X_i)$ , and is primitive since  $H \cdot C = 1$ , so this is impossible. We conclude that deformations of  $C$  must cover a locus of dimension at least 3 in  $X_i$ , as required.

To verify Claim 5.10 in the remaining cases, we need to find all the codimension-one faces of the cones  $\overline{A}(X_i)$  which intersect the interior of the movable cone. As usual, we will do this case-by-case.

- (1)  $Z = Gr(2, 5)$ . As explained above, here the SQMs  $X_i$  are obtained by flopping classes of the following kinds:

$$l - l_1, 2l - l_1 - l_i (i = 2, \dots, 5), 3l - l_1 - l_i - l_j (i, j = 2, \dots, 5, i \neq j).$$

Note that there are restrictions on which sets of curves can be flopped: the class  $l - l_1$  must be flopped first, cubics  $3l - l_1 - l_i - l_j$  cannot be flopped unless conics through  $p_i$  and  $p_j$  have already been flopped, and two cubics whose union passes through all points cannot both be flopped, because of the condition  $\sum_i a_i \leq 2$  in the definition of  $V_1$ . The following is a complete list (up to relabeling) of all possible sequences of flops of classes of these kinds; for brevity, we use the notation  $1 \cdots i$  to indicate the flop of the class  $kl - l_1 - \cdots - l_i$ .

- (1)  
 (1, 12), (1, 12, 13), (1, 12, 13, 14), (1, 12, 13, 14, 15),  
 (1, 12, 13, 123), (1, 12, 13, 14, 123) (1, 12, 13, 14, 15, 123)  
 (1, 12, 13, 14, 123, 124), (1, 12, 13, 14, 15, 123, 124)  
 (1, 12, 13, 14, 123, 124, 134) (1, 12, 13, 14, 15, 123, 124, 134)  
 (1, 12, 13, 14, 15, 123, 124, 125)

It is straightforward to calculate the nef cones of all these SQMs and observe that their codimension-one faces which intersect the interior on the movable cone are dual to fiber components, as required. Here is a list of the extremal rays of the nef cone in each case on the above list:

- (1):  $\{H - E_1 - \sum_i E_i\}, H - 2E_1$ .
- flops involving only line and conics, e.g. (1, 12): here the extremal rays are of the form  $\{H - E_1 - E_2 - \sum_i E_i\}, H - 2E_1, H - 2E_1 - E_2$ .
- flops involving lines, conics, and one cubic, e.g. (1, 12, 13, 123): here the extremal rays are of the form  $\{H - E_1 - E_2 - E_3 - \sum_i E_i\}, H -$



$$2E_1 - E_2, H - 2E_1 - E_3, 2H - 3E_1 - 2E_2 - 2E_3 - E_i (i = 4, 5), \\ 2H - 3E_1 - 2E_2 - 2E_3 - E_4 - E_5.$$

- flops involving lines, conics, and more than one cubic, for example, (1, 12, 13, 14, 123, 124): here the extremal rays are of the form  $\{H - E_1 - E_2 - E_3 - E_4 - \sum_i E_i\}$ ,  $H - 2E_1 - E_2$ ,  $H - 2E_1 - E_3$ ,  $H - 2E_1 - E_4$ ,  $2H - 4E_1 - E_2 - E_3 - E_4$ ,  $2H - 4E_1 - E_2 - E_3 - E_4 - E_5$ ,  $2H - 3E_1 - 2E_2 - 2E_3 - E_i (i = 4, 5)$ ,  $2H - 3E_1 - 2E_2 - 2E_3 - E_4 - E_5$ ,  $3H - 5E_1 - 3E_2 - 2E_3 - 2E_4$ ,  $3H - 5E_1 - 3E_2 - 2E_3 - 2E_4 - E_5$ ,  $4H - 7E_1 - 3E_2 - 3E_3 - 3E_4$ ,  $4H - 7E_1 - 3E_2 - 3E_3 - 3E_4 - 2E_5$ .

