RESTRICTION VARIETIES AND THE RIGIDITY PROBLEM

IZZET COSKUN

Abstract. This is a survey paper based on the author's lectures given at IMPAN in December 2013. We will discuss recent results on the restriction and rigidity problems following [C2], [C3], [C4] and [C5]. The purpose of the lectures was to develop a more geometric approach to the study of isotropic flag varieties. As an illustration of the techniques, we compute the map induced in cohomology of the inclusion of $OG(k,n)$ and $SG(k,n)$ in $G(k,n)$ via an explicit sequence of rational equivalences. We also discuss applications to classifying representatives of Schubert classes.

Contents

1. Introduction

Homogeneous varieties are ubiquitous in mathematics. Especially the Grassmannians and flag varieties associated to the classical Lie groups play a central role in geometry, representation theory and combinatorics. In this paper, we survey recent developments in two important problems, the restriction and rigidity problems, in the geometry and cohomology of homogeneous varieties following [C2], [C3], [C4] and [C5]. These notes grew out of lectures I gave at IMPAN in December 2013. The lectures were organized around the following three themes:

(1) Develop a concrete geometric theory of isotopic flag varieties in the spirit of the classical theory of Grassmannians, reducing the theory to a few simple principles of quadric geometry.
(2) Construct explicit rational equivalences between subvarieties of homogeneous varieties and unions of Schubert varieties.
(3) Use explicit rational equivalences and intersection theory to study rigidity of Schubert classes.

Let V be an n-dimensional $\mathbb{C}$ vector space. Let $G(k,n)$ denote the Grassmannian parameterizing k-dimensional subspaces of V. Let Q be a nondegenerate symmetric or skew-symmetric form. A subspace W is isotropic with respect to Q if $w^TQv = 0$ for every $v, w \in W$. The isotropic Grassmannians $OG(k,n)$ (respectively, $SG(k,n)$) parameterize k-dimensional subspaces of V isotropic with respect to the symmetric (respectively, skew-symmetric) form Q. The isotropic Grassmannians naturally include in

2010 *Mathematics Subject Classification.* Primary: 14M15. Secondary: 14J60, 13C14, 13D02, 14F05.

Key words and phrases. Homogeneous varieties, Schubert varieties, quadratic forms, restriction coefficients.

During the preparation of this article the author was partially supported by the NSF CAREER grant DMS-0950951535 and the NSF grant DMS 1500031.

the Grassmannian $G(k,n)$. The restriction problem asks to compute the induced map in cohomology in terms of the Schubert bases of the corresponding Grassmannians.

The restriction problem was solved for the symmetric case in [C3] and the skew-symmetric case in [C4]. The idea is to give a sequence of explicit rational equivalences that specialize the intersection of a Schubert variety in $G(k,n)$ with $OG(k,n)$ or $SG(k,n)$ into a union of Schubert varieties in $OG(k,n)$ or $SG(k,n)$. A similar strategy led to geometric Littlewood-Richardson rules for Grassmannians (see [V1], [C1]) and two-step flag varieties (see [C1], [CV]). The goal was to understand the cohomology of $OG(k,n)$ and $SG(k,n)$ in equally concrete terms. The varieties that occur during the specialization process are called *restriction varieties* and have since found several other applications. We will introduce restriction varieties and give the solution of the restriction problem in §4.

In §5, we will focus on the rigidity problem which asks to classify the Schubert classes in the cohomology of a homogeneous variety that can be represented by subvarieties other than Schubert varieties. This problem dates back at least to Borel and Haefliger [BH] in the 1960s and has recently been answered in the cominuscule case ([R], [RT], see also [CR], [HM] and [MZ]). We will explain geometric approaches to the rigidity problem and discuss the rigidity of Schubert classes in Grassmannians and isotropic Grassmannians. Restriction varieties play an important role here as well by providing explicit deformations of Schubert varieties in some cases.

I have tried to preserve the informal nature of the lectures by focusing on examples and important special cases, referring the reader to the literature for proofs and details. I have included many exercises throughout the text. These vary considerably in difficulty. I have also included a variety of open problems. I have used the heading 'Problem' to distinguish them from exercises.

Organization of the paper. In §2, we will review the background on Grassmannians, isotropic Grassmannians and their cohomology. In §3, we will discuss 4 basic principles that govern the geometry of quadratic forms. In §4, we introduce restriction varieties and describe the solution of the restriction problem. In §5, we introduce the rigidity problem and discuss recent progress towards its solution.

Acknowledgments. I would like to thank Piotr Pragacz, Jarek Buczyński, the participants of the seminar and IMPAN for their hospitality and giving me the opportunity to deliver the lectures that gave rise to these notes. I thank the referee for many useful suggestions.

2. Preliminaries on Schubert varieties

In this section, we review basic facts concerning Schubert varieties and the cohomology of homogeneous varieties. We refer the reader to [Bri], [EH2] and [GH] for more detailed introductions to the subject.

The Grassmannian. Let V be an n-dimensional $\mathbb{C}$ vector space. The Grassmannian $G(k,n)$ parameterizes k-dimensional subspaces of V. It is a smooth, projective variety of dimension $k(n-k)$ and embeds in $\mathbb{P}^{\binom{n}{k}-1}$ under the Plücker embedding given by

$$G(k,n) \to \mathbb{P}(\wedge^k V) \quad W \mapsto [\wedge^k W].$$

The ideal of the Grassmannian under the Plücker embedding is generated by an explicit set of quadratic relations called *Plücker relations.*

The cohomology of $G(k,n)$ has an additive $\mathbb{Z}$-basis given by the classes of Schubert varieties. An *admissible partition* for $G(k,n)$ is a partition λ with k parts such that

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

It is customary to omit the parts that are equal to 0 from the notation and we will follow this custom whenever it is unambiguous. Let $|\lambda| = \sum_{i=1}^k \lambda_i$ denote the *weight* of the partition.

The *Young diagram* associated to the partition λ is an array of k left-justified rows of unit squares with λ_i squares in the ith row. Young diagrams provide a convenient pictorial representation of partitions. A partition is admissible for $G(k,n)$ if and only if its Young diagram fits in a $k \times (n-k)$ rectangle.

Given an admissible partition λ and a flag

$$F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n = V,$$

we define the Schubert variety $\Sigma_\lambda(F_\bullet)$ as

$$\{W \in G(k,n) | \dim(W \cap F_{n-k+i-\lambda_i}) \geq i, 1 \leq i \leq k\}.$$

The Schubert variety $\Sigma_\lambda(F_\bullet)$ contains a Zariski dense open subset $\Sigma^0_\lambda(F_\bullet)$ isomorphic to affine space $\mathbb{A}^{k(n-k)-|\lambda|}$ called a *Schubert cell*. In particular, the Schubert variety $\Sigma_\lambda(F_\bullet)$ is irreducible of dimension $k(n-k)-|\lambda|$. The Schubert cell $\Sigma^0_\lambda(F_\bullet)$ parameterizes

$$\{W \in G(k,n) | \dim(W \cap F_{n-k+i-\lambda_i}) = i, \dim(W \cap F_{n-k+i-\lambda_i-1}) = i-1, 1 \leq i \leq k\}.$$

Exercise 2.1. Fix an ordered basis $e_1, \dots, e_n$ of V. Let F_i be the span of $e_1, \dots, e_i$ and let $a_i = n-k+i-\lambda_i$. Show that any subspace $W \in \Sigma^0_\lambda(F_\bullet)$ admits a unique basis of the form $v_1, \dots, v_k$, where

$$v_j = e_{a_j} + \sum_{\substack{i<a_j \\ i \neq a_l}} c_{j,i} e_i.$$

Conclude that the Schubert cell is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$.

There is a natural partial order on admissible partitions defined by $\mu \geq \lambda$ if and only if $\mu_i \geq \lambda_i$ for $1 \leq i \leq k$. There is an inclusion $\Sigma_\mu(F_\bullet) \subset \Sigma_\lambda(F_\bullet)$ if and only if $\mu \geq \lambda$ under this partial ordering. The Schubert cell is given by

$$\Sigma^0_\lambda(F_\bullet) = \Sigma_\lambda(F_\bullet) - \bigcup_{\mu > \lambda} \Sigma_\mu(F_\bullet).$$

The cohomology class σ_λ of $\Sigma_\lambda(F_\bullet)$ depends only on the partition λ and not on the flag $F_\bullet$. Furthermore, the Schubert cells with respect to a fixed flag give a cell decomposition of $G(k,n)$. Consequently, as λ varies over all admissible partitions, the cohomology classes σ_λ form an additive $\mathbb{Z}$-basis for the cohomology of $G(k,n)$.

Example 2.2. The Schubert varieties in $G(2,4)$ (other than all of $G(2,4)$ and a point) are $\Sigma_1(F_\bullet), \Sigma_{1,1}(F_\bullet)$, $\Sigma_2(F_\bullet)$ and $\Sigma_{2,1}(F_\bullet)$. Interpreting $G(2,4)$ as the space of lines in $\mathbb{P}^3$, these Schubert varieties parameterize the following geometric loci:

- $\Sigma_1(F_\bullet)$ parameterizes lines that intersect a fixed line $\mathbb{P}F_2$,
- $\Sigma_{1,1}(F_\bullet)$ parameterizes lines that are contained in a fixed plane $\mathbb{P}F_3$,
- $\Sigma_2(F_\bullet)$ parameterizes lines that contain a fixed point $\mathbb{P}F_1$,
- $\Sigma_{2,1}(F_\bullet)$ parameterizes lines that contain the fixed point $\mathbb{P}F_1$ and are contained in the fixed plane $\mathbb{P}F_3$.

In the Plücker embedding, Schubert varieties are cut out on the Grassmannian by linear equations. The image of $G(2,4)$ in the Plücker embedding is the quadric hypersurface

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

If $F_\bullet$ is the flag generated by the standard basis, then $\Sigma_1(F_\bullet)$ is defined on the Grassmannian by $p_{34} = 0$. Notice that this Schubert variety is isomorphic to a cone over a quadric surface with vertex $[1:0:0:0:0:0]$ corresponding to the line $\mathbb{P}F_2$. Hence, Schubert varieties are in general singular.

Similarly, $\Sigma_{1,1}(F_\bullet)$ and $\Sigma_2(F_\bullet)$ are defined by the linear equations $p_{14} = p_{24} = p_{34} = 0$ and $p_{23} = p_{24} = p_{34} = 0$, respectively. Their intersection is $\Sigma_{2,1}(F_\bullet)$.

Exercise 2.3. Show that in the Plücker embedding the Schubert variety $\Sigma_\lambda(F_\bullet)$ is cut out on $G(k,n)$ by the linear equations $p_{j_1,j_2,\dots,j_k} = 0$, where $j_1 < j_2 < \cdots < j_k$ and there exists at least one index l such that $j_l > n - k + l - \lambda_l$.

Exercise 2.4. Show that linear subspaces on $G(k,n)$ are Schubert varieties corresponding to partitions with $\lambda_k \geq n-k-1$ or $\lambda_{k-1} = n-k$. Geometrically, these are Schubert varieties that parameterize subspaces that either contain a fixed $(k-1)$-dimensional subspace or are contained in a fixed $(k+1)$-dimensional subspace.

A Schubert variety $\Sigma_\lambda(F_\bullet)$ is smooth if and only if it is isomorphic to a sub-Grassmannian of $G(k,n)$ parameterizing k-planes that contain a fixed linear space F_s and are contained in a fixed linear space F_m for $s \leq k \leq m$. Equivalently, the complement of the Young diagram of λ in the $k \times (n-k)$ rectangle is itself a rectangle (see [BiL], [BiC], [C2], [LS] for more on singularities of Schubert varieties). More generally, we can describe the singular locus of a Schubert variety and give an explicit resolution. Express the partition λ by $(\mu_1^{i_1}, \dots, \mu_j^{i_j})$, where $\mu_1 > \mu_2 > \cdots > \mu_j$ and the part μ_l occurs with multiplicity i_l. A *singular partition* associated to λ is a partition whose Young diagram is obtained from the Young diagram of λ by adding a single hook. These are the partitions of the form $(\mu_1^{i_1}, \dots, (\mu_l+1)^{i_l+1}, \mu_{l+1}^{i_{l+1}-1}, \dots, \mu_j^{i_j})$.

Example 2.5. The singular partitions associated to $\lambda = (4,4,3,1)$ in $G(4,9)$ are the partitions $(5,5,5,1)$, $(4,4,4,4)$.

Theorem 2.6. [LS] *The singular locus of the Schubert variety $\Sigma_\lambda(F_\bullet)$ is $\cup_\mu \Sigma_\mu(F_\bullet)$, where μ varies over all singular partitions associated to λ.*

Theorem 2.6 is easy to deduce from an explicit resolution of singularities. Set $b_s = \sum_{l=1}^s i_l$. Consider the variety parameterizing the following partial flags

$$\{W^{b_1} \subset W^{b_2} \subset \cdots \subset W^{b_j} | W_{b_l} \subset F_{n-k+b_l-\lambda_l}, 1 \leq l \leq j\}.$$

This variety may be constructed as an iterated bundle of Grassmannians, hence, it is smooth. The projection to W_j defines a surjective birational map onto $\Sigma_\lambda(F_\bullet)$. The exceptional locus of this map has codimension at least 2. Therefore, the image of the exceptional locus, which is easy to identify with $\cup_\mu \Sigma_\mu(F_\bullet)$, is precisely the singular locus of $\Sigma_\lambda(F_\bullet)$ (see [C2] for further details).

Isotropic Grassmannians. Let Q be a nondegenerate symmetric or skew-symmetric form on V. A symmetric form over $\mathbb{C}$ can be diagonalized and up to conjugacy is determined by its rank. In a suitable basis, we can write Q as

$$x_1^2 + \cdots + x_n^2.$$

The rank of a skew-symmetric form is always even. Hence, if Q is a nondegenerate skew-symmetric form on V, we must necessarily have $n = 2m$. Skew-symmetric forms can be written in the normal form

$$x_1 \wedge x_2 + \cdots + x_{2m-1} \wedge x_{2m}.$$

When Q is symmetric, the equation $Q = 0$ defines a smooth, quadric hypersurface in $\mathbb{P}V$. We will frequently refer to the geometry of this hypersurface. When Q is skew-symmetric, the form is harder to visualize.

A subspace $W \subset V$ is *isotropic* with respect to Q if $v^T Q w = 0$ for every $v, w \in W$. The locus in $G(k,n)$ parameterizing subspaces W that are isotropic with respect to Q is called the *isotropic Grassmannian* and denoted by $OG(k,n)$ and $SG(k,n)$ depending on whether Q is symmetric or skew-symmetric, respectively.

Proposition 2.7. *The orthogonal Grassmannian $OG(k,n)$ is a smooth variety of dimension*

$$\dim(OG(k,n)) = \frac{k(2n-3k-1)}{2}.$$

If $n \neq 2k$, then $OG(k,n)$ is irreducible. When $n = 2k$, $OG(k,n)$ has two isomorphic connected components. The Grassmannian $SG(k,n)$ is a smooth irreducible variety of dimension

$$\dim(SG(k,n)) = \frac{k(2n-3k+1)}{2}.$$

Exercise 2.8. Consider the incidence correspondence $I = \{(v,W) | v \in W\}$, where W is an isotropic subspace and v is a vector in W. Show that I is isomorphic to an $OG(k-1,n-2)$ bundle over the quadric hypersurface $Q = 0$ in the symmetric case and to an $SG(k-1,n-2)$ bundle over $\mathbb{P}V$ in the skew-symmetric case. Deduce Proposition 2.7 by induction on k and n. See [C3] and [C4] for more details.

When $n = 2k$, an individual component of $OG(k,2k)$ is called the *spinor variety*. Two k-dimensional subspaces W_1 and W_2 belong to the same connected component if and only if $\dim(W_1 \cap W_2) = k$ modulo 2.

Example 2.9. A quadric surface in $\mathbb{P}^3$ has two one-parameter families of lines. Two lines belong to the same connected component if and only if they are disjoint. Similarly, a quadric fourfold in $\mathbb{P}^5$ has two three-parameter families of planes. Two distinct planes belong to the same connected component if and only if they intersect in a point.

Exercise 2.10. Using the fact that $G(2,4)$ is a quadric fourfold in $\mathbb{P}^5$ under the Plücker embedding show that $OG(3,6)$ parameterizes planes in $G(2,4)$. Conclude that each component of $OG(3,6)$ is isomorphic to $\mathbb{P}^3$.

The cohomology of isotropic Grassmannians has an additive $\mathbb{Z}$-basis given by the classes of Schubert varieties. We now describe the Schubert varieties in each case.

Schubert varieties in $OG(k,n)$. First, we consider $OG(k,2m+1)$. Let $0 \leq s \leq k$ be an integer. Let λ and μ denote strictly decreasing sequences of integers

$$m \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s > 0, \quad m-1 \geq \mu_{s+1} > \cdots > \mu_k \geq 0$$

such that $\lambda_i + \mu_j \neq m$ for any i,j. Schubert varieties in $OG(k,2m+1)$ are indexed by sequences $(\lambda;\mu)$. Fix an isotropic flag

$$F_\bullet : F_1 \subset \cdots \subset F_m \subset F_{m-1}^\perp \subset \cdots \subset F_1^\perp \subset V,$$

where $F_i^\perp = \{w \in V \mid w^T Q v = 0 \text{ for all } v \in F_i\}$ denotes the orthogonal to F_i under the form Q. In geometric terms, $\mathbb{P}F_i^\perp$ is the linear space everywhere tangent to the hypersurface $Q = 0$ along the linear space $\mathbb{P}F_i$. The Schubert variety $\Sigma_{\lambda;\mu}(F_\bullet)$ is the Zariski closure of the locus

$$\{\Lambda \in OG(k,2m+1) | \dim(\Lambda \cap F_{m+1-\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(\Lambda \cap F_{\mu_j}^\perp) = j \text{ for } s+1 \leq j \leq k\}.$$

The class $\sigma_{\lambda;\mu}$ of $\Sigma_{\lambda;\mu}(F_\bullet)$ is independent of the isotropic flag and depends only on the partitions.

