Effective Calabi–Yau pairs of Picard rank 2

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The Morrison–Kawamata conjecture predicts that for varieties that are "close to Calabi–Yau", the cones of nef and movable divisor are simple, up to symmetries of the variety. (See Section 1 for defintions and a precise statement.) The basic difficulty in proving the conjecture in general is the need to produce birational automorphisms of a variety: in simple terms, given X with $\mathrm{Nef}(X)$ or $\mathrm{Mov}(X)$ not rational polyhedral, one must show that X has an infinite group of automorphisms or birational automorphisms.

In this short note, we prove the Morrison–Kawamata conjecture for klt Calabi–Yau pairs (X, Δ) where X has Picard rank 2 and Δ is a nonzero effective divisor. In this situation the difficulty explained above disapears: the movable and nef cones are both rational polyhedral, spanned by effective divisors. The proof is an application of the Birkar–Cascini–Hacon–McKernan theorem that klt pairs of log general type have minimal models.

The situation of Calabi–Yau pairs of Picard number 2 was also considered by Zhang [Zh]. Building on work of Oguiso and Lazić–Peternell [LP], he showed that if the movable cone has one rational extremal ray, then the group of birational automorphisms must be finite. In light of the Morrison–Kawamata conjecture, this points the way towards our theorem, but Zhang's paper does not appear to contain the corresponding statement.

We also observe (Section 3) that the standard conjectures of the minimal model programme imply the stronger result that X is a Mori dream space.

Here is the main result.

Theorem 0.1. *Let (X, Δ) be a klt Calabi–Yau pair with $\rho(X) = 2$ and $\Delta \neq 0$. Then the closed movable cone $\mathrm{Mov}(X)$ and the nef cone $\mathrm{Nef}(X)$ are rational polyhedral, spanned by effective divisors.*

1 Preliminaries

We use the standard terminology from biratonal geometry and singularities of pairs.

A *klt Calabi–Yau* pair is a $\mathbf{Q}$-factorial klt pair (X, Δ) such that $K_X + \Delta \equiv 0$. (Note that Δ may be an $\mathbf{R}$-divisor.) For a $\mathbf{Q}$-factorial projective variety X, a *small $\mathbf{Q}$-factorial modification* (SQM for short) of X is a rational map $X \dashrightarrow X'$ to another $\mathbf{Q}$-factorial projective variety X' which is an isomorphism in codimension 1. A *pseudo-automorphism* of X means an SQM $X \dashrightarrow X$. The group of pseudo-automorphisms of X is denoted $\mathrm{PSAut}(X)$.

The nef cone of X is denoted $\mathrm{Nef}(X)$; the closed movable cone (that is, the closed cone generated by movable divisors) by $\mathrm{Mov}(X)$. The intersections of these cones with the cone of effective divisors are denoted $\mathrm{Nef}^e(X)$ and $\mathrm{Mov}^e(X)$. Note that $\mathrm{PSAut}(X)$ acts on $\mathrm{Mov}(X)$, preserving the subcone $\mathrm{Mov}^e(X)$. The pseudoeffective cone of X — that is, the closure of the cone of effective divisors — is denoted $\mathrm{PsEff}(X)$. Its interior, the big cone, is denoted $\mathrm{Big}(X)$.

Inspired by mirror symmetry, Morrison [Mor] proposed the following conjecture for strict Calabi–Yau manifolds. It was modified and generalised to terminal Calabi–Yau fibre spaces by Kawamata [Ka], and further extended to klt Calabi–Yau pairs in [Tot]. We state the conjecture in a simplified form here, sufficient for our purposes.

Conjecture 1.1 (Morrison–Kawamata)**.** *Let (X, Δ) be a klt Calabi–Yau pair. Then the actions of $Aut(X)$ and $PsAut(X)$ on the cones $\mathrm{Nef}^e(X)$ and $\mathrm{Mov}^e(X)$ have rational polyhedral fundamental domains.*

We will use the followinng celebrated theorem of Birkar–Cascini–Hacon–McKernan [BCHM, Theorem 1.2] repeatedly in our proof. (Our choice of notation for pairs is slightly unusual; the motivation is to avoid confusion between the fixed divisor Δ appearing in our klt pair and various other divisors Θ which will appear in the proof.)