Let us verify that these classes are indeed nef on the appropriate SQMs:

- Classes in  $\overline{A}(X)$ : we saw in Section 3 that all nef divisors on  $X$  are effective. A class of this type only appears in a nef cone where we have flopped curves which are disjoint from some representative of that class. So it remains nef on such a class.
- The class  $H - 2E_1$ : this class is represented by any hyperplane section of  $Gr(2, 5)$  which contains  $T_{p_1}Gr(2, 5)$ , so the base locus of this class is  $T_{p_1} \cap Gr(2, 5)$ , which is the locus of lines on the Grassmannian through  $p_1$ , that is the locus of curves with class  $l - l_1$ . Flopping this locus, this class becomes base-point-free, in particular nef, and remains so if we flop loci of curves which are disjoint from some representative of the class.
- Classes of the form  $H - 2E_1 - E_i$ : such a class is represented by a hyperplane section of  $Gr(2, 5)$  which contains  $T_{p_1}Gr(2, 5)$  and the point  $p_i$ . The linear space  $T_{p_1}$  and the point  $p_i$  together span a subspace  $\mathbb{P}^7 \subset \mathbb{P}^9$ , so there is a 1-parameter family of such hyperplane sections. The base locus of this family certainly contains the locus of curves with classes  $l - l_1$  and  $2l - l_1 - l_i$ : on the other hand, by Pieri's rule [7, I.5], it has class  $\sigma_1^2 = \sigma_{1,1} + \sigma_2$ , so it must be equal to this locus. Flopping these loci, this class becomes base-point-free, in particular nef, and again remains so on further flops.
- Classes of the form  $2H - 3E_1 - 2E_2 - 2E_3 - E_i$ : we can decompose such a class into effective classes in different ways, as follows:

$$H - 2E_1 - E_2 - E_4 + H - E_1 - E_2 - 2E_3, \\ H - 2E_1 - E_3 - E_4 + H - E_1 - 2E_2 - E_3.$$

(Here each class is effective since there is a unique  $\mathbb{P}^8 \subset \mathbb{P}^9$  containing the tangent space  $T_{p_1}X$  and two other general points.) These decompositions show that the base locus of this class on  $X$  is exactly the locus of curves  $l - l_1$ ,  $2l - l_1 - l_i$ ,  $3l - l_1 - l_i - l_j$ . Moreover, these hyperplanes have distinct normal directions along the curves to be flopped, so after flopping all the curves in the base locus, the proper transforms of the hyperplanes become disjoint. Therefore, this class becomes base-point-free after performing the appropriate sequence of flops.

- The class  $D = 2H - 3E_1 - 2E_2 - 2E_3 - E_4 - E_5$ : First, one can see that this class is effective by writing it in the form

$$-\frac{1}{5}K_X + (H - 2E_1 - E_2 - E_3).$$

To show this class is nef on the appropriate flop is somewhat more complicated. Applying the element  $\varphi = (E_1 - E_4) \in \text{MW}(f)$  to our class transforms it to a class of the form  $-\frac{m}{5}K_X + (H - E_1 - E_2 - E_3 - E_4)$  for some integer  $m$ . By restricting to the preimage  $S$  of a general line in  $\mathbb{P}^5$  and using invariance of intersection numbers under automorphisms, we find that  $m = 0$ : that is,  $\varphi(D) = D' = H - E_1 - E_2 - E_3 - E_4$ . We saw in Section 3 that the class  $D'$  is semi-ample on  $X$ ; replacing  $D$  and  $D'$  by appropriate multiples, we can assume it is base-point-free. Since  $\varphi$  and hence  $\varphi^{-1}$  are elements of  $\text{MW}(f) \subset \text{PsAut}(X/\mathbb{P}^5)$ , it follows that the base locus of  $D$  is a union of fiber components. If we perform any sequence of flops of fiber components and take the proper transform the same is true: that is, on any SQM of  $X$  obtained by flopping fiber components, the proper transform of  $D$  is nonnegative on any curve that is not a fiber component. On the other hand, we know that after performing the appropriate sequence of flops,  $D$  becomes relatively nef, i.e. nef on all fiber components. Hence,  $D$  is nef on the appropriate SQM of  $X$ .