Given λ there is an associated sequence

$$m-1 \geq \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \geq 0$$

of strictly decreasing integers defined by requiring that there does not exist an index $1 \leq i \leq s$ and an index $s+1 \leq j \leq m$ such that $\lambda_i + \tilde{\lambda}_j = m$. In other words, $\tilde{\lambda}$ is the sequence obtained from $m-1, m-2, \ldots, 1, 0$ by omitting the integers $m-\lambda_s, \ldots, m-\lambda_1$. The sequence μ is a subsequence of $\tilde{\lambda}$, hence we have that $\mu_l = \tilde{\lambda}_{i_l}$. Define the discrepancy of the pair (λ,μ) by

$$\operatorname{dis}(\lambda,\mu) = (m-k)s + \sum_{l=s+1}^{k} (m-k+l-i_l).$$

Then the codimension of $\Sigma_{\lambda;\mu}(F_\bullet)$ in $OG(k, 2m+1)$ is

$$\sum_{i=1}^{s} \lambda_i + \operatorname{dis}(\lambda, \mu).$$

When $k = m$, μ is uniquely determined by λ and authors often omit μ in this case. We will not follow this convention.

Example 2.11. The Schubert varieties in $OG(2, 5)$ (other than the whole space and a point) are $\Sigma_{1;0}$ and $\Sigma_{2;1}$. They parameterize lines on a quadric threefold that intersect a fixed line $\mathbb{P}F_2$ and lines that contain a point $\mathbb{P}F_1$ (and are automatically contained in the tangent hyperplane $\mathbb{P}F_1^\perp$ at that point), respectively.

Exercise 2.12. By induction on k, calculate the dimension of $\Sigma_{\lambda;\mu} \subset OG(k, n)$ and verify that its codimension is $\sum_{i=1}^{s} \lambda_i + \operatorname{dis}(\lambda, \mu)$.

Next, we consider $OG(k, 2m)$. In this case, m-dimensional linear spaces form two connected components. We can specify incidence conditions with respect to m-dimensional linear spaces in either component. Our notation needs to reflect this difference. We denote the linear spaces in one connected component by F_m and the linear spaces in the other connected component by $F_{m-1}^\perp$. Technically, the intersection of a linear space tangent along F_{m-1} intersects $Q = 0$ in the union of two half-dimensional linear spaces belonging to different connected components. However, this abuse will simplify the notation greatly. Let $0 \le s \le k$ be an integer and let λ and μ denote sequences of strictly decreasing integers

$$m - 1 \ge \lambda_1 > \cdots > \lambda_s \ge 0 \quad m - 1 \ge \mu_{s+1} > \cdots > \mu_k \ge 0$$

such that $\lambda_i + \mu_j \ne m - 1$ for any i, j. Schubert varieties in $OG(k, 2m)$ are indexed by such sequences $(\lambda; \mu)$. In order to isolate the Schubert varieties in the spinor variety, in addition we need to assume that when $k = m$ and m is even (respectively, odd), s is even (respectively, odd). The Schubert variety $\Sigma_{\lambda;\mu}(F_\bullet)$ is defined as the Zariski closure of the locus

$$\{\Lambda \in OG(k, 2m) \mid \dim(\Lambda \cap F_{m-\lambda_i}) = i \text{ for } 1 \le i \le s, \dim(\Lambda \cap F_{\mu_j}^\perp) = j \text{ for } s < j \le k\}.$$

Example 2.13. In $OG(2, 6)$, the Schubert varieties (other than the whole space or a point) are as follows:

1. The Schubert varieties $\Sigma_{;2,0}, \Sigma_{0;0}$ parameterize lines on a quadric fourfold that intersect a fixed plane. The class depends on the type of plane.
2. The Schubert varieties $\Sigma_{;2,1}, \Sigma_{0;1}$ parameterize lines that are contained in the linear space $\mathbb{P}F_1^\perp$ and intersect a plane.
3. The Schubert variety $\Sigma_{1;0}$ parameterizes lines that intersect a fixed line.
4. The Schubert varieties $\Sigma_{1;2}$, $\Sigma_{1,0;}$ parameterize lines that are contained in a fixed plane.
5. The Schubert varieties $\Sigma_{2,0;}$, $\Sigma_{2;2}$ parameterize lines that are contained in a plane and contain a fixed point $\mathbb{P}F_1$.

As in the previous case, given λ, there is a corresponding partition

$$m - 1 \ge \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \ge 0$$

satisfying the condition that there does not exist indices $1 \le i \le s$ and $s + 1 \le j \le m$ such that $\lambda_i + \tilde{\lambda}_j = m - 1$. The sequence $\tilde{\lambda}$ is obtained from

$$m - 1, m - 2, \dots, 1, 0$$

by removing the integers $m - 1 - \lambda_s, \dots, m - 1 - \lambda_1$. The partition (λ, μ) is a subpartition of $(\lambda, \tilde{\lambda})$ of total length m. Hence, $\mu_l = \tilde{\lambda}_{i_l}$. Define the discrepancy $\operatorname{dis}(\lambda, \mu) = s(m - k) + \sum_{l=s+1}^{k} (m - k + l - i_l)$. The codimension of $\Sigma_{\lambda;\mu}$ in $OG(k, 2m)$ is given by

$$\sum_{i=1}^{s} \lambda_i + \operatorname{dis}(\lambda, \mu).$$

Schubert varieties in $SG(k,n)$**.** Since Q is a nondegenerate skew-symmetric form, we must have $n = 2m$. Let $0 \leq s \leq k$ be an integer. Let λ and μ be strictly decreasing sequences

$$m \geq \lambda_1 > \cdots > \lambda_s > 0 \quad m > \mu_{s+1} > \cdots > \mu_k \geq 0$$

such that $\lambda_i + \mu_j \neq m$ for any i, j. Let $F_\bullet$ be an isotropic flag

$$F_1 \subset \cdots \subset F_m \subset F_{m-1}^\perp \subset \cdots \subset F_1^\perp \subset V.$$

Then the Schubert variety $\Sigma_{\lambda,\mu}(F_\bullet)$ is the Zariski closure of the locus

$$\{\Lambda \in SG(k,n) | \dim(\Lambda \cap F_{m+1-\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(\Lambda \cap F_{\mu_j}^\perp) = j \text{ for } s < j \leq k\}.$$

The discrepancy is defined as in $OG(k, 2m+1)$ and the codimension of the Schubert variety in $SG(k,n)$ is given by

$$\sum_{i=1}^{s} \lambda_i + \mathrm{dis}(\lambda, \mu).$$

Cell decompositions. In all three cases, there is a partial ordering on the sequences $(\lambda; \mu)$. For $SG(k,n)$ and $OG(k, 2m+1)$, a sequence $(\lambda'; \mu') \geq (\lambda; \mu)$ if and only if $s' \geq s$, $\lambda'_i \geq \lambda_i$ for $1 \leq i \leq s$ and $\mu'_j \geq \mu_j$ for $s' < j \leq k$. For $OG(k, 2m)$ in addition we need to require that if $\mu_{s+1} = m-1$ and $s' = s+1$, then $\lambda'_{s+1} > 0$. A Schubert variety $\Sigma_{\lambda';\mu'}(F_\bullet) \subset \Sigma_{\lambda;\mu}(F_\bullet)$ if and only if $(\lambda'; \mu') \geq (\lambda; \mu)$. The complement

$$\Sigma_{\lambda;\mu}(F_\bullet) - \bigcup_{(\lambda';\mu') > (\lambda;\mu)} \Sigma_{\lambda';\mu'}(F_\bullet)$$

is called a *Schubert cell* and is isomorphic to affine space. The Schubert cells give a cell decomposition of $OG(k,n)$ and $SG(k,n)$. Consequently, Schubert classes give an additive $\mathbb{Z}$-basis of the cohomology of $OG(k,n)$ and $SG(k,n)$.

Exercise 2.14. When $n = 2m+1$, after a change of variables, write Q as $x_1x_2 + \cdots + x_{2m-1}x_{2m} + x_{2m+1}^2$. Show that the complement of the locus $x_1 = x_3 = \cdots = x_{2i+1} = 0$ in the locus $x_1 = x_3 = \cdots = x_{2i-1} = 0$ is isomorphic to affine space. Generalize this to show that Schubert cells give a cell decomposition of $OG(k, 2m+1)$. Further generalize to $OG(k, 2m)$ and $SG(k,n)$.

Flag varieties. Let $k_1 < k_2 < \cdots < k_t < n$ be a sequence of t increasing positive integers. For notational convenience, set $k_0 = 0$ and $k_{t+1} = n$. The flag variety $F(k_1, \ldots, k_t; n)$ parameterizes partial flags

$$W_1 \subset W_2 \subset \cdots \subset W_t \subset V,$$

where W_i has dimension k_i.

For any set of subindices $i_1, \ldots, i_l$ of $\{1, \ldots, t\}$, the flag variety admits a forgetful morphism

$$\pi_{i_1,\ldots,i_l} : F(k_1, \ldots, k_t; n) \to F(k_{i_1}, \ldots, k_{i_l}; n) \quad (W_1, \ldots, W_t) \mapsto (W_{i_1}, \ldots, W_{i_l}).$$

In particular, the projection $\pi_t : F(k_1, \ldots, k_t; n) \to G(k_t, n)$ realizes the partial flag variety $F(k_1, \ldots, k_t; n)$ as a $F(k_1, \ldots, k_{t-1}; k_t)$ bundle over the Grassmannian $G(k_t, n)$. By induction,

$$\dim(F(k_1, \ldots, k_t; n)) = \sum_{i=1}^{t} k_i(k_{i+1} - k_i).$$

The cohomology of a partial flag variety is generated by the classes of Schubert varieties. We use a notation for Schubert varieties in $F(k_1, \ldots, k_t; n)$ which is well-adapted for the forgetful morphism π_t. A *coloring* c associated to such a sequence is a sequence of k_t integers $c_1, \ldots, c_{k_t}$ such that exactly $k_i - k_{i-1}$ of the integers in the sequence are equal to i. Let λ be an admissible partition for $G(k_t, n)$ and c a coloring for the sequence $k_1, \ldots, k_t$. Schubert varieties in $F(k_1, \ldots, k_t; n)$ are parameterized by colored partitions $(\lambda; c)$. Let $F_\bullet$ be a complete flag, then the Schubert variety $\Sigma_{\lambda;c}(F_\bullet)$ is the Zariski closure of the locus

$$\{W_1 \subset \cdots \subset W_t \in F(k_1, \dots, k_t; n) | \dim(W_j \cap F_{n-k_t+i-\lambda_i}) = \#\{c_l | c_l \leq j, l \leq i\}\}.$$

For $1 \leq u < t$, define *the codimension of the color* u $\mathrm{cdim}(u)$ by

$$\mathrm{cdim}(u) = \sum_{1 \leq i \leq k_t, \ c_i \leq u} \#\{j > i | c_j = u + 1\}.$$

Define the codimension of the coloring $\mathrm{cdim}(c)$ by

$$\mathrm{cdim}(c) = \sum_{u=1}^{t-1} \mathrm{cdim}(u).$$

Exercise 2.15. By analyzing the projection π_t show that the codimension of the Schubert variety $\Sigma_{\lambda;c}$ in the flag variety is given by $|\lambda| + \mathrm{cdim}(c)$.

Exercise 2.16. Show that $F(1,3;4)$ is isomorphic to $OG(2,6)$. Describe the correspondence between the Schubert varieties under this isomorphism.

Similarly, the isotropic partial flag varieties $OF(k_1, \dots, k_t; n)$ and $SF(k_1, \dots, k_t; n)$ parameterize isotropic partial flags

$$W_1 \subset \cdots \subset W_t \subset V,$$

where W_i is an isotropic subspace of dimension k_i with respect to a symmetric, respectively, skew-symmetric, non-degenerate quadratic form. The forgetful morphisms $\pi_t : OF(k_1, \dots, k_t; n) \to OG(k_t, n)$ and $\pi_t : SF(k_1, \dots, k_t; n) \to SG(k_t, n)$ realize these varieties as $F(k_1, \dots, k_{t-1}, ; k_t)$-bundles over the isotropic Grassmannians. Consequently, we have

$$\dim(OF(k_1, \dots, k_t; n)) = \frac{k_t(2n - 3k_t - 1)}{2} + \sum_{i=1}^{t-1} k_i(k_{i+1} - k_i),$$

$$\dim(SF(k_1, \dots, k_t; n)) = \frac{k_t(2n - 3k_t + 1)}{2} + \sum_{i=1}^{t-1} k_i(k_{i+1} - k_i).$$

Schubert varieties in these varieties can be parameterized by triples $(\lambda; \mu; c)$, where $(\lambda; \mu)$ is a pair of partitions admissible for $OG(k_t, n)$ or $SG(k_t, n)$ and c is a coloring of the sequence $k_1, \dots, k_t$. The Schubert variety $\Sigma_{\lambda;\mu;c}(F_\bullet)$ with respect to the isotropic flag $F_\bullet$ is the Zariski closure of the locus of flags satisfying the following incidence conditions

$$\dim(W_h \cap F_{\lceil \frac{n}{2} \rceil - \lambda_i}) = \#\{c_l | c_l \leq h, l \leq i\} \quad \dim(W_h \cap F_{\mu_j}^\perp) = \#\{c_l | c_l \leq h, l \leq j\}.$$

As before, the codimension of the Schubert variety in the flag variety is given by $\sum_{i=1}^s \lambda_i + \mathrm{dis}(\lambda, \mu) + \mathrm{cdim}(c)$.

Dual Schubert classes. Every Schubert class in a rational homogeneous variety has a dual Schubert class. Two Schubert classes of complementary dimension have intersection number equal to one if and only if they are dual Schubert classes. Otherwise, their intersection is zero. For the reader's convenience, we will explicitly describe the dual Schubert class in every classical case.

The dual classes for $G(k,n)$ have the following description. Given a partition λ, the dual partition λ^* is defined by $\lambda_i^* = n - k - \lambda_{k-i+1}$. The dual partition is the partition corresponding to the 180 degree rotation of the complement of λ in the $k \times (n-k)$ rectangle. Then the dual of σ_λ is σ_{λ^*}.

Example 2.17. The classes $\sigma_{3,3,2,1}$ and $\sigma_{4,3,2,2}$ are dual in $G(4,9)$.

Similarly, in $OG(k, 2m+1)$ or $SG(k, 2m)$ the dual of the Schubert class $\sigma_{\lambda;\mu}$ is given by $\sigma_{\lambda*,\mu^*}$, where $\lambda_i^* = m - \mu_{k-i+1}$ and $\mu_j^* = m - \lambda_{k-j+1}$.

Example 2.18. The classes $\sigma_{2;0}$ and $\sigma_{3,1}$ are duals of each other in $OG(2, 7)$.

Exercise 2.19. Describe the dual Schubert classes for $OG(k, 2m)$. (Hint: As long as m is even; or $\lambda_s \neq 0$ and $\mu_{s+1} \neq m-1$, you can define $\lambda_i^* = m - 1 - \mu_{k-i+1}$ and $\mu_j^* = m - 1 - \lambda_{k-j+1}$. When m is odd, you need to modify the half dimensional linear spaces.)

Given a coloring c for $k_1, \dots, k_t$, the dual coloring c^* has $c_i^* = c_{k_t-i+1}$. In other words, the dual coloring reverses the order of c. Given a Schubert class $\sigma_{\lambda;c}$ in a partial flag variety or a Schubert class $\sigma_{\lambda;\mu;c}$ in an isotropic flag variety, the dual class is given by $\sigma_{\lambda^*;c^*}$ and $\sigma_{\lambda^*;\mu^*;c^*}$, respectively.

Proposition 2.20. *Let Y be a subvariety of a rational homogeneous variety. Then the cohomology class of Y is a nonnegative linear combination of Schubert classes. In particular, for rational homogeneous varieties, the cones of effective and nef cycles coincide and are generated by Schubert classes.*

Proof. Since Schubert classes give an additive basis of the cohomology, we can express the class of Y as a linear combination of Schubert classes $[Y] = \sum a_\lambda \sigma_\lambda$. For each Schubert class σ_λ, we can pair $[Y]$ by the dual Schubert class σ_λ^*. By Kleiman's Transversality Theorem [K], a general translate of a Schubert variety with class σ_λ^* intersects Y transversely in finitely many points. Hence, $\sigma_\lambda^* \cdot [Y] = a_\lambda \geq 0$. ☐

As in the proof of Proposition 2.20, we can compute the classes of subvarieties of rational homogeneous varieties by pairing with dual Schubert classes.

Exercise 2.21. Compute the cohomology class of $OG(k, n)$ and $SG(k, n)$ in $G(k, n)$.

Exercise 2.22. Compute the intersection products in the cohomology of $G(2, 4)$ and $G(2, 5)$.

Exercise 2.23. Compute the intersection products in the cohomology of $OG(2, 5)$, $OG(3, 6)$ and $OG(2, 6)$.