Theorem 1.2 (BCHM)**.** *Let (X, Θ) be a klt pair of log general type, meaning that $K_X + \Theta$ is big. Then there exists a minimal model for $K_X + \Theta$.*

A consequence of this theorem is that if (X, Δ) is a klt Calabi–Yau pair, then for any big divisor Θ and ϵ sufficiently small (depending on Θ) the pair $(X, \Delta + \epsilon\Theta)$ is klt and of log general type, so there is a minimal model for Θ.

2 Proof of the main theorem

Recall that a $\mathbf{Q}$-factorial variety X is a *Mori dream space* if $b_1(X) = 0$ and the Cox ring $\mathrm{Cox}(X)$ is finitely generated. By the theorem of Hu–Keel (see Section 3), this implies that $\mathrm{Nef}(X)$ and $\mathrm{Mov}(X)$ are rational polyhedral, spanned by effective divisors. Birkar–Cascini–Hacon–McKernan [BCHM, Corollary 1.3.2] showed that dlt log Fano pairs, in particular all klt weak Fano varieties, are Mori dream spaces.

Proposition 2.1. *If $\Delta \in \mathrm{Mov}(X) \cap \mathrm{Big}(X)$, then Theorem 0.1 is true.*

Proof. Let X' be a minimal model for Δ: then since $\Delta \in \mathrm{Mov}(X)$ the rational map $X \dashrightarrow X'$ is an SQM. But then $-K_{X'} \equiv \Delta$ is nef and big on X', hence X' is a Mori dream space, so $\mathrm{Mov}(X') = \mathrm{Mov}(X)$ and $\mathrm{Nef}(X)$ are rational polyhedral, spanned by effective divisors. □

Proposition 2.2. *Any ray of $\mathrm{Mov}(X)$ that lies in $\mathrm{Big}(X)$ is rational, spanned by an effective divisor.*

Proof. Let M be a ray of $\mathrm{Mov}(X)$ lying in $\mathrm{Big}(X)$. Let Θ be a big divisor lying outside M; running the MMP for Θ, we see that M must correspond to a divisorial contraction of X. □

Corollary 2.3. *If $\mathrm{Mov}(X) \subseteq \mathrm{Big}(X)$, the statement about $\mathrm{Mov}(X)$ in Theorem 0.1 is true.*

So we can restrict attention to the case that $\mathrm{Mov}(X)$ has at least one ray that is not big.

Proposition 2.4. *At least one ray of $\mathrm{Mov}(X)$ is rational.*

Proof. Suppose neither ray is rational; we will obtain a contradiction.

If $\mathrm{Mov}(X) \neq \mathrm{PsEff}(X)$, then $\mathrm{Mov}(X)$ has a ray in $\mathrm{Big}(X)$, but then that ray must be rational by Proposition 2.2.

If $\mathrm{Mov}(X) = \mathrm{PsEff}(X)$ and neither ray is rational, then $\Delta \in \mathrm{Mov}(X) \cap \mathrm{Big}(X)$, but then by Proposition 2.1 $\mathrm{Mov}(X)$ is rational, a contradiction. □

By Proposition 2.1 we need only consider the case that Δ does not lie in the interior of $\mathrm{Mov}(X)$. Denote by M_1 the ray of $\mathrm{Mov}(X)$ closest to Δ.

Proposition 2.5. *The ray M_1 is rational, spanned by the class of an effective divisor.*

Proof. If M_1 lies in $\mathrm{Big}(X)$, it is rational and effective by Proposition 2.2. If not, then Δ must lie on M_1. Since Δ is an effective $\mathbf{Q}$-divisor, this proves the claim. □

So it remains to consider M_2.

Proposition 2.6. *The ray M_2 is rational, spanned by the class of an effective divisor.*

Proof. If M_2 lies in the big cone, then again it is rational and effective by Proposition 2.2. So assume M_2 is a boundary ray of $\mathrm{PsEff}(X)$.

Let $\Theta = (1-r)\Delta$ for any $0 < r < 1$. Then (X, Θ) is a klt pair and $K_X + \Theta \equiv -r\Delta$.