- Classes of the form  $2H - 4E_1 - E_2 - E_3 - E_4$  or  $2H - 4E_1 - E_2 - E_3 - E_4 - E_5$ : the second class decomposes into a sum of effective divisors in two different ways as follows:  $(H - 2E_1 - E_2 - E_3) + (H - 2E_1 - E_4 - E_5)$ ,  $(H - 2E_1 - E_2 - E_4) + (H - 2E_1 - E_3 - E_5)$ . Since these are all distinct irreducible prime divisors, the base locus of this class has codimension 2. Moreover, one can see that the intersection of these two representatives of the class is exactly the locus to be flopped. A similar (easier) argument works for the first class.
  - Classes of the form  $3H - 5E_1 - 2E_3 - 2E_4$  or  $3H - 5E_1 - 2E_3 - 2E_4 - E_5$ : as in a previous case, there are pseudo-automorphisms of  $X$  taking these classes to  $H - E_1 - E_2$  or  $H - E_1 - E_2 - E_5$ . The same argument as before allows us to conclude that these classes are nef on the appropriate flops.
  - Classes of the form  $4H - 7E_1 - 3E_2 - 3E_3 - 3E_4$  or  $4H - 7E_1 - 3E_2 - 3E_3 - 3E_4 - 2E_5$ : again there are pseudo-automorphisms taking these classes to  $H - E_5$  and  $H - 2E_1 - E_5$  respectively. The first class is base-point-free on  $X$ , and the base locus of the second is known. We can then argue as before to conclude that these classes are nef on the appropriate flop.
- (2)  **$Z$  is the intersection of two quadrics in  $\mathbb{P}^{n+2}$ .** Here the SQMs  $X_i$  are obtained by flopping classes of the following kinds:

$$l - l_1, 2l - l_1 - l_i (i = 2, \dots, 4).$$

Here it is possible to perform any sequence of flops starting with the flop of  $l - l_1$ . The calculation of the nef cones of the resulting SQMs is formally identical to the previous case, so again we get the nef cones