3. The Golden Rules of quadric geometry

There are four general principles that govern the geometry of isotropic linear spaces with respect to a quadratic form. In this section, we will recall these principles. In the next section, these principles will dictate the geometry of restriction varieties and their specializations.

We begin by discussing symmetric forms. Let Q be a nondegenerate symmetric form. Let Q_d^r denote a d-dimensional linear space such that the restriction of Q is a quadratic form of corank r, equivalently rank $d - r$. Up to conjugacy, the form in Q_d^r is $x_1^2 + \cdots + x_{d-r}^2$. The singular locus of the corresponding quadric hypersurface in $\mathbb{P}Q_d^r$ is defined by $x_1 = \cdots = x_{d-r} = 0$. This is the *kernel* or *vertex* K of the quadratic form restricted to Q_d^r.

The corank bound. Let $Q_{d_2}^{r_2} \subset Q_{d_1}^{r_1}$ be two linear spaces such that $K_1 \subset K_2$. Then

$$r_2 - r_1 \leq d_2 - d_1.$$

In particular, the corank r of a linear section of a nondegenerate quadric hypersurface is bounded by its codimension $n - d$.

Exercise 3.1. Verify that the tangent hyperplane section $T_pQ \cap Q$ to a smooth quadric hypersurface Q is singular only along p. More generally, let Q be a quadric hypersurface of corank r and vertex W. Let p be a smooth point on Q. Then the tangent hyperplane section $T_pQ \cap Q$ is singular along the span of p and W. Using this verify the corank bound.

The linear space bound. The largest dimensional isotropic subspace contained in Q^r_d has dimension $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an isotropic subspace of dimension j intersects the kernel of the quadratic form in Q^r_d in a subspace of dimension at least $\max(0, j - \lfloor \frac{d-r}{2} \rfloor)$.

Exercise 3.2. By taking $x_{2j-1} = \sqrt{-1}\, x_{2j}$ for $j = 1, \dots, \lfloor \frac{d-r}{2} \rfloor$ and $x_{d-r} = 0$ if $d-r$ is odd, show that there are isotropic subspaces of dimension $\lfloor \frac{d+r}{2} \rfloor$ in Q^r_d. Either by induction on dimension or by the Lefschetz hyperplane theorem, show that a smooth quadric hypersurface does not contain any linear spaces of more than half the dimension. Let Q^0_{d-r} be the linear space of Q^r_d defined by $x_{d-r+1} = \cdots = x_d = 0$. Notice that the quadratic form restricted to Q^0_{d-r} is nondegenerate, therefore, the largest dimensional isotropic subspace in Q^0_{d-r} has dimension $\lfloor \frac{d-r}{2} \rfloor$. Deduce the linear space bound.

Irreducibility. The quadratic form restricted to Q^{d-2}_d is a product of two linear forms, which define two isotropic subspaces of dimension $d-1$. If $n = 2k$, then the two linear spaces belong to distinct connected components of $OG(k, 2k)$. The quadratic form restricted to Q^{d-1}_d is the square of a linear form.

The variation of tangent spaces. Assume that the kernel of the quadratic form restricted to Q^r_d intersects a linear space L in codimension j. Then the image of the Gauss map restricted to the points of L along which the quadric hypersurface in $\mathbb{P}Q^r_d$ is smooth has dimension at most $j-1$. In other words, the tangent spaces to the quadric hypersurface in $\mathbb{P}Q^r_d$ along the points of L where it is smooth vary in a $(j-1)$-dimensional family.

Exercise 3.3. Consider the quadric hypersurface $xy - z^2$ in $\mathbb{P}^3$. Show that the tangent space to the surface along the line $x - az = y - \frac{1}{a}z = 0$ is constant. Generalize the calculation to arbitrary dimension and rank to deduce the principle of variation of tangent spaces.

There are similar basic principles for skew-symmetric forms. We explain them next. Let Q^r_d denote a d-dimensional subspace of V such that the restriction of the skew-form Q to Q^r_d has corank r. Let $\mathrm{Ker}(Q^r_d)$ denote the kernel of the restriction of Q to Q^r_d.

Evenness of rank. The rank of a skew-symmetric form is even. Consequently, $d-r$ is even.

In particular, after a change of variables, we can write the restriction of the form Q to Q^r_d as

$$x_1 \wedge x_2 + \cdots + x_{d-r-1} \wedge x_{d-r}.$$

in these coordinates, the kernel is given by $x_1 = \cdots = x_{d-r} = 0$.

The corank bound. Let $Q^{r_2}_{d_2} \subset Q^{r_1}_{d_1}$ and let $r_1' = \dim(\mathrm{Ker}(Q^{r_1}_{d_1}) \cap Q^{r_2}_{d_2})$. Then $r_2 - r_1' \leq d_1 - d_2$. In particular, $d + r \leq n$ for any Q^r_d.

The linear space bound. The dimension of an isotropic subspace of Q^r_d is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. A j-dimensional isotropic linear subspace of Q^r_d satisfies $\dim(L \cap \mathrm{Ker}(Q^r_d)) \geq j - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be a j-dimensional isotropic subspace such that $\dim(L \cap \mathrm{Ker}(Q^r_d)) = j-1$. Then an isotropic linear subspace that intersects $L - \mathrm{Ker}(Q^r_d)$ is contained in $L^{\perp}$.

Exercise 3.4. Prove the corank bound, the linear space bound and the kernel bound following a similar strategy to the symmetric case.

4. Restriction problem

Let G be a group, H a subgroup of G and V an irreducible representation of G. Then V is also a representation of H. The *restriction problem* asks for the decomposition of V into irreducible representations of H. In this section, we study geometric analogues of the restriction problem. We can formulate the main problem as follows.

Problem 4.1. *Let $f : X \to Y$ be a morphism between two rational, projective homogeneous varieties. Compute the map induced in cohomology $f^* : H^*(Y,\mathbb{Z}) \to H^*(X,\mathbb{Z})$. In particular, Schubert classes provide additive $\mathbb{Z}$-bases for both $H^*(Y,\mathbb{Z})$ and $H^*(X,\mathbb{Z})$. Given a Schubert class $\sigma_\lambda \in H^*(Y,\mathbb{Z})$,*

$$f^*(\sigma_\lambda) = \sum a_\mu^\lambda \sigma_\mu \in H^*(X,\mathbb{Z}),$$

where a_μ^λ are nonnegative, integer coefficients. Find positive, geometric algorithms for computing the coefficients a_μ^λ.

Problem 4.1 encompasses many important special cases. For example, when $Y = X \times X$ and f is the embedding of X as the diagonal, the problem specializes to finding the structure constants of the cohomology of X. In this section, we will study the case when f is the inclusion $i : OG(k,n) \to G(k,n)$ or the inclusion $i : SG(k,n) \to G(k,n)$. The structure constants of the map induced by these inclusions are called *restriction coefficients* and the *restriction problem* asks for an algorithm to compute them. We devote the rest of the section to answering this problem. We refer the reader to [FP], [Pr1] and [Pr2] for alternative approaches. The reader who would like to explore the connection between the geometric and representation theoretic problems may consult [BS] and [P].

The basic strategy is to give explicit rational equivalences between a subvariety of $OG(k,n)$ or $SG(k,n)$ and a union of Schubert classes. This strategy has been very fruitful in obtaining geometric Littlewood-Richardson rules for Grassmannians and two-step flag varieties (see [C1], [CV] and [V1]) and for computing Gromov-Witten invariants and quantum cohomology (see [C1], [C7], [C8] and [V2]). We will not discuss these rules here and refer the reader to the literature.

We start with two fundamental examples which capture the main aspects of the algorithm for computing restriction coefficients.

Example 4.2. Consider a general hyperplane section $H \cap Q$ of a quadric in $\mathbb{P}^3$. This is a smooth conic curve and is not a Schubert variety in $OG(1,4)$. Vary H in a one-parameter family H_t until H_0 becomes tangent to Q. Then $H_0 \cap Q = L \cup L'$ is a union of two lines one from each ruling. These are Schubert varieties in $OG(1,4)$. The class of the conic $H \cap Q$ is equal to the sum of the classes of L and L'.

Example 4.3. Consider the locus of lines in a quadric Q in $\mathbb{P}^6$ such that the lines intersect a fixed line L and are contained in a general hyperplane section $H \cap Q$ containing L. This is not a Schubert variety in $OG(2,7)$. Take a one-parameter family of hyperplanes H_t such that H_0 is tangent to Q along a point $p \in L$. Consider the varieties of lines that are contained in H_t and intersect L. In the limit $t = 0$, the lines may contain p. In that case, the lines are contained in H_0 and contain p. This is a Schubert variety with class $\sigma_{3;1}$. Otherwise, the lines intersect L away from p. In that case the lines must be contained in $L^\perp$ by the principle of variation of tangent spaces. This locus is a Schubert variety with class $\sigma_{2;2}$. We conclude that the original variety has class $\sigma_{3;1} + \sigma_{2;2}$.

Exercise 4.4. Verify the details of the previous example.

4.1. **The symmetric case.**

We will compute the restriction coefficients by a sequence of specializations. We begin with the intersection of a general Schubert variety $\Sigma_\lambda(F_\bullet)$ with $OG(k,n)$. Initially, every linear space in the flag $F_\bullet$ defining Σ_λ is transverse to the quadric Q. We will successively change the flag by making the linear spaces less and less transverse to Q until the flag becomes isotropic. In the process, the variety will break into a union of Schubert varieties for $OG(k,n)$. Restriction varieties are the components of the limits that occur during the process. The order of specialization and the limits are dictated by the principles discussed in the previous section. We will now introduce restriction varieties and discuss their basic geometric properties.

Notation 4.5. Let Q be a symmetric nondegenerate quadratic form on an n-dimensional vector space V. Let L_{n_i} denote an isotropic subspace of dimension n_i. When $2m = n$, let L_m and L'_m denote isotropic subspaces in different connected components. Let Q^r_d denote a d-dimensional subspace of V such that the restriction of Q to Q^r_d has corank r. Let K denote the kernel of the restriction of Q to $Q^{r_j}_{d_j}$. The dimension of K is r.

Definition 4.6. An *orthogonal sequence*

$$(L_\bullet, Q_\bullet) = L_{n_1} \subset \cdots L_{n_s} \subset Q^{r_{k-s}}_{d_{k-s}} \subset \cdots \subset Q^{r_1}_{d_1}$$

is a sequence of isotropic subspaces L_{n_i} of dimension n_i (or possibly L'_{n_s} if $2n_s = n$) and linear spaces $Q^{r_j}_{d_j}$ of dimension d_j such that the restriction of Q to $Q^{r_j}_{d_j}$ has kernel K_j and corank r_j satisfying the following conditions.

(1) The singular loci of $Q^{r_j}_{d_j}$ are ordered by inclusion $K_j \subset K_{j+1}$ for $1 \leq j < k-s$.
(2) For every $1 \leq i \leq s$ and $1 \leq j \leq k-s$, $\dim(K_j \cap L_{n_i}) = \min(n_i, r_j)$.
(3) If $r_j = r_{j+1} > 0$ for some j, either $r_1 = r_j$ and $n_{r_1} = r_1$; or $r_l - r_{l-1} = d_{l-1} - d_l$ for every $l \geq j$ and $d_j - d_{j+1} = 1$.

Remark 4.7. The fact that L_{n_i} is an isotopic subspace contained in $Q^{r_j}_{d_j}$ implies certain inequalities among n_i, d_j and r_j. The corank bound implies that

$$r_j + d_j \leq r_{j-1} + d_{j-1} \leq n \quad \text{for } 1 < j \leq k-s.$$

The linear space bound implies that

$$2n_s \leq d_{k-s} + r_{k-s}.$$

We will always implicitly assume these inequalities. The first two conditions say that the kernels K_j are in as special a position as possible. They are totally ordered by inclusion and they have the maximal possible intersection with the isotropic subspaces in the sequence. The third condition is a consequence of the order of specialization. We will not specialize a linear space $Q^{r_j}_{d_j}$ until the kernels K_l for $l > j$ are as large as possible given the corank bound.

Notation 4.8. Let x_j denote the number of isotropic subspaces in the sequence that are contained in K_j, equivalently, x_j is the number of n_i for $1 \leq i \leq s$ such that $n_i \leq r_j$.

Definition 4.9. We call the sequence $(L_\bullet, Q_\bullet)$ *admissible* if it satisfies the following conditions.

(A1) For every $1 \leq j \leq k-s$, we have

$$x_j \geq k - j + 1 - \frac{d_j - r_j}{2}.$$

(A2) There does not exist integers i, j such that $r_j + 1 = n_i$.
(A3) We have $r_{k-s} \leq d_{k-s} - 3$.

Definition 4.10. The *restriction variety* $V(L_\bullet, Q_\bullet)$ associated to an admissible orthogonal sequence $(L_\bullet, Q_\bullet)$ is the subvariety of $OG(k,n)$ defined as the Zariski closure of the following locus

$$\begin{aligned} V(L_\bullet, Q_\bullet)^0 = &\{W \in OG(k,n) \mid \dim(W \cap L_{n_i}) = i, \dim(W \cap Q^{r_j}_{d_j}) = k - j + 1, \\ &\dim(W \cap K_j) = x_j\} \end{aligned}$$

Remark 4.11. Condition (A1) is a consequence of the linear space bound. The isotropic subspace of dimension $k-j+1$ has to intersect the kernel K_j in a subspace of dimension at least $k-j+1-\frac{d_j-r_j}{2}$. Condition (A3) guarantees that the restriction of Q to $Q_{d_{k-s}}^{r_{k-s}}$ is not a nonreduced isotropic space or the union of two isotopic subspaces. Condition (A2) is a consequence of variations of tangent spaces. If $r_j+1=n_i$, then any linear space intersecting L_{n_i} away from K_j is contained in $L_{n_i}^{\perp}$.

Exercise 4.12. The restriction varieties do not need to be irreducible. Show that $V(Q_3^0 \subset Q_4^0)$ in $OG(2,5)$ has two connected components.

We introduced the notion of a marked sequence to describe the irreducible components of restriction varieties.

Definition 4.13. Let $(L_\bullet, Q_\bullet)$ be an admissible orthogonal sequence. An index j such that

$$x_j = k-j+1-\frac{d_j-r_j}{2}$$

is called a *special index*. A marking $m_\bullet$ of $(L_\bullet, Q_\bullet)$ for each special index j designates one of the irreducible components of the $(\frac{d_j+r_j}{2})$-dimensional isotropic subspaces of $Q_{d_j}^{r_j}$ as even and the other one as odd such that:

- If $d_{j_1}+r_{j_1}=d_{j_2}+r_{j_2}$ for two special indices $j_1<j_2$, then the component containing a linear space W is assigned the same parity for both indices.
- If $2n_s=d_j+r_j$ for a special index j, the component which contains L_{n_s} is assigned the parity of s.
- If $n=2k$, then the component containing L_k is assigned the parity that characterizes the spinor variety containing L_k.

A *marked restriction variety* $V(L_\bullet, Q_\bullet, m_\bullet)$ is the subvariety of $V(L_\bullet, Q_\bullet)$ parameterizing k-dimensional isotropic subspaces W such that for each special index j, W intersects isotropic subspaces of dimension $\frac{d_j+r_j}{2}$ designated even (respectively, odd) by $m_\bullet$ in a subspace of even (respectively, odd) dimension.

Exercise 4.14. Show that for the restriction variety in Exercise 4.12 there are two possible markings and these distinguish the two irreducible components. Show that for $V(Q_4^2 \subset Q_5^1 \subset Q_6^2 \subset Q_{10}^0)$ there are also two possible markings distinguishing the two components.

Proposition 4.15. *The marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ is an irreducible subvariety of $OG(k,n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet, m_\bullet)) = \sum_{i=1}^{s}(n_i-i) + \sum_{j=1}^{k-s}(d_j+x_j-2s-2j)$$

Sketch of proof. The dimension and the irreducibility can be checked by induction on k. When $k=1$, $V(L_\bullet, Q_\bullet, m_\bullet)$ is isomorphic to either a projective space of dimension n_1-1 or an irreducible quadric surface of dimension d_1-2. In this case, the proposition is true. Suppose that the proposition holds for $k<k_0$. Suppose $V(L_\bullet, Q_\bullet, m_\bullet)$ is a marked restriction variety in $OG(k_0,n)$. If $s=k_0$, then it is isomorphic to a Schubert variety in $G(k_0, L_{n_s})$. Consequently, it is irreducible of the claimed dimension. Otherwise, we can define a new sequence $(L'_\bullet, Q'_\bullet, m'_\bullet)$ by omitting $Q_{d_1}^{r_1}$. The reader can easily check that the resulting sequence is still an admissible orthogonal sequence. Sending $W \in V^0(L_\bullet, Q_\bullet, m_\bullet)$ to $W \cap Q_{d_2}^{r_2}$ defines a morphism onto $V^0(L'_\bullet, Q'_\bullet, m'_\bullet)$, where the generic fiber over W' consists of choosing $W' \subset W \subset Q_{d_1}^{r_1}$. The proposition follows from this description and the theorem on the dimension of fibers. $\square$

Remark 4.16. The dimension of a marked restriction variety does not depend on the marking.

Example 4.17. Set $u = \lceil \frac{n}{2} \rceil$. If the sequence $(L_\bullet, Q_\bullet)$ satisfies $d_j + r_j = n$ for $1 \leq j \leq k-s$, then $(L_\bullet, Q_\bullet)$ is an isotropic flag. In that case the restriction variety $V(L_\bullet, Q_\bullet)$ is simply the Schubert variety $\Sigma_{u-n_1,\dots,u-n_s;r_{k-s},\dots,r_1}$ defined with respect to this isotropic flag.

Example 4.18. The intersection of a general partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$ leads to the sequence $Q^0_{a_1} \subset \cdots \subset Q^0_{a_k}$. If $a_i < 2i-1$ for some i, then the intersection is empty. If $a_i \geq 2i$ for every $1 \leq i \leq k$, then the intersection is nonempty of the expected dimension.

Restriction varieties are in general singular. S. Adalı[Ad] has given an explicit resolution and described their singular loci.

Quadric diagrams. It is convenient to record admissible sequences in terms of combinatorial objects called quadric diagrams. Consider a sequence of n integers written from left to right. The place of the integer is its order in the sequence counted from the left. We say that a bracket or a brace is in position i if i of the integers are to the left of the bracket or brace.

Definition 4.19. An *orthogonal sequence of brackets and braces* of type (k,n) is a sequence of n natural numbers, $s \leq k$ right brackets] and $k-s$ right braces } such that

- Every bracket or brace occupies a positive position and each position is occupied by at most one bracket or brace. If $n = 2m$, a bracket at position m may be decorated by a prime $]'$.
- Every number i in the sequence satisfies $0 \leq i \leq k-s$. The positive integers are nondecreasing from left to right and are to the left of every zero in the sequence.
- Every bracket is to the left of every brace.

Notation 4.20. By convention, the brackets are indexed from left to right and the braces are indexed from right to left. Let $]^i$ and $\}^i$ denote the ith brace and bracket and let $p(]^i)$ and $p(\}^i)$ denote their positions, respectively. In a sequence of brackets and braces of type s for $OG(k,n)$, we make the convention that $\}^{k-s+1}$ denotes $]^s$ and $k-s+1$ should be read as 0. This convention will allow us to state certain combinatorial rules more succinctly.

Let $l(i)$ and $l(\leq i)$ denote the number of *positive* integers in the sequence that are equal to i and less than or equal to i, respectively. For $i > j > 0$, let $\rho(i,j)$ denote the number of integers between $\}^i$ and $\}^j$. Let $\rho(j,0)$ denote the number of integers to the right of $\}^j$. For example, for

$$1]1]122]33]00000\}00\}00\}0000$$

we have $p(]^1) = 1$, $p(]^2) = 2$, $p(]^3) = 5$, $p(]^4) = 7$, $p(\}^3) = 12$, $p(\}^2) = 14$, $p(\}^1) = 16$. We also have $l(1) = 3$, $l(2) = 2$, $l(3) = 2$, $\rho(3,2) = 2$, $\rho(2,1) = 2$, $\rho(1,0) = 4$.

When we are discussing more than one sequence, we will write p_D, ρ_D and l_D to indicate that we are referring to the invariants of D.

Definition 4.21. An *orthogonal quadric diagram* for $OG(k,n)$ is an orthogonal sequence of brackets and braces of type (k,n) with s brackets such that the following conditions hold.

(D1) $l(j) \leq \rho(j, j-1)$ for $1 \leq j \leq k-s$.
(D2) $2p(]^s) \leq p(\}^{k-s}) + l(\leq k-s)$.
(D3) Suppose that integer $1 \leq j < k-s$ occurs in the sequence. If $j+1$ does not occur in the sequence, either $j = 1$ and every position after a 1 is occupied with a]; or $l(i) = \rho(i, i-1)$ for every $j+1 < i \leq k-s$ and $\rho(j+1, j) = 1$.

Definition 4.22. An orthogonal quadric diagram is *admissible* if it satisfies the following additional conditions.

(A1) Let x_j denote the number of brackets such that $p(]^i) \leq l(\leq j)$. Then

$$x_j \geq k - j + 1 - \frac{p(\}^j) - l(\leq j)}{2}.$$

(A2) The two integers to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is 1.
(A3) There are at least three zeros to the left of $\}^{k-s}$.

Example 4.23. In the diagrams 12]000}00}00 and 0]0000}0 condition (A2) is violated. In the diagram 200}00}00 condition (A3) is violated.

The translation between admissible sequences $(L_\bullet, Q_\bullet)$ and quadric diagrams is straightforward. Given an admissible sequence $(L_\bullet, Q_\bullet)$, the corresponding quadric diagram $D(L_\bullet, Q_\bullet)$ is obtained as follows. The sequence of integers starts with r_1 integers equal to 1, followed by $r_i - r_{i-1}$ integers equal to i in increasing order. The sequence ends with $n - r_{k-s}$ integers equal to 0. The are s brackets in positions n_i and $k - s$ braces in positions d_j.

Conversely, given an admissible quadric diagram D, the associated admissible sequence $(L_\bullet, Q_\bullet)(D)$ has an isotropic linear space $L_{p(]^i)}$ of dimension $p(]^i)$ for each bracket $]^i$ in D and a linear space $Q_{p(\}^j)}^{l(\leq j)}$ of dimension $p(\}^j)$ of corank $l(\leq j)$ for each brace $\}^j$ in D.

Exercise 4.24. Show that admissible sequences correspond to admissible quadric diagrams (see [C3]).

Definition 4.25. An admissible quadric diagram is *saturated* if $l(j) = \rho(j, j-1)$ for every $1 \leq j \leq k - s$.

Exercise 4.26. Show that Example 4.17 translates to the following lemma.

Lemma 4.27. *A saturated admissible quadric diagram represents a Schubert variety.*

Now we will describe a combinatorial process for computing the class of a restriction variety in terms of Schubert classes. The algorithm will be recorded in terms of quadric diagrams. Every quadric diagram will be the root of a tree of quadric diagrams where each leaf terminates in a saturated admissible quadric diagram. The class of the restriction variety will be the sum of the Schubert classes summed over the leaves of this tree.

We begin by defining two new quadric diagrams.

Definition 4.28. Fix an admissible quadric diagram D. Assume that D is not saturated, then there exists an index j such that $l(j) \leq \rho(j, j-1)$. Let

$$\kappa = \max\{j \mid l(j) < \rho(j, j-1)\}.$$

Define D^a to be the sequence of brackets and braces obtained from D by changing the $(l(\leq \kappa)+1)$st integer in D to κ.

If $p_{D^a}(]^s) > l_{D^a}(\leq \kappa)$, let

$$\eta = \min\{i \mid p_{D^a}(]^i) > l_{D^a}(\leq \kappa)\}.$$

Define D^b to be the sequence of brackets and braces obtained from D^a by moving the bracket $]^\eta$ to position $p_{D^a}(\leq \kappa)$. Otherwise, D^b is not defined.

Example 4.29. Let $D = 00]0000\}00\}00$. Then $\kappa = 2$ and

$$D^a = 20]0000\}00\}00 \quad D^b = 2]00000\}00\}00.$$

Let $D = 22]33]000\}00\}00\}00$. Then $\kappa = 1$ and

$$D^a = 12]33]000\}00\}00\}00 \quad D^b = 1]233]000\}00\}00\}00.$$

Let $D = 000\}00\}00$. Then $\kappa = 2$, D^b is not defined and $D^a = 200\}00\}00$.
Let $D = 00]00\}00$. Then $\kappa = 1$, $D^a = 10]00\}00$ and $D^b = 1]000\}00$.

We see that D^a and D^b may fail to be admissible. Notice that D^a may fail all three conditions in Definition 4.22, whereas D^b may fail only (A2). We first give an algorithm that will replace D^a and D^b with a set of admissible quadric diagrams.

Algorithm 4.30. Input: A quadric diagram D.

(1) If D does not satisfy condition (A1), discard D and do not return any diagrams. If D satisfies condition (A1), proceed to Step 2.
(2) If D does not satisfy condition (A2), let i_0 be the index of the minimal index bracket for which (A2) fails and let ν be the integer immediately to the left of $]^{i_0}$. If $\nu > 0$ (resp. $\nu = 0$), replace ν by $\nu - 1$ (resp. $k-s$), move $\}^{\nu-1}$ (resp. $\}^{k-s}$) one unit to the left provided that the position $p(\}^{\nu-1}) - 1$ (resp. $p(\}^{k-s}) - 1$) is unoccupied. Return the resulting diagram to Step 1. If position $p(\}^{\nu-1}) - 1$ (resp. $p(\}^{k-s}) - 1$) is occupied, discard the diagram and do not return any diagrams. If condition (A2) is satisfied, proceed to Step 3.
(3) If D does not satisfy (A3), replace D with two diagrams obtained from D by replacing $\}^{k-s}$ in position $p_D(\}^{k-s})$ with $]^{s+1}$ in position $p = p_D(\}^{k-s}) - 1$. If n is even and $2p = n$, then in one of the diagrams let $]^{s+1}$ be decorated with a prime. Return the resulting diagrams to Step 1. If D satisfies (A3), output the diagram.

Example 4.31. We apply Algorithm 4.30 to the diagrams in Example 4.29 which are not admissible.

$$20]0000\}00\}00 \to 22]000\}000\}00 \quad 2]00000\}00\}00 \to 1]00000\}0\}000.$$

$$12]33]000\}00\}00\}00 \to 11]33]000\}00\}0\}000.$$

$$200\}00\}00 \xrightarrow{\times 2} 00]000\}00.$$

In order to save space, when we replace a diagram with two identical diagrams we will write $\times 2$ over the arrow rather than drawing the diagram twice.

$$10]00\}00 \to$$

The diagram is discarded since it does not satisfy (A1).

Exercise 4.32. Run Algorithm 4.30 on the following diagrams

$$0]000\}00\}0 \quad 22]3]3]0000\}000\}00\}00.$$

Exercise 4.33. Show that Algorithm 4.30 terminates and outputs a (possibly empty) set of admissible quadric diagrams. (Hint: At each stage either the diagram is discarded, the number of braces decreases or a brace moves to the left. None of these steps can be repeated infinitely often; see [C3]).

The set of admissible quadric diagrams *derived from* D^a (resp. D^b) are the set of quadric diagrams output by Algorithm 4.30 with input D^a (resp. D^b).

Definition 4.34. Let D be a sequence of brackets and braces such that $p(]^s) > l(\kappa)$. If $l(\leq i) < p(]^{x_\kappa+1})$ for $1 \leq i \leq k-s$, then set $y_{x_\kappa+1} = k-s+1$. Otherwise, let

$$y_{x_\kappa+1} = \max\{i \mid l(\leq i) \geq p(]^{x_\kappa+1})\}.$$

Recall that $]^{x_\kappa+1}$ is the first bracket in a position greater than $l(\leq \kappa)$. If the integer to the immediate left of $]^{x_\kappa+1}$ is 0, then $y_{x_\kappa+1} = k-s+1$. Otherwise, $y_{x_\kappa+1}$ is the integer to the immediate left of $]^{x_\kappa+1}$. We are now ready to state the main algorithm.

Algorithm 4.35. Input: An admissible quadric diagram D.

(1) If D is saturated, return D and stop. Otherwise, proceed to Step 2.
(2) If $p(]^s) \leq l(\kappa)$ or $p(]^{x_\kappa+1}) - l(\leq \kappa) - 1 > y_{x_\kappa+1} - \kappa$, then return the quadric diagrams derived from D^a to Step 1. Otherwise, return the quadric diagrams derived from both D^a and D^b to Step 1.

Successive runs of Algorithm 4.35 either decrease the number of braces or increase the number of positive integers in the sequence. Since neither of these can go on for ever, the algorithm terminates in a set of saturated, admissible quadric diagrams. It is easy to check that if D is an admissible quadric diagram, then each run of the algorithm outputs at least one admissible quadric diagram.

Definition 4.36. A *degeneration path* is a sequence of admissible symplectic diagrams

$$D_1 \to D_2 \to \cdots \to D_r$$

such that D_{i+1} is one of the outputs of running Algorithm 4.35 on D_i for $1 \leq i < r$.

Theorem 4.37. [C3, Theorem 5.12] *Let $V(L_\bullet, Q_\bullet)$ be a restriction variety and let D be the corresponding admissible quadric diagram. Then*

$$[V(L_\bullet, Q_\bullet)] = \sum c_{\lambda;\mu} \sigma_{\lambda;\mu},$$

where $c_{\lambda;\mu}$ is the number of degeneration paths starting with D and ending with the quadric diagram associated to the Schubert class $\sigma_{\lambda;\mu}$.

The proof of Theorem 4.37 is obtained by interpreting Algorithm 4.35 as a specialization. Consider the one-parameter family of partial flags $(L_\bullet, Q_\bullet)(t)$, where all the flag elements are fixed except for $Q^{r_\kappa}_{d_\kappa}$ and $Q^{r_\kappa}_{d_\kappa}$ varies in a pencil that becomes tangent to Q in one larger dimension at $t = 0$. The algorithm describes the flat limit of the restriction varieties defined with respect to the flags $(L_\bullet, Q_\bullet)(t)$ at $t = 0$.

Exercise 4.38. Carry out this specialization for the Examples 4.2 and 4.3. Show that the descriptions in these examples agree with Algorithm 4.35.

We now discuss several consequences of Theorem 4.37. The first corollary is a solution of the restriction problem. The pullback of a Schubert class in $G(k,n)$ can be expressed as the class of a sum of restriction varieties. Suppose that the Schubert variety in $G(k,n)$ is defined with respect to a general partial flag

$$F_{a_1 = n-k+1-\lambda_1} \subset F_{a_2 = n-k+2-\lambda_2} \subset \cdots \subset F_{a_k = n-\lambda_k}.$$

The restriction of Q to the flag elements F_i is a non-degenerate quadratic form $Q^0_{a_i}$. If $a_i \leq 2i-1$ for some i, then $i^*\sigma_\lambda = 0$. Suppose that $a_i \geq 2i$ for $1 \leq i \leq k$. We associate the sequence

$$Q^0_{a_1} \subset Q^0_{a_2} \subset \cdots \subset Q^0_{a_k}$$

to the class σ_λ. Running Algorithm 4.30 on the corresponding quadric diagram produces a collection of admissible quadric diagrams. The class $i^*\sigma_\lambda$ is the sum of the Schubert classes obtained by running Algorithm 4.35 on this collection of admissible quadric diagrams. We conclude the following theorem.

Theorem 4.39. *Algorithm 4.35 gives a positive, geometric rule for computing restriction coefficients.*

Example 4.40. Our first example computes the class $i^*(\sigma_{2,1}) = [V(Q^0_3 \subset Q^0_5)] = 2\sigma_{2;1} + 2\sigma_{1,0;} + 2\sigma_{1;2}$ in $OG(2,6)$.

$$000\}00\}0 \xrightarrow{\times 2} 00]000\}0 \to \underline{1]0000\}0}$$

$$\swarrow \quad \searrow$$

$$\underline{00]0]000} \qquad \underline{00]0]'000}$$

Example 4.41. The next example calculates $i^*(\sigma_{3,2,1}) = [V(Q^0_4 \subset Q^0_6 \subset Q^0_8)] = 2\sigma_{4;3,1} + 2\sigma_{3;3,2} + 4\sigma_{2,1;1}$ in $OG(3,9)$.

$$0000\}00\}00\}0 \to 3000\}00\}00\}0 \xrightarrow{\times 2} 000]000\}00\}0 \to 200]000\}00\}0 \xrightarrow{\times 2} 000]0]0000\}0 \to \underline{100]0]0000\}0}$$

$$\downarrow$$

$$22]0000\}00\}0 \to 1]20000\}00\}0 \to \underline{1]22000\}00\}0}$$

$$\downarrow$$

$$11]0000\}0\}00 \to \underline{11]2000\}0\}00}$$

Exercise 4.42. Compute all the restriction coefficients for the inclusions $i : OG(2,5) \to G(2,5)$, $i : OG(2,6) \to G(2,6)$ and $i : OG(3,7) \to G(3,7)$.

As an another application of Theorem 4.37, one can compute the classes of the moduli space of vector bundles of rank two on hyperelliptic curves in $OG(g-1, 2g+2)$. Let Q_1 and Q_2 be two general quadric hypersurfaces in $\mathbb{P}^{2g+1}$ and consider the pencil they generate. Let I be the incidence correspondence consisting of pairs (Q, L), where Q is a quadric of this pencil and L is a connected component of the space of g-dimensional projective linear spaces on Q. Since the half-dimensional linear spaces on Q have two components, I is a double cover of $\mathbb{P}^1$ ramified over the points in the pencil where the rank of the quadric drops. Since the discriminant is a hypersurface of degree $2g+2$ and the pencil is general, we conclude that I is ramified over $2g+2$ points. An easy local calculation shows that I is a hyperelliptic curve of genus g. Furthermore, every hyperelliptic curve arises via this construction.

Let C be a smooth hyperelliptic curve of genus g. Let $M_{2,o}(C_g)$ denote the moduli space of vector bundles of rank 2 with a fixed odd-degree determinant on C. Realize C as above. By a celebrated theorem of Desale and Ramanan [DR], $M_{2,o}(C_g)$ is isomorphic to the space of $(g-2)$-dimensional projective linear spaces contained in the pencil of quadrics. In particular, $[M_{2,o}(C_g)] = 2^{g-1} i^* \sigma_{g-1,g-2,\dots,2,1}$. Using this description, Theorem 4.37 allows us to compute the class of $M_{2,o}(C_g)$ in $OG(g-1, 2g+2)$. The first four cases are given in the following table.