- (1):  $\{H - E_1 - \sum_i E_i\}$ ,  $H - 2E_1$ .
- (1, 12):  $\{H - E_1 - E_2 - \sum_i E_i\}$ ,  $H - 2E_1$ ,  $2H - 3E_1 - 2E_2$ ,  $2H - 3E_1 - 2E_2 - E_i$ ,  $2H - 3E_1 - 2E_2 - E_3 - E_4$ .
- (1, 12, 13):  $\{H - E_1 - E_2 - E_3 - \sum_i E_i\}$ ,  $H - 2E_1$ ,  $2H - 3E_1 - 2E_2 - E_3$ ,  $2H - 3E_1 - 2E_2 - E_3 - E_4$ ,  $2H - 3E_1 - 2E_3$ ,  $3H - 5E_1 - 2E_2 - 2E_3$ ,  $3H - 5E_1 - 2E_2 - 2E_3 - E_4$ .
- (1, 12, 13, 14):  $H - E_1 - E_2 - E_3 - E_4$ ,  $H - 2E_1$ ,  $2H - 3E_1 - 2E_i - E_j - E_k$  ( $i, j, k = 2, 3, 4$ ),  $3H - 5E_1 - 2E_i - 2E_j - E_k$  ( $i, j, k = 2, 3, 4$ ),  $4H - 6E_1 - 3E_2 - 3E_3 - 3E_4$ ,  $5H - 8E_1 - 4E_i - 3E_j - 3E_k$  ( $i, j, k = 2, 3, 4$ ),  $6H - 10E_1 - 4E_i - 4E_j - 3E_k$  ( $i, j, k = 2, 3, 4$ ),  $7H - 12E_1 - 4E_2 - 4E_3 - 4E_4$ .
- Classes in  $\overline{A}(X)$  are effective; those with representatives disjoint from flopping locus remain nef on the flops.
- The class  $H - 2E_1$ : again this is represented by any hyperplane section of  $Z$  which contains  $T_{p_1}$ . Again the base locus is  $T_{p_1} \cap Z$ , which is the locus of lines through  $Z$ . The argument from the previous case goes through unchanged.
- Classes  $2H - 3E_1 - 2E_i$ : this decomposes into effective divisors as  $(H - E_1 - 2E_2) + (H - 2E_1)$  and  $(H - 2E_1 - E_2) + (H - E_1 - E_2)$ . In both decompositions the second term is a movable class, and the fixed parts of the two decompositions intersect properly, so this class is movable. Moreover, one can read off the base locus from the decompositions and see that the class becomes nef on the appropriate flop.
- Classes  $2H - 3E_1 - 2E_i - E_j$ : again we have two decompositions not sharing a divisor, namely  $(H - 2E_1 - E_3) + (H - E_1 - 2E_2)$  and  $(H - 2E_1 - E_2) + (H - E_1 - E_2 - E_3)$ . The standard argument shows these classes become nef on the appropriate flop.
- Classes  $2H - 3E_1 - 2E_i - E_j - E_k$ : there is a pseudo-automorphism taking this class to the class  $H - E_1 - E_i - E_j$ , which is base-point-free, so we can argue as before to conclude that it is nef on the appropriate flop.
- Classes  $3H - 5E_1 - 2E_i - 2E_j$ ,  $3H - 5E_1 - 2E_i - 2E_j - E_k$ : there are pseudo-automorphisms taking these classes to  $H - 2E_1$  and  $H - E_1 - E_2$ , respectively. The latter two classes are base-point-free on  $X$ , so we argue as before.
- The class  $4H - 6E_1 - 3E_2 - 3E_3 - 3E_4$ : there is a pseudo-automorphism taking this class to the base-point-free class  $H - E_1 - E_2 - E_3$ .
- The class  $5H - 8E_1 - 4E_i - 3E_j - 3E_k$ : there is a pseudo-automorphism taking this to the base-point-free class  $H - E_1 - E_2$ .
- The class  $6H - 10E_1 - 4E_i - 4E_j - 3E_k$ : there is a pseudo-automorphism taking this to the class  $2H - 3E_1 - 2E_2$ . We have already shown that the latter class is movable and identified its base locus, so we can use the same arguments as before to conclude that the original class becomes nef on the appropriate flop.
- The class  $7H - 12E_1 - 4E_2 - 4E_3 - 4E_4$ : first we apply a pseudo-automorphism to take this to the class  $3H - 4E_1 - 4E_2$ . The latter class can be split up into effective classes in two ways, as  $(2H - 3E_1 -$

$2E_2) + (H - E_1 - 2E_2)$  and  $(2H - 2E_1 - 3E_2) + (H - 2E_1 - E_2)$ . The first term in each is movable, as already shown, and the two fixed divisors intersect in codimension 2. We can then argue as before to conclude that this class becomes nef on the appropriate flop.

- (3)  **$Z$  is a cubic hypersurface in  $\mathbb{P}^{n+1}$ .** In this case the set  $\{X_i\}$  consists of a single SQM, namely  $X$  itself. The nef cone  $\overline{A(X)}$  is described in Theorem 3.1, and one reads off that the faces which intersect the interior of the movable cone are dual to fiber components.
- (4)  **$Z \rightarrow \mathbb{P}^n$  is a double cover branched over a quartic.** Again in this case the set  $\{X_i\}$  consists only of  $X$  itself, and Theorem 3.1 tells us that the faces of the cone of curves which intersect the interior of the movable cone are dual to fiber components.