(1) $[M_{2,o}(C_2)] = 2\sigma_{;1}$
(2) $[M_{2,o}(C_3)] = 4\sigma_{0;1} + 4\sigma_{;3,1}$
(3) $[M_{2,o}(C_4)] = 16\sigma_{2;3,1} + 16\sigma_{1,0;1} + 16\sigma_{1;4,1}$
(4) $[M_{2,o}(C_5)] = 64\sigma_{3,1;3,1} + 64\sigma_{2,1,0;1} + 64\sigma_{2,1;5,1} + 32\sigma_{3,0;4,1} + 32\sigma_{3;5,4,1} + 32\sigma_{5,0;3,1} + 32\sigma_{5;5,3,1} + 32\sigma_{4,0;3,2} + 32\sigma_{4;5,3,2}$.

When $g=2$, $M_{2,o}(C_2)$ is a complete intersection of two quadric fourfolds. A consequence of this computation is that for $g > 2$, these moduli spaces are not complete intersections of ample divisors in $OG(g-1, 2g+2)$ (see [C3]).

Exercise 4.43. Compute $[M_{2,o}(C_6)]$ and $[M_{2,o}(C_7)]$.

An algorithm similar to Algorithm 4.35 computes the restriction coefficients for isotropic flag varieties. We refer the reader to [C3] for details.

There are other natural maps that one can define between orthogonal flag varieties and ordinary flag varieties. Perhaps the most interesting is the following

$$\phi : OG(k,n) \to F(k, n-k; n) \quad W \mapsto (W, W^{\perp})$$

and more generally

$$\phi : OF(k_1, \dots, k_t; n) \to F(k_1, \dots, k_t, n-k_1, \dots, n-k_t; n) \quad (W_1, \dots, W_t) \mapsto (W_1, \dots, W_t, W_1^{\perp}, \dots, W_t^{\perp}).$$

Problem 4.44. *Generalize Algorithm 4.35 to obtain a positive geometric algorithm for computing the map induced on cohomology by ϕ in terms of the Schubert bases of these flag varieties.*

4.2. **The skew-symmetric case.** The discussion for $SG(k,n)$ is similar, but the definitions have to be adapted for alternating forms. The fact that skew-symmetric forms have even rank constrains the geometry. Consequently, this case is more delicate. We preserve the notation from the previous section.

Notation 4.45. Let L_{n_i} denote an isotropic subspace of dimension n_i. Let $Q_{d_j}^{r_j}$ denote a d_j-dimensional subspace of V such that the restriction of the skew-symmetric form Q to $Q_{d_j}^{r_j}$ has corank r_j. Let K_j denote the kernel of the restriction of Q to $Q_{d_j}^{r_j}$. The dimension of K_j is r_j.

It is no longer possible to ensure that the kernels of the restriction of Q to linear spaces are nested. This complicates the definition of a symplectic sequence. Throughout let $(L_\bullet, Q_\bullet)$ be a partial flag

$$L_{n_1} \subset \cdots L_{n_s} \subset Q_{d_{k-s}}^{r_{k-s}} \subset \cdots \subset Q_{d_1}^{r_1}$$

satisfying

- $\dim(K_j \cap K_l) \geq r_j - 1$ for $1 \leq j < l \leq k - s$,
- $\dim(L_{n_i} \cap K_j) \geq \min(n_i, \dim(K_j \cap K_{k-s}) - 1)$ for every $1 \leq i \leq s$ and $1 \leq j \leq k - s$.

Definition 4.46. We say that the partial flag $(L_\bullet, Q_\bullet)$ is *in order* if

- $K_j \cap K_l = K_j \cap K_{j+1}$ for $1 \leq j < l \leq k - s$,
- $\dim(L_{n_i} \cap K_j) = \min(n_i, \dim(K_j \cap K_{k-s}))$ for $1 \leq i \leq s$ and $1 \leq j \leq k - s$.

The sequence is *in perfect order* if

- $K_j \subset K_{j+1}$ for $1 \leq j < k - s$,
- $\dim(L_{n_i} \cap K_j) = \min(n_i, r_j)$ for $1 \leq i \leq s$ and $1 \leq j \leq k - s$.

Definition 4.47. The partial flag $(L_\bullet, Q_\bullet)$ is a *symplectic sequence* if it satisfies the following conditions

(1) Either the sequence is in order or there exists at most one index $1 \leq \eta \leq k - s$ such that

$$K_j \subseteq K_l \text{ for } l \geq j > \eta \text{ and } K_j \cap K_l = K_j \cap K_{j+1} \text{ for } j < \eta, l > j.$$

Furthermore, if $K_\eta \subset K_{k-s}$, then

$$\dim(L_{n_i} \cap K_j) = \min(n_i, \dim(K_j \cap K_{k-s})) \text{ for } 1 \leq i \leq s \text{ and } 1 \leq j < \eta;$$
$$\dim(L_{n_i} \cap K_j) = \min(n_i, \dim(K_j \cap K_{k-s}) - 1) \text{ for } 1 \leq i \leq s \text{ and } \eta \leq j \leq k - s.$$

If $K_\eta \not\subset K_{k-s}$, then $\dim(L_{n_i} \cap K_j) = \min(n_i, \dim(K_j \cap K_{k-s}))$ for $1 \leq i \leq s$ and $1 \leq j < k - s$.

(2) If $\alpha = \dim(K_j \cap K_{k-s}) > 0$, then either $j = 1$ and $n_\alpha = \alpha$ or there exists an index $1 \leq j_0 \leq k - s$ such that for $j_0 \neq l > \min(j, \eta)$, we have $r_l - r_{l-1} = d_{l-1} - d_l$,

$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap K_{j_0})$$

and $K_\eta \not\subset K_{j_0}$.

Remark 4.48. As in the orthogonal case, n_i, r_j and d_j automatically satisfy certain inequalities. The corank bound implies that $r_j - \dim(K_j \cap K_{j-1}) \leq d_{j-1} - d_j$ for every $1 < j \leq k - s$. The linear space bound implies that $2(n_s + r_j - \dim(K_j \cap L_{n_s})) \leq r_j + d_j$ for every $1 \leq j \leq k - s$. These inequalities will be implicitly assumed. More importantly, $d_j - r_j$ has to be even since the rank of a skew-symmetric form is even. In the orthogonal case, we could require the kernels to be nested. Due to the parity restrictions we cannot do this in symplectic case, which leads to the more convoluted condition (1) in the definition of symplectic sequences. Condition (2) is a consequence of the order of specialization. These conditions will automatically be satisfied in practice, so the reader can ignore them in a first reading.

Notation 4.49. Let x_j denote the number of isotropic subspaces in the sequence that are contained in K_j.

Definition 4.50. We call the sequence $(L_\bullet, Q_\bullet)$ *admissible* if it satisfies the following conditions.

(SA1) For every $1 \leq j \leq k - s$, we have

$$x_j \geq k - j + 1 - \frac{d_j - r_j}{2}.$$

(SA2) There does not exist integers i, j such that $n_i = \dim(K_j \cap L_{n_i}) + 1$.

Definition 4.51. The *symplectic restriction variety* $V(L_\bullet, Q_\bullet)$ associated to an admissible symplectic sequence $(L_\bullet, Q_\bullet)$ is the subvariety of $SG(k, n)$ defined as the Zariski closure of the following locus

$$\begin{aligned} V(L_\bullet, Q_\bullet)^0 = \{ &W \in SG(k,n) \mid \dim(W \cap L_{n_i}) = i, \dim(W \cap Q^{r_j}_{d_j}) = k - j + 1, \\ &\dim(W \cap K_j) = x_j \} \end{aligned}$$

Remark 4.52. As in the orthogonal case, condition (SA1) is a consequence of the linear space bound. The isotropic subspace of dimension $k-j+1$ has to intersect the kernel K_j in a subspace of dimension at least $k-j+1-\frac{d_j-r_j}{2}$. Condition (SA2) is a consequence of variations of tangent spaces. If $r_j+1=n_i$, then any linear space intersecting L_{n_i} away from K_j is contained in $L_{n_i}^{\perp}$.

Exercise 4.53. As in the orthogonal case, use induction on k to prove the following proposition.

Proposition 4.54. *The symplectic restriction variety $V(L_\bullet, Q_\bullet)$ is an irreducible subvariety of $SG(k,n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet)) = \sum_{i=1}^{s}(n_i - i) + \sum_{j=1}^{k-s}(d_j + x_j - 1 - 2k + 2j)$$

Definition 4.55. A symplectic sequence $(L_\bullet, Q_\bullet)$ is *saturated* if $r_j + d_j = n$ for $1 \leq j \leq k-s$.

Schubert varieties in $SG(k,n)$ are examples of restriction varieties. As in the orthogonal case, we have the following.

Lemma 4.56. *Let $(L_\bullet, Q_\bullet)$ be a saturated, admissible symplectic sequence in perfect order and let $n=2m$. Then $V(L_\bullet, Q_\bullet)$ is a Schubert variety with class $\sigma_{m-n_1+1,\dots,m-n_s+1;r_{k-s},\dots,r_1}$. Conversely, every Schubert variety arises as such a symplectic restriction variety.*

Let $a_i = n-k+i-\lambda_i > 2i-1$ for $1 \leq i \leq k$. Then the intersection of a general Schubert variety Σ_λ in $G(k,n)$ with $SG(k,n)$ is also a symplectic restriction variety associated to the sequence $Q_{a_1}^{r_1} \subset \cdots \subset Q_{a_k}^{r_k}$, where $0 \leq r_i \leq 1$ satisfies $a_i = r_i \mod 2$ and $K_j \cap K_{j-1} = 0$ for $1 < j \leq k-s$. Hence, symplectic restriction varieties interpolate between these two kinds of varieties.

Symplectic diagrams. We can record symplectic restriction varieties by symplectic diagrams. Their definition and properties are similar to orthogonal diagrams.

Definition 4.57. Let $0 \leq s \leq k$ be an integer. A *sequence of brackets and braces of type s* for $SG(k,n)$ consists of a sequence of natural numbers of type s, s brackets and $k-s$ braces such that

- Every bracket or brace occupies a positive position and each position is occupied by at most one bracket or brace.
- Every bracket is to the left of every brace.
- Every positive integer greater than or equal to j is to the left of the jth brace.
- The total number of integers equal to zero or greater than j to the left of the jth brace is even.

Notation 4.58. We use the same conventions as in the case of $OG(k,n)$. We count the brackets from left to right and denote the ith bracket by $]^i$. We count the braces from right to left and denote the jth brace by $\}^j$.

Definition 4.59. Two sequences of brackets and braces are *equivalent* if the lengths of their sequence of numbers are equal, the brackets and braces occur at the same positions and the collection of digits that occur between any two consecutive brackets and/or braces are the same. We can depict an equivalence class of sequences by the representative where the digits are listed so that between any two consecutive brackets and/or braces the positive integers precede the zeros and are listed in non-decreasing order. We will always use this *canonical representative* and blur the distinction between the equivalence class and this representative.

Example 4.60. The sequences 1221]01000}001}00 and 1122]10000}100}00 are equivalent. The latter is the canonical representative.

Due to the evenness of rank, in the symplectic case it is not possible to keep the positive integers in increasing order and all at the beginning of the sequence.

Definition 4.61. A sequence of brackets and braces is *in order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros except possibly for one j immediately to the right of $\}^{j+1}$ for $1 \leq j < k-s$. Otherwise, we say that the sequence is not in order. A sequence is *in perfect order* if the sequence of integers consists of non-decreasing positive integers followed by zeros. As in the orthogonal case, we call a sequence of brackets and braces *saturated* if $l(j) = \rho(j, j-1)$ for every $1 \leq j \leq k-s$.

Example 4.62. The sequence 22]00]0000}100}0 is in order. The sequence 11]22]0000}00}00 is in perfect order. The sequences 22]10000}00}0 and 122]3300}100}00}00 are not in order. In the first sequence the 1 in place 3 and in the second sequence the 1 in place 8 violate the order.

Definition 4.63. A *symplectic diagram* for $SG(k,n)$ is a sequence of brackets and braces that satisfies the following conditions.

(1) $l(j) \leq \rho(j, j-1)$ for $1 \leq j \leq k-s$.

(2) Let τ_j be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^j$. Then
$$2\tau_j \leq p(\}^j) + r_j \quad \text{for } 1 \leq j \leq k-s.$$

(3) Either the sequence is in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of i after $\}^{i+1}$.

(4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If an integer i occurs to the left of all the zeros, then either $i = 1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j, j-1) = l(j)$ for $j_0 \neq j > \min(i, \eta)$ and $\rho(j_0, j_0 - 1) \leq l(j_0) + 2 - \xi_{j_0}$. Moreover, any integer η violating order occurs to the right of $\}^{j_0}$.

Definition 4.64. A symplectic diagram is called *admissible* if it satisfies the following two conditions.

(SA1) Let x_j denote the number of brackets $]^i$ such that every integer to the left of $]^i$ is positive and less than or equal to j. Then
$$x_j \geq k - j + 1 - \frac{p(\}^j) - r_j}{2}.$$

(SA2) The two integers to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is 1.

Exercise 4.65. Choose a symplectic basis of V and define a correspondence between symplectic diagrams and symplectic sequences. Show that every admissible symplectic sequence can be represented by an admissible symplectic diagram. Conversely, every admissible symplectic diagram is the diagram associated to an admissible symplectic sequence (see [C4]).

As in the orthogonal case, we have the following easy lemma.

Lemma 4.66. *Admissible, saturated symplectic diagrams in perfect order correspond to Schubert cycles.*

We are now ready to describe the algorithm. The goal is to transform an admissible symplectic diagram into a collection of admissible, saturated symplectic diagrams in perfect order. The fact that the rank of a skew-symmetric form is even constrains the possible degenerations. Consequently, describing D^a in this case is trickier.

Definition 4.67. We make the convention that an integer equal to $k-s+1$ is 0 and $\}^{k-s+1}$ refers to $]^s$. Let D be an admissible symplectic diagram.

(1) If D is not in order, let η be the integer in condition (3) violating the order.
   (a) If every integer $\eta < j \leq k-s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta + 1$ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by interchanging η and ν.

(b) If an integer $\eta < j \leq k-s$ does not occur to the left of η, let ν be the leftmost integer equal to $j+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{j+1}$ not equal to ν and changing ν to j.

(2) If D is in order but is not a saturated admissible diagram in perfect order, let κ be the largest index j for which $l(j) < \rho(j, j-1)$.
   (a) If $l(\kappa) < \rho(\kappa, \kappa-1)$, let ν be the leftmost digit equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained from D by changing ν and the leftmost 0 to the right of $\}^{\kappa+1}$ not equal to ν to κ.
   (b) If $l(\kappa) = \rho(\kappa, \kappa-1)$, let η be the integer equal to $\kappa-1$ immediately to the right of $\}^{\kappa}$.
      (i) If κ occurs to the left of η, let ν be the leftmost integer equal to κ in the sequence of D. Let D^a be the canonical representative of the diagram obtained from D by changing ν to $\kappa-1$ and η to 0.
      (ii) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost zero to the right of $\}^{\kappa+1}$ not equal to ν and changing ν to κ.

Let p be the position in D immediately to the right of ν. If there exists a bracket at a position $p' > p$, let $q > p$ be the minimal position occupied by a bracket. Let D^b be the diagram obtained from D^a by moving the bracket in position q to position p. Otherwise, D^b is not defined.

Example 4.68. Let $D = 2300\}10\}0\}0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}20\}0\}0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]200\}00\}$, then the second 2 violates the order and $D^a = 220]000\}00\}$, $D^b = 22]0000\}00\}$.

Let $D = 124400\}00\}1\}0\}00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii) and $D^a = 123400\}10\}0\}0\}00$.

Let $D = 22]00\}00\}00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1,0) - 1$, we are in case (2)(i) and $D^a = 12]00\}10\}00$ and $D^b = 1]200\}10\}00$.

Let $D = 3300\}200\}0\}$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3,2) - 1$, we are in case (2)(ii)(a) and $D^a = 2300\}000\}0\}$.

Finally, let $D = 330000\}00\}1\}0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2,1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b) and $D^a = 230000\}10\}0\}0$.

It may be that neither D^a nor D^b is admissible. As in the orthogonal case, there are algorithms for transforming them into admissible diagrams.

Algorithm 4.69. Input: A symplectic diagram D.

Step 1. If D satisfies condition (SA1), proceed to the next step. Otherwise, let j be the largest index for which (SA1) fails. Let $\pi_1 < \pi_2$ be the places of the two rightmost integers equal to j. Let D^c be the diagram obtained from D as follows. Delete $\}^j$ and the j in place π_2. Move the brackets, braces and integers in positions $\pi_1 < p < \pi_2$ to position $p+1$ and add a j in place $\pi_1 + 1$. Add a bracket in position $\pi_1 + 1$. Subtract one from the integers $j < h \leq k-s$; and if $j = k-s$, change the integers equal to $k-s$ to 0. Return D^c to the beginning

Step 2. If D satisfies condition (SA2), output D. Otherwise, let $]^i$ be the smallest index bracket for which (SA2) fails. Let j be the integer immediately to the left of $]^i$. Let D^c be the symplectic diagram obtained from D as follows. Replace this j with $j-1$ ($k-s$ if $j=0$) and move $\}^{j-1}$ ($\}^{k-s}$ if $j=0$) one position to the left unless that position is occupied. If the position is occupied, discard the diagram and stop. Otherwise, return D^c to Step 2.

We say that D is a diagram derived from D^a (respectively, D^b) if D is an output of running Algorithm 4.69 on D^a (respectively, D^b).

Example 4.70. Let $D = 22]33]00\}00\}00\}00$. Then the diagram $D^a = 12]33]00\}00\}10\}00$ fails condition (SA1) since $x_1 = 0 < 1 = 5 - (10-2)/2$. Hence, according to Step 1 of Algorithm 4.69, we replace D^a with the admissible diagram $11]1]22]00\}00\}000$.

Let $D = 00\}00\}00$. Then $D^a = 22\}00\}00$ fails condition (SA1) since $x_2 = 0 < 1 - (2-2)/2$. Hence, Step 1 of Algorithm 4.69 replaces D^a with $00]00\}00$ which is admissible.

Similarly, if $D = 11]33]00\}00\}00\}00$, then the diagram $D^a = 11]23]00\}20\}00\}00$ fails condition (SA1) since $x_2 = 1 < 2$. Hence, according to Step 1 of Algorithm 4.69, we replace D^a with $11]22]2]00\}000\}00$, which is admissible.

If $D = 22]2]200\}0000\}00$, then the diagram $D^a = 12]2]200\}1000\}00$ is not admissible since it fails condition (SA2) for $]^1$. Step 2 of Algorithm 4.69 replaces D^a first with $11]2]200\}100\}000$. Note that this diagram fails condition (SA2) for $]^2$. Hence, Step 2 replaces it with $11]1]200\}10\}0000$. This diagram is admissible, hence it is the diagram derived from D^a.

Exercise 4.71. Show that Algorithm 4.69 terminates and replaces D^a or D^b derived from an admissible symplectic diagram D with a collection of admissible symplectic diagrams.

Finally, we can state the main algorithm for computing the classes of symplectic restriction varieties. Let D be an admissible symplectic diagram and let ν be as in Definition 4.67. Let $\pi(\nu)$ denote the place of ν in the sequence of integers. If $p(]^s) > \pi(\nu)$, then $]^\alpha$ be the first bracket to the right of ν. If the integer to the immediate left of $]^\alpha$ is positive, let y_ν be this integer. Otherwise, let $y_\nu = k - s + 1$.

Algorithm 4.72. Input: Let D be an admissible symplectic diagram.

(1) If D is saturated and in perfect order, return D and stop. Otherwise, proceed to Step 2.
(2) If $p(]^s) \leq \pi(\nu)$ or $p(]^\alpha) - \pi_\nu - 1 > y_\nu - \nu$ in D, return the admissible symplectic diagrams derived from D^a to Step 1. Otherwise, return the admissible symplectic diagrams derived from both D^a and D^b to Step 1.

Definition 4.73. A *degeneration path* is a sequence of admissible symplectic diagrams

$$D_1 \to D_2 \to \cdots \to D_r$$

such that D_{i+1} is one of the outputs of running Algorithm 4.72 on D_i for $1 \leq i < r$.

As in the orthogonal case, we have the following theorem.

Theorem 4.74. [C4, Theorem 3.33] *Let D be an admissible symplectic diagram and let $V(D)$ be the corresponding symplectic restriction variety. Then*

$$[V(D)] = \sum c_{\lambda;\mu} \sigma_{\lambda;\mu},$$

where $c_{\lambda;\mu}$ is the number of degeneration paths starting with D and ending in the symplectic diagram $D(\sigma_{\lambda;\mu})$.

As a corollary, we obtain a positive geometric rule for computing the restriction coefficients for $SG(k,n)$.

Theorem 4.75. [C4, Theorem 5.2] *Theorem 4.74 provides a positive, geometric rule for computing the restriction coefficients for the inclusion $i : SG(k,n) \subset G(k,n)$.*

There is a generalization of the algorithm to symplectic flag varieties [C6]. As in the orthogonal case, the following variant is an interesting open problem.

Problem 4.76. *Let $\phi : SF(k_1, \dots, k_t; n) \to F(k_1, \dots, k_t, n-k_1, \dots, n-k_t; n)$ be the map given by*

$$(W_1, \dots, W_t) \mapsto (W_1, \dots, W_t, W_1^\perp, \dots, W_t^\perp).$$

Compute the map induced in cohomology in terms of the Schubert bases via a similar sequence of specializations.

We conclude this section with several examples.

Example 4.77.

$$\begin{tikzcd}
00\}00\}00 \arrow[r] & 00]00\}00 \arrow[r] \arrow[d] & 00]0]000 \\
& 1]100\}00 &
\end{tikzcd}$$

We conclude that $i^*\sigma_{3,2} = \sigma_{2,1;} + \sigma_{3;2}$ in $SG(2,6)$.

Finally, we give a larger example in $SG(3,10)$ that illustrates the inductive structure of the algorithm.

Example 4.78.

$$\begin{tikzcd}
300\}20\}10\}000 \arrow[r] & 200\}00\}10\}000 \arrow[r] & 200]00\}10\}000 \arrow[r] \arrow[d] & 1]0000\}00\}000 \arrow[d] \\
& & 100]00\}00\}000 \arrow[dl] \arrow[d] & 1]2200\}00\}000 \arrow[d] \\
& 100]0]000\}000 \arrow[dl] \arrow[d] & 11]200\}0\}0000 \arrow[d] & 1]1200\}10\}000 \arrow[d] \\
\underline{000]0]0]00000} & 11]00]100\}000 \arrow[dl] \arrow[d] & \underline{11]11]00\}0000} & 1]1100\}00\}000 \arrow[d] \\
\underline{11]1]0000\}000} & \underline{11]11]00\}0000} & & \underline{1]1100]00\}000}
\end{tikzcd}$$

This calculation shows that $i^*\sigma_{5,4,3} = \sigma_{3,2,1;} + \sigma_{4,3;3} + 2\sigma_{4,2;4} + \sigma_{5,1;3}$ in $H^*(SG(3,10),\mathbb{Z})$.

5. The rigidity problem

In 1961, Borel and Haefliger [BH] asked whether Schubert classes in rational homogeneous varieties can be represented by projective subvarieties other than Schubert varieties. In this section, we discuss recent progress in answering this problem.

A Schubert class σ is called *rigid* if Schubert varieties are the only projective subvarieties representing σ. More generally, given a positive integer m, the class $m\sigma$ can be represented by the unions of m Schubert varieties. The class σ is called *multi rigid* if unions of m Schubert varieties are the only closed algebraic sets representing the class $m\sigma$. As we have seen in §2, Schubert varieties are often singular. If the class of a singular Schubert variety is rigid, then that class cannot be represented by a smooth subvariety.

The rigidity of Schubert classes has been studied by many authors [Br], [C2], [C5], [CR],[Ho1], [Ho2], [R], [RT], [W]. The multi rigid Schubert classes have been classified in cominuscule homogeneous varieties. We recall that the cominuscule homogeneous varieties coincide with the Compact Complex Hermitian Symmetric Spaces and are the following varieties:

1. Grassmannians $G(k,n)$,
2. Smooth quadric hypersurfaces in $\mathbb{P}^n$,
3. The isotropic Grassmannians $SG(m,2m)$,
4. The spinor varieties, i.e., the irreducible components of $OG(m,2m)$,
5. Two exceptional varieties: The Cayley plane $\mathbb{OP}^2$ (E_6/P_6) and the Freudenthal variety $G(\mathbb{O}^3,\mathbb{O}^6)$ (E_7/P_7).

Robles and The in [RT] building on the work of R. Bryant, J. Hong and M. Walters identified an obstruction $\mathcal{O}_\sigma$ in Lie algebra cohomology and showed that if $\mathcal{O}_\sigma$ vanishes, then σ is multi rigid. They classified the Schubert classes in cominuscule homogeneous varieties with $\mathcal{O}_\sigma = 0$. Furthermore, for classes with $\mathcal{O}_\sigma \neq 0$, Robles [R] constructed explicit irreducible representatives for either 2σ or 4σ. Robles and the author in [CR] proved the following sharpening.

Theorem 5.1. [CR, Theorem 1.1] *Let σ be a Schubert class with $\mathcal{O}_\sigma \neq 0$ in the cohomology of a cominuscule homogeneous variety X and let m be a positive integer. Then there exists an irreducible subvariety of X that represents $m\sigma$.*

Hong and Mok [HM] (see also [MZ]) have an alternative approach to proving the rigidity of certain smooth Schubert varieties in homogeneous varieties of Picard rank one. Their method uses the varieties of minimal rational tangents.

The complete classification of rigid and multi rigid Schubert classes for more general rational homogeneous varieties is largely open. In this section, we discuss the problem in greater detail, give several applications of restriction varieties and pose several open problems.

Rigidity in $G(k, n)$. We begin by describing rigid and multi rigid Schubert classes in $G(k, n)$. In §2, we described the smooth Schubert varieties of $G(k, n)$ as the linearly embedded sub-Grassmannians. Even when a Schubert variety is singular, there may be representatives of σ_λ which are smooth.

Example 5.2. A Schubert variety with class σ_1 in $G(k, n)$ is singular if k and $n-k$ are both greater than 1. However, σ_1 is the class of a hyperplane under the Plücker embedding of $G(k, n)$. By Bertini's Theorem, we can always find smooth hyperplane sections that represent σ_1. Hence, the class σ_1 is not rigid and can be represented by smooth subvarieties of $G(k, n)$.

On the other hand, the theory of minimal degree varieties can be used to prove the rigidity of certain classes.

Exercise 5.3. Show that the degree d of an irreducible, nondegenerate variety of dimension m in $\mathbb{P}^n$ satisfies the inequality $d \geq n - m + 1$. The varieties where the equality $d = n - m + 1$ holds are called *varieties of minimal degree.*

The varieties of minimal degree have been classified by Bertini [Be]. Eisenbud and Harris have given a modern account of the classification in [EH1].

Theorem 5.4. [EH1] *A variety of minimal degree is one of the following:*

1. *A quadric hypersurface.*
2. *The Veronese surface in $\mathbb{P}^5$.*
3. *A rational normal scroll, that is the projectivization $\mathbb{P}E$ of a vector bundle E on $\mathbb{P}^1$ embedded by the complete linear series $|\mathcal{O}_{\mathbb{P}E}(1)|$.*
4. *A cone over one of the previous varieties.*

Example 5.5. The Schubert class σ_2 in $G(2, 5)$ is rigid. Let X be a projective subvariety of $G(2, 5)$ representing σ_2. Then in the Plücker embedding X has dimension 4 and degree 3. We claim that X must span a $\mathbb{P}^6$ and be a variety of minimal degree. Otherwise, X would span a $\mathbb{P}^5$ and be a cubic hypersurface in this $\mathbb{P}^5$. The Grassmannian $G(2, 5)$ is cut out by quadratic Plücker relations. By Bezout's Theorem, $G(2, 5)$ cannot contain X without containing its linear span. However, the largest dimensional linear space in $G(2, 5)$ has dimension 3. This is a contradiction. Hence, X must be a variety of minimal degree. From the classification of these varieties, we conclude that X must be a cone with vertex p over a cubic scroll. Hence, X must be the intersection of $G(2, 5)$ with its tangent plane at p. Since this is a Schubert variety with class σ_2, we conclude that X is a Schubert variety.

The following fundamental theorem of Thom, Grauert-Kerner, Schelssinger, Kleiman-Landolfi allows one to prove the rigidity of many other Schubert classes in $G(k,n)$.

Theorem 5.6. [KL, Theorem 2.2.8] *Let K be a field and let $X = \mathbb{P}^n_K \times \mathbb{P}^m_K$ be embedded in projective space by the Segre morphism. If $n \geq 1$ and $m \geq 2$, then the cone over X is rigid over K in the sense that any small deformation is isomorphic to the cone over X.*

Exercise 5.7. Let λ be the partition $\lambda_1 = \cdots = \lambda_{k-1} = n-k-1$ and $\lambda_k = 0$. Show that in the Plücker embedding a Schubert variety Σ_λ is a cone over the Segre embedding of $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. Deduce that the Schubert class σ_λ is rigid unless $n = 2k = 4$.

By induction building on Example 5.5 and Exercise 5.7, [C2] characterizes rigid Schubert classes in $G(k,n)$. The Schubert subvarieties contained in a variety $Y \subset G(k,n)$ carry a lot of geometric information about Y (see [AC] for a related discussion). Let Y be a representative of a Schubert class. The singular locus of Y and the intersection of Y with various Schubert varieties are often enough to force Y to be a Schubert variety. For our purposes, it is convenient to record a partition λ by grouping the parts that are equal. More concretely, write $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ with $\mu_1 > \mu_2 > \cdots > \mu_j$ and μ_s occurs i_s times in the partition λ. In particular, $\sum_{s=1}^j i_s = k$.

Definition 5.8. A partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ is a *rigid partition* for $G(k,n)$ if there does not exist an index $1 \leq s < j$ with $i_s = 1$ and $n-k > \mu_s = \mu_{s+1} + 1$.

Theorem 5.9. [C2, Theorem 1.3] *A Schubert class σ_λ in $G(k,n)$ is rigid if and only if λ is a rigid partition for $G(k,n)$.*

A multi rigid class by definition is rigid. However, a rigid class does not have to be multi rigid.

Example 5.10. The class σ_2 in $G(2,5)$ is rigid (see Example 5.5) but not multi rigid. To describe deformations of $m\sigma_2$, it is more convenient to work projectively. Fix an irreducible curve C of degree m in $\mathbb{P}^4$. Consider the subvariety X of $G(2,5)$ parameterizing lines that intersect C. Consider the incidence correspondence

$$I = \{(p,l) \mid p \in l, p \in C\} \subset C \times G(2,5).$$

The variety I is irreducible by the theorem on the dimension of fibers and dominates X. Hence, X is irreducible and has dimension 4. We claim that the class of X is $m\sigma_2$. To determine the class, we can intersect with complementary dimensional Schubert cycles. A general Schubert variety $\Sigma_{3,1}$ intersects X in m points. The variety $\Sigma_{3,1}$ parameterizes lines in $\mathbb{P}^4$ that pass through a point q and lie in a hyperplane P containing q. The hyperplane P intersects C in m points $p_1, \dots, p_m$. Hence, the lines parameterized by $X \cap \Sigma_{3,1}$ are the m lines spanned by q and p_i. On the other hand, a general $\Sigma_{2,2}$ is disjoint from X. The Schubert variety $\Sigma_{2,2}$ parameterizes lines contained in a plane Π. Since a general plane Π is disjoint from C, $X \cap \Sigma_{2,2} = \emptyset$. We conclude that X is an irreducible variety with class $m\sigma_2$.

Hong [Ho1], [Ho2] and Robles and The [RT] have classified the multi rigid Schubert classes in $G(k,n)$.

Theorem 5.11. [Ho2] [RT] *A Schubert class σ_λ is multi rigid if and only if $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ satisfies the following conditions*

- $i_s \geq 2$ *for* $1 < s < j$,
- $\mu_{s-1} \leq \mu_s - 2$ *for* $1 < s \leq j$,
- $i_1 \geq 2$ *if* $\mu_1 < n-k$; *and* $i_j \geq 2$ *if* $\mu_j > 0$.

If a Schubert class σ_λ is rigid and the corresponding Schubert variety is singular, then σ_λ cannot be represented by a smooth subvariety of $G(k,n)$. However, even when σ_λ is not rigid, it might not be possible to represent σ_λ by a smooth subvariety of $G(k,n)$.

Example 5.12. Consider the class $\sigma_{3,2}$ in $G(3,7)$. This class is not rigid. Fix a four-dimensional subspace $V' \subset V$. Let Y be a smooth hyperplane section of the variety $G(2,V')$ (in the Plücker embedding). Consider the incidence variety

$$X = \{(W_2, W_3) | W_2 \in Y, W_2 \subset W_3\} \subset G(2,V') \times G(3,7).$$

The second projection is a variety in $G(3,7)$ with class $\sigma_{3,2}$. To see this, we can pair the variety with complementary dimensional Schubert cycles. Consider a general Schubert cycle $\Sigma_{a,b,c}$ such that $a+b+c = 7$ and $b \geq 3$. Then the linear spaces parameterized by this Schubert cycle have two-dimensional subspaces contained in a linear space W of dimension at most 3. Since $W \cap V' = 0$, we conclude that $\Sigma_{a,b,c} \cap X = \emptyset$. Similarly, the linear spaces parameterized by a $\Sigma_{3,2,2}$ are contained in a 5-dimensional linear space U. For a general such Schubert variety the two-dimensional $U \cap V'$ will not be a point of Y. Hence, $\Sigma_{3,2,2} \cap X = \emptyset$. Finally, for a general $\Sigma_{4,2,1}$, the intersection $\Sigma_{4,2,1} \cap X$ consists of one point. We conclude that X represents $\sigma_{3,2}$. It is clear that X is not a Schubert variety. On the other hand, $\sigma_{3,2}$ cannot be represented by a smooth subvariety of $G(3,7)$. Suppose it were represented by a smooth subvariety Y. Then the intersection of Y with a general Schubert variety $\sigma_{1,1,1}$ would be smooth by Kleiman's transversality theorem. However, this intersection has class $\sigma_{4,3,1}$, which is a rigid and singular class. We conclude that $\sigma_{3,2}$ cannot be represented by a smooth subvariety.

Definition 5.13. A partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ is a *non-smoothable* partition for $G(k,n)$ if either there exists an index $1 \leq s < j$ such that $i_s \neq 1$ and $n-k > \mu_s$ or there exists an index $1 \leq s < j$ such that $n - k > \mu_s \neq \mu_{s+1} + 1$.