This completes the proof of Claim 5.10. As explained above, this shows that for each of our examples  $X$ , there is a finite collection  $\{X_i\}$  of SQMs of  $X$  such that each nef cone  $\overline{A(X_i)}^e$  is rational polyhedral and spanned by effective divisors, and such that

$$\mathrm{PsAut}^*(X/\mathbb{P}^{n-1}) \cdot \left( \bigcup_i \overline{A(X_i)} \right) = \overline{M(X)}^e.$$

Given this, it is then straightforward to produce a precise rational polyhedral fundamental domain for the action of the larger group  $\mathrm{PsAut}^*(X, \Delta)$  on  $\overline{M(X)}^e$ , as required.  $\square$

## 6. APPENDIX: SEXTIC HYPERSURFACE IN WEIGHTED PROJECTIVE SPACE

In this appendix, we show that the Morrison–Kawamata conjecture also holds for case (5) on the list of Fano manifolds of index  $n - 1$  in Section 1. The proof here is rather simpler than in the other cases, since it turns out the the movable cone is rational polyhedral. For facts about weighted projective spaces, see for example [9].

**Proposition 6.1.** *Let  $Z$  be a general hypersurface of degree 6 in the weighted projective space  $\mathbb{P}(3, 2, 1, \dots, 1)$  of dimension  $n + 1$ . Then there is a line bundle  $H$  on  $Z$  such that  $-K_Z = (n - 1)H$  and  $H^n = 1$ . Let  $X$  be the blowup of  $Z$  in the point  $p$ , defined as the base locus of  $H$ . Then Conjecture 1.1 holds for  $X$ .*

*Proof.* The canonical class of the weighted projective space  $\mathbb{P} = \mathbb{P}(a_0, \dots, a_{n+1})$  (where  $\mathrm{gcd}(a_0, \dots, a_{n+1}) = 1$ ) is  $K_{\mathbb{P}} = \mathcal{O}(-\sum a_i)$ . So by adjunction a hypersurface  $Z$  of degree  $d$  in this space has canonical class  $K_Z = \mathcal{O}(d - \sum_{i=0}^{n+1} a_i)$ . Applying this in our case where  $n$  of the weights equal 1, we get  $-K_Z = \mathcal{O}(-6 + 3 + 2 + n) = \mathcal{O}(n - 1)$ . So  $H = \mathcal{O}(1)|_Z$ . By the weighted form of Bézout’s Theorem, we get  $H^n = \frac{1}{3 \cdot 2 \cdot 1 \cdots 1} \cdot 6 \cdot 1^n = 1$ , as claimed.

Now let  $X$  be the blowup of  $Z$  in the base locus of  $H$ . Note that  $\rho(X) = 2$ . As before we get an elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ . To prove the first statement of the cone conjecture for  $X$ , it suffices to exhibit two contractions of  $X$  which contract different curves. To see that this is enough, note that two line bundles  $L_1, L_2$  corresponding to the contractions must span the two edges of  $\overline{A(X)}$ ; moreover

$L_1$  and  $L_2$  must be semiample, in particular effective, so  $\overline{A(X)}^e = \overline{A(X)}$ , a rational polyhedral cone. The contractions we need are the blow-down map  $\pi : X \rightarrow Z$  and the elliptic fibration  $f : X \rightarrow \mathbb{P}^{n-1}$ .

The statement about the movable cone follows immediately: in fact  $\overline{M(X)}^e = \overline{A(X)}^e$ . To see this, observe that by the previous paragraph, the cone of curves  $\overline{\text{Curv}(X)}$  is spanned by  $C_1$ , the class of a line in the exceptional divisor of  $\pi$ , and  $C_2$ , the class of a fiber of  $f$ . So a non-nef divisor on  $X$  must have negative degree on either  $C_1$  or  $C_2$ . Since curves in each of these classes fill up a locus of codimension  $\leq 1$  in  $X$ , no such divisor can be movable.  $\square$

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