Theorem 5.14. [C2, Theorem 1.6] *If λ be a non-smoothable partition for $G(k,n)$, then σ_λ cannot be represented by a smooth subvariety of $G(k,n)$.*

A complete characterization of smoothable Schubert classes is not known except in $G(2,n)$ and $G(3,n)$ (see [C2, Corollaries 4.5 and 4.6]).

Problem 5.15. *Classify smoothable Schubert classes in $G(k,n)$. For example, can $\sigma_{3,2,1,0}$ be represented by smooth subvarieties of $G(4,8)$?*

There are several other notions of smoothability. We will not say anything about them other than giving a few examples and raising some questions.

Problem 5.16. *When is (a multiple of) a Schubert class a positive linear combination of classes of smooth subvarieties of $G(k,n)$?*

Since Schubert classes span extremal rays of the effective cone, if $m\sigma_\lambda = \sum a_i[Z_i]$ with Z_i smooth subvarieties and $a_i > 0$, then each Z_i must have class proportional to σ_λ. Consequently, if σ_λ is multi rigid and a Schubert variety with class σ_λ is not smooth, then the class σ_λ cannot be a positive linear combination of classes of smooth subvarieties of $G(k,n)$. More generally, one can ask the following more interesting problem.

Problem 5.17. *When is (a multiple of) a Schubert class a linear combination of classes of smooth subvarieties of $G(k,n)$?*

Example 5.18. The class σ_2 in $G(2,5)$ can be expressed as $\sigma_2 = \sigma_1^2 - \sigma_{1,1}$. A Schubert variety $\Sigma_{1,1}$ is isomorphic to the Grassmannian $G(2,4)$ and is smooth. On the other hand, the class σ_1^2 can be represented by a codimension 2 linear section of the Grassmannian. By Bertini's Theorem, such a linear section can be chosen to be smooth. Hence, σ_2 can be represented as a linear combination of classes of smooth subvarieties of $G(2,5)$, even though it is rigid.

Remark 5.19. In contrast, Hartshorne, Rees and Thomas [HRT] have shown that σ_2 in $G(3,6)$ cannot be represented as a linear combination of the classes of closed smooth submanifolds of $G(3,6)$. In particular, it is not a linear combination of classes of smooth subvarieties of $G(k,n)$.

It is also interesting to ask these problems for classes other than Schubert classes.

Problem 5.20. *Given an effective cohomology class c in $G(k,n)$, determine when c can be represented by an irreducible subvariety. Determine when c can be represented by a smooth subvariety. Determine when c is a positive linear combination of classes of smooth subvarieties. Determine when c is a linear combination of classes of smooth subvarieties.*

Using geometric constructions it is possible to give examples of classes that can be represented by irreducible or smooth subvarieties of $G(k,n)$. However, at present a complete classification is far from known. One may ask the representative cycles to have other geometric properties.

Problem 5.21. *When can a cohomology class c in $G(k,n)$ be represented by a rational subvariety? When can it be represented by a rationally connected subvariety? When can it be represented by a smooth, rational or rationally connected subvariety?*

Remark 5.22. Projective space has the remarkable property that every effective cohomology class can be represented by a smooth irreducible subvariety. To represent m times the class of a linear space of dimension k, simply take a smooth hypersurface of degree m in a linear space of dimension $k+1$. Similarly, every effective cohomology class of $\mathbb{P}^n$ can be represented by a rational subvariety by taking the hypersurface to have an ordinary $(m-1)$-fold point. These are false for other Grassmannians since a large multiple of a multi rigid Schubert class cannot be represented by an irreducible subvariety. In particular, it cannot be represented by a rational subvariety.

Rigidity in $OG(k,n)$. In this subsection, we summarize some of the results on the rigidity and multi rigidity of Schubert classes in $OG(k,n)$ following [C5]. The next example describes the classification for $OG(1,n)$.

Example 5.23. The variety $OG(1,n)$ is a smooth quadric hypersurface Q in $\mathbb{P}^{n-1}$. The Schubert classes are

(1) Isotropic linear spaces $\mathbb{P}L_j$ for $0 \leq j < \frac{n}{2}$,
(2) If n is even, isotropic linear spaces $\mathbb{P}L_{\frac{n}{2}}$ and $\mathbb{P}L'_{\frac{n}{2}}$,
(3) The quadric sections $Q \cap \mathbb{P}Q_d^{n-d}$ for $n \geq d > \frac{n}{2}+1$.

The linear spaces are smooth and their classes are rigid. When $d < n$, the quadric sections $Q \cap \mathbb{P}Q_d^{n-d}$ are singular with the singular locus isomorphic to $\mathbb{P}^{n-d-1}$. The cohomology class of the Schubert variety $Q \cap \mathbb{P}Q_d^{n-d}$ is the same as the cohomology class of any linear section $Q \cap \mathbb{P}Q_d^r$ with $r \leq n-d$. The singular locus of this variety is isomorphic to $\mathbb{P}^{r-1}$. Hence, this variety is not isomorphic to a Schubert variety if $r < n-d$. Therefore, these classes are not rigid. In particular, since $Q \cap \mathbb{P}Q_d^0$ is a smooth quadric, every Schubert class in $OG(1,n)$ can be represented by a smooth subvariety of $OG(1,n)$. The classes of the linear spaces $\mathbb{P}L_j$, for $1 < j \leq \frac{n-1}{2}$, are rigid but not multi rigid. For example, twice the class of $\mathbb{P}L_j$ can be represented by a smooth quadric of the same dimension. If $2k=n$, the classes of the Schubert varieties $\mathbb{P}L_k$ and $\mathbb{P}L'_k$ are multi rigid [Ho1].

Example 5.23 shows that by deforming the quadrics to less singular quadrics, one can obtain deformations of Schubert varieties. Similarly, while the isotropic spaces are rigid, their multiples may be deformed to quadrics. One can use these two facts systematically to prove the failure of rigidity or multi rigidity for many Schubert classes. Therefore, restriction varieties play an important role in the study of rigidity. The next example illustrates the point.

Example 5.24. The orthogonal Grassmannian $OG(2,5)$ is isomorphic to $\mathbb{P}^3$. The codimension two Schubert varieties $\Sigma_{1;1}$ in $OG(2,5)$ are lines; however, not all lines in $OG(2,5)$ are Schubert varieties.

A Schubert variety $\Sigma_{1;1}$ is determined by specifying a point p on $Q \subset \mathbb{P}^4$. Projectively, the Schubert variety parameterizes lines in Q that contain the point p. In particular, the space of Schubert varieties with class $\sigma_{1;1}$ is a quadric threefold. Let $Q' \subset Q$ be a codimension one smooth quadric and let $l \subset Q'$ be a line. Then the space of lines that are contained in Q' and intersect l is also a line in $OG(2,5)$. Note that this is the restriction variety $V(L_2 \subset Q_4^0)$. The lines parameterized by the Schubert variety $\Sigma_{1;1}$ sweep out the singular quadric surface $T_pQ \cap Q$, whereas the lines parameterized by the restriction variety sweep out the smooth quadric surface Q'.

Since $OG(2,5)$ is isomorphic to $\mathbb{P}^3$, the space of lines in $OG(2,5)$ is isomorphic to the Grassmannian $G(2,4)$ parametrizing lines in $\mathbb{P}^3$. The Grassmannian $G(2,4)$ admits a map to $(\mathbb{P}^4)^*$ sending a point $q \in G(2,4)$ to the hyperplane in $\mathbb{P}^4$ spanned by the linear spaces parameterized by the line corresponding to q. This is a two-to-one map branched over the locus of Schubert varieties. This is one of the first examples where restriction varieties provide an explicit deformation of Schubert varieties. This example also shows that a Schubert class may be represented by a variety that is isomorphic, even projectively equivalent (under $GL(n)$ but not $SO(n)$), to a Schubert variety but is not a Schubert variety.

Exercise 5.25. Generalize the previous example to show that the Schubert classes $\sigma_{m,m-1,\dots,2;m-1}$ are not rigid in $OG(m, 2m+1)$. Describe the space of lines in $OG(m, 2m+1)$. Discuss the case of higher dimensional linear spaces.

We now use restriction varieties to give deformations of Schubert varieties more systematically.

Definition 5.26. A Schubert class $\sigma_{\lambda;\mu}$ in $OG(k,n)$ is of *Grassmannian type* if $s = k$ (equivalently, μ has length 0) and $\lambda_k > 0$. A Schubert class $\sigma_{\lambda;\mu}$ is of *quadric type* if $s = 0$ and $\mu_1 < \frac{n}{2} - 1$.

Notation 5.27. For discussing rigidity, it is more convenient to record the partitions $(\lambda;\mu)$ for $OG(k,n)$ slightly differently. Given λ define a sequence $a(\lambda)$ by setting $a_i = \lfloor \frac{n-1}{2} \rfloor - \lambda_i - i$. Record the sequence $a(\lambda)$ by grouping the equal terms $(\alpha_1^{i_1}, \dots, \alpha_t^{i_t})$ so that $\alpha_1 < \alpha_2 < \cdots < \alpha_t$ and α_j occurs with multiplicity i_j in the sequence $a(\lambda)$. Similarly, given μ define a sequence $b(\mu)$ by setting $b_j = n - \mu_j - j$. Record the sequence $b(\mu)$ also by grouping the equal terms $(\beta_1^{j_1}, \dots, \beta_u^{j_u})$ so that $\beta_1 < \cdots < \beta_u$ and β_l occurs with multiplicity j_l in the sequence $b(\mu)$.

Example 5.28. Given the partitions $(4,2,1;3,0)$ for $OG(5,13)$, we have $a(\lambda) = (1,2,2) = (1^1, 2^2)$ and $b(\mu) = (6,8)$.

The next two theorems characterize the rigidity of Schubert classes of Grassmannian and quadric type.

Theorem 5.29. [C5, Theorem 1.4] *Let $\sigma_{\lambda;}$ be a Schubert class of Grassmannian type in the cohomology of $OG(k,n)$. Express the associated partition $a(\lambda)$ by grouping the equal terms as $(\alpha_1^{i_1}, \dots, \alpha_t^{i_t})$. Then:*

1. *The class $\sigma_{\lambda;}$ is rigid if and only if there does not exist an index $1 \leq u < t$ such that $i_u = 1$ and $0 < \alpha_u = \alpha_{u+1} - 1$.*
2. *The class $\sigma_{\lambda;}$ is multi rigid if and only if $i_u \geq 2$ for every $2 \leq u \leq t$, $i_1 \geq 2$ unless $\alpha_1 = 0$, and $\alpha_u \leq \alpha_{u+1} - 2$ for every $1 \leq u < t$.*
3. *The class $\sigma_{\lambda;}$ is not smoothable if there exists an index $1 \leq u < t$ such that $0 < \alpha_u < \alpha_{u+1} - 1$, or an index $1 \leq u < t$ such that $\alpha_u > 0$ and $i_u > 1$.*

Exercise 5.30. Let $i : OG(k,n) \to G(k,n)$ denote the natural inclusion. Identify the image of a Schubert variety of Grassmannian type in $OG(k,n)$ as a Schubert variety in $G(k,n)$ under i. Show that the class of Grassmannian type $\sigma_{\lambda;}$ in $OG(k,n)$ is rigid (respectively, multi rigid) if and only if the corresponding class in $G(k,n)$ is rigid (respectively, multi rigid). Deduce parts (1) and (2) of the Theorem 5.29 from Theorems 5.9 and 5.11. Show that if the class is smoothable in $OG(k,n)$, then the corresponding class is smoothable in $G(k,n)$ and deduce part (3).

Theorem 5.31. [C5, Theorem 1.5] *Let $\sigma_{;\mu}$ be a Schubert class of quadric type in the cohomology of $OG(k,n)$. Express the partition $b(\mu)$ by grouping the equal terms as $(\beta_1^{i_1}, \dots, \beta_t^{i_t})$. Then:*

1. *The class $\sigma_{;\mu}$ is not rigid unless $t = 1$ and $\beta_1 = n - k$.*

(2) *The class $\sigma_{;\mu}$ is not smoothable if there exists an index $1 \leq u < t$ such that $\lfloor \frac{n+1}{2} \rfloor < \beta_u < \beta_{u+1} - 1$ or an index $1 \leq u < t$ such that $\lfloor \frac{n+1}{2} \rfloor < \beta_u$ and $i_u > 1$.*

Sketch of proof. Let $\sigma_{;\mu}$ be a Schubert class of quadric type in $OG(k,n)$. We then have $\mu_1 < \frac{n}{2} - 1$. Consequently, the dimension of $F_{\mu_1}^{\perp}$ is greater than $\frac{n}{2} + 1$. Since the corank of a quadric is bounded by its codimension, for a quadric $Q_{n-\mu_1}^{r_1}$, we have that $r_1 \leq \mu_1 \leq n - \mu_1 - 3$. In particular, $Q_{n-\mu_1}^{r_1}$ is irreducible. Let V be the restriction variety defined by the sequence $(Q_\bullet)$

$$Q_{n-\mu_1}^0 \subset Q_{n-\mu_2}^0 \subset \cdots \subset Q_{n-\mu_k}^0.$$

The i-th linear space in this sequence has the same dimension as the i-th linear space in the sequence defining the Schubert variety $\Sigma_{;\mu}$ but the restriction of the quadratic form is nondegenerate instead.

Exercise 5.32. Show that the sequence $(Q_\bullet)$ is admissible and V is a restriction variety. Using Algorithm 4.35 compute the class of V to see that it is $\sigma_{;\mu}$.

The exception $t = 1$ and $\beta_1 = n - k$ in the statement of the theorem corresponds to the fundamental class of $OG(k,n)$. In all other cases, we now show that the restriction variety constructed in the previous paragraph gives a non-trivial deformation of the Schubert variety. The linear spaces parameterized by V sweep out the quadric $Q_{n-\mu_k}^0$. Hence, if $\mu_k \neq 0$, the restriction variety cannot be projectively equivalent to a Schubert variety since for a Schubert variety the linear spaces sweep out a quadric of corank μ_k. If $\mu_k = 0$, since $t \neq 1$, there exists μ_u such that $\mu_u > k - u$. Let v be $\max\{u | b_u > k - u\}$. The smallest dimensional quadric that contains a v-dimensional subspace of every linear space parameterized by a Schubert variety $\Sigma_{b_\bullet}$ has corank b_v. In the restriction variety this quadric has the same dimension and has full rank. Therefore, we conclude that the restriction variety cannot be projectively equivalent to a Schubert variety. This concludes the proof that unless $t = 1$ and $\mu_k = 0$, a Schubert cycle of quadric type is not rigid. In fact, we have proved that such a class can always be represented by the intersection of a general Schubert variety in $G(k,n)$ with the orthogonal Grassmannian $OG(k,n)$.

Exercise 5.33. Consider the Schubert variety Σ in $OG(k,n)$ parameterizing isotropic subspaces contained in a maximal isotropic subspace. Notice that this Schubert variety is smooth and isomorphic to a Grassmannian $G(k, \lfloor \frac{n}{2} \rfloor)$. Deduce part (2) of the theorem from Theorem 5.29 by intersecting a representative of $\sigma_{;\mu}$ by a general translate of Σ. (Hint: if $\sigma_{;\mu}$ can be represented by a smooth subvariety, then, by Kleiman's Transversality Theorem, this intersection would be smooth.)

$\square$

Exercise 5.34. Using the fact that Schubert classes of quadric type in $OG(k,n)$ can be represented by the intersection of a Schubert variety in $G(k,n)$ with $OG(k,n)$ deduce the following corollary of the proof.

Corollary 5.35. *Let $\sigma_{;\mu}$ be a Schubert class of quadric type in $OG(k,n)$. If $t = 1$, then $\sigma_{;\mu}$ is smoothable.*

More generally, show that if the corresponding Schubert class in $G(k,n)$ is smoothable, then so is the Schubert class of quadric type in $OG(k,n)$.

We now turn our attention to more general cohomology classes.

Example 5.36. The quadric diagram associated to the Schubert class $\sigma_{4;3,1}$ in $OG(3,9)$ is 1]22000}00}0. By Algorithm 4.35, the restriction variety associated to the sequence 1]00000}00}0 has the same class but is not isomorphic to a Schubert variety. Similarly, the quadric diagram associated to the Schubert class $\sigma_{5;5,4,3,0}$ in $OG(5,13)$ is 22]234000}0}0}000}. The restriction variety associated to the quadric diagram 22]340000}0}0}000} has the same cohomology class but is not isomorphic to a Schubert variety.

The quadric diagram associated to the Schubert class $\sigma_{2;1}$ in $OG(2,5)$ is 1]000}0. The restriction variety associated to the sequence 00]00}0 also has the same class. More generally, the quadric diagram associated to the Schubert class $\sigma_{5,3,1;3,1}$ in $OG(5,11)$ is 1]22]00]000}00}0. The restriction variety associated to the sequence 1]200]0]000}00}0 has the same class. The latter varieties are not isomorphic to Schubert varieties.

By analyzing Algorithm 4.35, we can find restriction varieties that have the same class as (or a multiple of) a Schubert variety but are not isomorphic to a Schubert variety. The next two theorems use this strategy to show that the class is not rigid or multi rigid.

Theorem 5.37. [C5, Theorem 1.7] *Let $\sigma_{\lambda;\mu}$ be a Schubert class in $OG(k,n)$. Express the sequences $a(\lambda), b(\mu)$ by grouping the equal terms $(\alpha_1^{i_1}, \dots, \alpha_t^{i_t}; \beta_1^{j_1}, \dots, \beta_u^{j_u})$. Assume that one of the following conditions holds for $(\lambda;\mu)$:*

1. *$\beta_1 < n-k$ and $\mu_{s+j_1} \neq \lfloor \frac{n-1}{2} \rfloor - \lambda_i$ for any $1 \leq i \leq s$.*
2. *There exists an index $1 \leq u < t$ such that $i_u = 1$, $0 < \alpha_u = \alpha_{u+1} - 1$ and $\lfloor \frac{n-1}{2} \rfloor - \lambda_{i_1 + \dots + i_u} \neq \mu_j$ for any $s < j \leq k$.*
3. *$\#\{i | \lfloor \frac{n-1}{2} \rfloor - \lambda_i \leq n - \mu_{s+1}\} = s + \mu_{s+1} - \frac{n-3}{2}$ and there exists an index $1 \leq h \leq s$ such that $\lfloor \frac{n-1}{2} \rfloor - \lambda_h = n - \mu_{s+1}$.*

Then $\sigma_{\lambda;\mu}$ is not rigid.

Sketch of proof. The idea is to use restriction varieties to obtain deformations of Schubert varieties. Let $\sigma_{\lambda;\mu}$ be a Schubert class in the cohomology of $OG(k,n)$. For simplicity, set $\nu_i = \lfloor \frac{n-1}{2} \rfloor - \lambda_i$. First, assume that $\beta_1 < n-k$ and $\mu_{s+j_1} \neq \nu_i$ for any $1 \leq i \leq s$. By assumption, we have that $\mu_{s+j_1} = \mu_{s+j_1-1} - 1 = \dots = \mu_{s+1} - j_1 + 1$. We must have that either $\nu_i < b_{s+j_1}$ or $\nu_i > b_{s+1} + 1$ for every $1 \leq i \leq s$. Define an admissible sequence $(L_\bullet, Q_\bullet)$ by

$$L_{\nu_1} \subset \dots \subset L_{\nu_s} \subset Q^{\mu_{s+1}-1}_{n-\mu_{s+1}} \subset Q^{\mu_{s+2}-1}_{n-\mu_{s+2}} \subset \dots \subset Q^{\mu_{s+j_1}-1}_{n-\mu_{s+j_1}} \subset \dots \subset Q^{\mu_k}_{n-\mu_k}.$$

This sequence differs from the sequence defining the Schubert variety with class $\sigma_{\lambda;\mu}$ only in that the ranks of the quadrics $Q^{\mu_{s+i_1}-1}_{n-\mu_{s+i_1}}, \dots, Q^{\mu_{s+j_1}-1}_{n-\mu_{s+j_1}}$ are one more than the corresponding quadrics in the sequence associated to the Schubert variety.

Exercise 5.38. Show that this sequence is admissible and using Algorithm 4.35 compute that the cohomology class of the restriction variety $V(L_\bullet, Q_\bullet)$ is $\sigma_{\lambda;\mu}$. Show that $V(L_\bullet, Q_\bullet)$ is not isomorphic to the Schubert variety $\Sigma_{\lambda;\mu}$ and conclude that the class $\sigma_{\lambda;\mu}$ is not rigid.

Next, assume that there exists an index $1 \leq u < t$ such that $i_u = 1$, $\alpha_u = \alpha_{u+1} - 1$ and $\nu_{i_1 + \dots + i_u} \neq b_j$ for any $s < j \leq k$. For simplicity, set $h = \sum_{l=1}^u i_l$. There exists a subvariety Y of $G(h+1, F_{\nu_{h+1}})$ parameterizing $(h+1)$-dimensional subspaces $\Lambda \subset F_{\nu_{h+1}}$ that satisfy $\dim(\Lambda \cap F_{\nu_i}) \geq i$ for $i < h$ but is not a Schubert variety. Let Z be the Zariski closure of the following quasi-projective variety

$$\{W \in OG(k,n) \mid W \cap F_{\nu_{h+1}} \in Y, \dim(W \cap F_{\nu_i}) = i, \text{for } i \neq h, \dim(W \cap F^\perp_{\mu_j}) = j\}.$$

Then the class of Z is $\sigma_{\lambda;\mu}$ since specializing Y to a Schubert variety specializes Z to a Schubert variety. Furthermore, Z is not a Schubert variety. Therefore, the class $\sigma_{\lambda;\mu}$ is not rigid.

Finally, assume that Condition (3) of the theorem holds. Consider the restriction variety V associated to the sequence

$$L_{\nu_1} \subset \dots \subset L_{\nu_{h-1}} \subset L_{\nu_h + 1} \subset L_{\nu_{h+1}} \subset \dots \subset Q^{\mu_{s+1}-1}_{n-\mu_{s+1}} \subset \dots \subset Q^{\nu_k}_{n-\nu_k}.$$

Notice that this sequence differs from the sequence defining the Schubert variety in that the dimension of the h-th isotropic linear space is one larger and the corank of the smallest dimensional quadric is one smaller.

Exercise 5.39. Show that this sequence is admissible. Using Algorithm 4.35 compute that V has class $\sigma_{\lambda;\mu}$. Deduce that $\sigma_{\lambda;\mu}$ is not rigid.

□

One can use similar arguments to conclude the following.

Theorem 5.40. [C5, Theorem 1.8] *Let $\sigma_{\lambda;\mu}$ be a Schubert class in $OG(k,n)$. Express the sequences $a(\lambda), b(\mu)$ by grouping the equal terms $(\alpha_1^{i_1}, \dots, \alpha_t^{i_t}; \beta_1^{j_1}, \dots, \beta_u^{j_u})$. Assume that either one of the conditions in Theorem 5.37 or one of the following conditions holds for $(\lambda;\mu)$:*

1. *There exists an index $1 \leq u \leq t$ such that $\lfloor \frac{n-1}{2} \rfloor - \lambda_{i_1+\cdots+i_u} \neq \mu_j$ for any $s+1 \leq j \leq k$ and either $i_u = 1$ with $0 < \alpha_u < \frac{n}{2}$ or $\alpha_u = \alpha_{u+1} - 1$.*
2. *$\lambda_{s-1} > \lambda_s + 1 > 1$ and either $a_s + s > \mu_{s+1}$ or $a_s + s = \mu_j$ for some $s+1 \leq j \leq k$ and $\mu_j = \mu_{j-1} + 2 = \mu_{s+1} + j - s - 1$.*

Then $\sigma_{\lambda;\mu}$ is not multi rigid.

Example 5.41. Consider the Schubert variety $\sigma_{4,2;2,0}$ in $OG(4;11)$ with quadric diagram 22]00]00000}00}. Then the variety corresponding to the sequence 22]000}0000}00} has class $2\sigma_{4,2;2,0}$. The corresponding Schubert variety is not multi rigid.

It is also possible to prove the rigidity of certain Schubert classes. We refer the reader to [C5, Theorem 1.10]. As a consequence one can characterize rigid Schubert classes in $OG(2,n)$ if $n > 8$. Finally, Robles and The have classified the multi rigid Schubert classes in the spinor variety. Recall that in this case the sequence μ is uniquely determined from λ, so it suffices to specify the conditions on λ.

Theorem 5.42. [RT, Theorem 8.1] *Let $\sigma_{\lambda;\mu}$ be a Schubert class in the cohomology of the spinor variety $Sp(m,2m)$. Express the associated sequence $a(\lambda)$ by grouping the equal terms $(\alpha_1^{i_1}, \dots, \alpha_t^{i_t})$. Then $\sigma_{\lambda;\mu}$ is multi rigid if and only if*

1. *$i_l \geq 2$ and $\alpha_{l-1} \leq \mu_l - 2$ for all $1 < l \leq t$,*
2. *$i_1 \geq 2$ if $\alpha_1 > 0$ and $\lambda_s > 1$ if $\lambda_s > 0$.*

The following problem remains open in its full generality.

Problem 5.43. *Characterize rigid and multi rigid Schubert classes in all orthogonal Grassmannians $OG(k,n)$.*

Rigidity in $SG(k,n)$. As in the case of orthogonal Grassmannians, restriction varieties give deformations of Schubert varieties in $SG(k,n)$ under suitable numerical assumptions. Recall that in this case $n = 2m$ is even. The following example is typical.

Example 5.44. The Grassmannian $SG(1,n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, all the Schubert varieties $\mathbb{P}L_{n_j}$ are linear spaces. However, not all linear spaces are Schubert varieties. Points and codimension one linear spaces are always Schubert varieties. The restriction of Q to a codimension one linear space has a one-dimensional kernel W, hence it is of the form $W^{\perp}$. We conclude that points and codimension one linear spaces are rigid. Linear spaces $\mathbb{P}M$ with $1 < \dim(M) < n-1$ do not have to be isotropic, hence the corresponding Schubert classes are not rigid since they can be deformed to non-isotropic linear spaces.

The following theorem generalizes this example and can be proved by exhibiting explicit restriction varieties that have the same class as the Schubert variety but are not Schubert varieties.

Theorem 5.45. [C4, Theorem 6.2] *Let $\sigma_{\lambda_\bullet;\mu_\bullet}$ be a Schubert class in the cohomology of $SG(k,n)$.*

1. *If $s = 0$ and $\mu_j > k - j + 1$ for some j, then $\sigma_{\lambda_\bullet;\mu_\bullet}$ is not rigid.*
2. *If $s \geq 1$ and $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1} + 1)$, then $\sigma_{\lambda_\bullet;\mu_\bullet}$ is not rigid.*

Corollary 5.46. [C4, Corollary 6.3]

(1) *If the Schubert class $\sigma_{\mu_1-k+1,\mu_2-k+2,\dots,\mu_k}$ in the cohomology of $G(k,n)$ can be represented by a smooth subvariety of $G(k,n)$, then the Schubert class $\sigma_{;\mu_1,\dots,\mu_k}$ can also be represented by a smooth subvariety of $SG(k,n)$.*
(2) *If there exists an index $i < k$ such that $m - i - 1 > \mu_i > \mu_{i+1} + 2$ or if there exists an index $1 < i < k$ such that $m - i > \mu_{i-1} = \mu_i + 1 > \mu_{i+1} + 2$, then $\sigma_{;\mu_1,\dots,\mu_k}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*
(3) *If the Schubert class $\sigma_{\lambda_1,\dots,\lambda_k}$ in the cohomology of $G(k,m)$ can be represented by a smooth subvariety of $G(k,m)$, then the Schubert class $\sigma_{\lambda_1,\dots,\lambda_k;}$ in the cohomology of $SG(k,n)$ can be represented by a smooth subvariety of $SG(k,n)$.*
(4) *If there exists an index $i < k$ such that $i < \lambda_i < \lambda_{i+1} + 2$ or an index $1 < i < k-1$ such that $i - 1 < \lambda_{i-1} = \lambda_i - 1 < \lambda_{i+1} - 2$, then $\sigma_{\lambda_1,\dots,\lambda_k;}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*

Robles and The [RT] have characterized the multi rigid Schubert classes in Lagrangian Grassmannians. Express $a(\lambda)$ as $(\alpha_1^{i_1},\dots,\alpha_t^{i_t})$. They show that a Schubert class $\sigma_{\lambda;\mu}$ in $SG(m,2m)$ is multi rigid if and only if

(1) $i_l \geq 2$ and $\alpha_{l-1} \leq \alpha_l - 2$ for all $1 < l \leq t$,
(2) $i_1 \geq 2$ if $\alpha_1 > 0$ and $\lambda_s \geq 3$ if $\lambda_s > 1$.

The following problem remains largely open.

Problem 5.47. *Characterize the rigid, multi rigid and smoothable Schubert classes in flag varieties and isotropic flag varieties.*

References

[AC] R. Abdelkerim and I. Coskun, Parameter spaces of Schubert varieties in hyperplane sections of Grassmannians, J. Pure Appl. Algebra, **216** (2012), 800–810.
[Ad] S. Adalı, Singularities of restriction varieties in $OG(k,n)$, preprint.
[BS] A. Berenstein and R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion, J. Amer. Math. Soc., **13** no. 2 (2000), 433–466.
[Be] E. Bertini, *Introduzione alla geometria proiettiva degli iperspazi*, Enrico Spoerri, Pisa, 1907.
[BiC] S. Billey and I. Coskun, Singularities of generalized Richardson varieties, Comm. Alg., **40** no. 2 (2012), 1466–1495.
[BiL] S. Billey and V. Lakshmibai, *Singularities of Schubert varieties*, Springer, 2000.
[BH] A. Borel and A. Haefliger, La classe d'homologie fondamental d'un espace analytique, Bull. Soc. Math. France, **89** (1961), 461–513.
[Br] R. Bryant, *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces,* Princeton University Press Ann. Math. Studies AM–153, 2005.
[Bri] M. Brion, Lectures on the geometry of flag varieties, Trends in Mathematics: *Topics in cohomological studies of algebraic varieties*, Birkhäuser, 2005, 33–85.
[C1] I. Coskun, A Littlewood-Richardson rule for two-step flag varieties, Invent. Math., **176** no. 2 (2009), 325–395.
[C2] I. Coskun, Rigid and non-smoothable Schubert classes, J. Differential Geom., **87** no.3 (2011), 493–514.
[C3] I. Coskun, Restriction varieties and geometric branching rules, Adv. Math., **228** no.4 (2011), 2441–2502.
[C4] I. Coskun, Symplectic restriction varieties and geometric branching rules, *Clay Mathematics Proceedings*, **18** (2013), 205–239.
[C5] I. Coskun, Rigidity of Schubert classes in orthogonal Grassmannians, Israel J. Math., **200** (2014), 85–126.
[C6] I. Coskun, Symplectic restriction varieties and geometric branching rules II, J. Combin. Theory Ser. A, **125** (2014), 47–97.
[C7] I. Coskun, Degenerations of surface scrolls and Gromov-Witten invariants of Grassmannians, J. Algebraic Geom., **15** (2006), 223–284.
[C8] I. Coskun, The enumerative geometry of del Pezzo surfaces via degenerations, Amer. J. Math., **128** no. 3 (2006), 751–786.
[CR] I. Coskun and C. Robles, Flexibility of Schubert classes, Differ. Geom. Appl., **31** no. 6 (2013), 759–774.
[CV] I. Coskun and R. Vakil, Geometric positivity in the cohomology of homogeneous varieties and generalized Schubert calculus, *Algebraic Geometry, Seattle 2005*, volume 1, 77–124.

[DR] U. V. Desale and S. Ramanan, Classification of vector bundles of rank 2 on hyperelliptic curves, Invent. Math. **38**(1976), 161–185.

[EH1] D. Eisenbud, and J. Harris, On varieties of minimal degree (a centennial account), *Algebraic geometry, Bowdoin 1985*, Proc. Sympos. Pure Math., **46** no. 1, Amer. Math. Soc., Providence, RI, 1987.

[EH2] D. Eisenbud and J. Harris, *3264 & all that: Intersection theory in algebraic geometry*, to appear.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience, 1978.

[FP] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, volume 1689 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1998. Appendix J by the authors in collaboration with I. Ciocan-Fontanine.

[HRT] R. Hartshorne, E. Rees, and E. Thomas, Nonsmoothing of algebraic cycles on Grassmann varieties, Bull. Amer. Math. Soc., **80** no.5 (1974), 847–851.

[Ho1] J. Hong, Rigidity of smooth Schubert varieties in Hermitian symmetric spaces, Trans. Amer. Math. Soc., **359** (2007), 2361–2381.

[Ho2] J. Hong, Rigidity of singular Schubert varieties in $Gr(m,n)$, J. Differential Geom., **71** no. 1 (2005), 1–22.

[HM] J. Hong and N. Mok, Characterization of smooth Schubert varieties in rational homogeneous manifolds of Picard number 1, J. Algebraic Geom., **22** no.2 (2013), 333–362.

[K] S. Kleiman, The transversality of a general translate, Compos. Math., **28** (3) (1974), 287–297.

[KL] S. Kleiman and J. Landolfi, Geometry and deformation of special Schubert varieties, Compos. Math., **23** no. 4 (1971), 407–434.

[LS] V. Lakshmibai and C.S. Seshadri, Singular locus of a Schubert variety, Bull. Amer. Math. Soc., **11** no. 2 (1984), 363–366.

[MZ] N. Mok and Y. Zhang, Rigidity of pairs of rational homogeneous spaces of Picard number 1 and analytic continuation of geometric substructures on uniruled projective manifolds, preprint.

[P] K. Purbhoo, Vanishing and non-vanishing criteria in Schubert calculus, Int. Math. Res. Not., 2006, Article ID 24590, 38 pp.

[Pr1] P. Pragacz, A generalization of the Macdonald-You formula, J. Algebra, **204** no. 2 (1998), 573–587.

[Pr2] P. Pragacz, Addendum to "A generalization of the Macdonald-You formula", J. Algebra, **226** no. 1 (2000), 639–648.

[R] C. Robles, Schur flexibility of cominuscule Schubert varieties, Comm. Anal. Geom., **21** (2013), no. 5, 979–1013.

[RT] C. Robles and D. The, Rigid Schubert varieties in compact Hermitian symmetric spaces, Selecta Math., **18** no. 2 (2012), 717–777.

[V1] R. Vakil, A geometric Littlewood-Richardson rule, Annals of Math., **164** (2006), 371-422.

[V2] R. Vakil, The enumerative geometry of rational and elliptic curves in projective space, J. Reine Angew. Math., **529** (2000), 101–153.

[W] M. Walters, Geometry and uniqueness of some extreme subvarieties in complex Grassmannians, Ph. D. thesis, University of Michigan, 1997.

Department of Mathematics, Statistics and CS, University of Illinois at Chicago, Chicago, IL 60607
E-mail address: `coskun@math.uic.edu`