□

The final step is to prove that the nef cone of X is rational polyhedral. (Since it is a subcone of $\mathrm{Mov}(X)$, effectivity will then be automatic.)

Proposition 2.7. *The nef cone* $\mathrm{Nef}(X)$ *is rational polyhedral.*

Proof. Since $\mathrm{Mov}(X)$ is rational polyhedral, we need only prove that a ray R of $\mathrm{Nef}(X)$ which lies in the interior of $\mathrm{Mov}(X)$ is rational. But any such ray lies in the big cone, so we can find a big Cartier divisor D which lies on the opposite side of R from $\mathrm{Nef}(X)$. As before, we can choose a number $\epsilon > 0$ such that $(X, \Delta + \epsilon D)$ is klt; then running the $(K_X + \Delta + \epsilon D)$-MMP, we see that R corresponds to a flipping contraction, hence is rational. □

3 Cox ring

Hu and Keel [HK] proved that properties of the nef and movable cone are closely linked to the question of finite generation of the Cox ring:

Theorem 3.1 (Hu–Keel)**.** *Let X be a $\mathbf{Q}$-factorial projective variety with $b_1(X) = 0$. Then X is a Mori dream space if and only if the following conditions hold:*

(i) $\mathrm{Nef}(X)$ *is rational polyhedral, spanned by semi-ample line bundles.*

(ii) There is a finite collection of SQMs $X \dashrightarrow X_i$ such that $\mathrm{Mov}(X)$ *is the union of the nef cones* $\mathrm{Nef}(X_i)$*, and each nef cone satisfies the previous condition.*

We have proven that for (X, Δ) a klt Calabi–Yau pair, $\mathrm{Mov}(X)$ and $\mathrm{Nef}(X)$ are rational polyhedral, spanned by effective divisors; it seems reasonable to expect that X might in fact be a Mori dream space. We observe that this expectation is justified, insofar as it follows from our main theorem assuming the truth of the Minimal Model Conjecture and the Abundance Conjecture:

Theorem 3.2. *Let (X, Δ) be a klt Calabi–Yau pair with $b_1(X) = 0$ and $\rho(X) = 2$. Assuming the Minimal Model Conjecture and the Abundance Conjecture for klt pairs, X is a Mori dream space.*

Proof. We must verify the two conditions in Theorem 3.1.

First we prove that $\mathrm{Mov}(X)$ decomposes into a finite union of nef cones. Let $D_1, D_2 \in \mathrm{Mov}(X)$ be generators of the extremal rays M_1 and M_2 . Then each D_i is effective, so the minimal model conjecture implies there is a sequence of flips $X \dashrightarrow X_1^i \dashrightarrow X_2^i \dashrightarrow X_{n_i}^i$ such that D_i is nef on $X_{n_i}^i$. But then $\mathrm{Mov}(X)$ is the union of the finitely many nef cones $\mathrm{Nef}(X_k^i)\,(k = 1, \ldots, n_i,\, i = 1, 2)$.

Next we prove that a nef divisor on an SQM X_i is semi-ample. First, if D is any Cartier divisor in $\mathrm{Mov}(X) \cap B(X)$, when we pass to an SQM on which D is nef, then it is semi-ample by the Basepoint-free Theorem. In particular, any D in the interior of $\mathrm{Mov}(X)$ becomes semi-ample on an appropriate SQM.

For the extremal rays M_1 and M_2, we argue as follows. Note that for ϵ sufficiently small, both $(X, (1-\epsilon)\Delta)$ and $(X, (1+\epsilon)\Delta)$ are klt pairs.

Now $D_2 - (K_X + (1-\epsilon)\Delta) \equiv D_2 + \epsilon\Delta$ is big, and for ϵ small it lies in the same nef cone as D_2. So passing to an SQM on which D_2 is nef, the Basepoint-Free Theorem again says that D_2 is semi-ample.

As for M_1, again we can assume that it is an extremal ray of $\mathrm{PsEff}(X)$. By Proposition 2.1 we can assume that M_1 is spanned by Δ. Passing to a model X' on which Δ is nef, M_1 is the extremal ray of $\mathrm{Nef}(X')$ that does not lie in $\mathrm{Big}(X)$.

Now applying the abundance conjecture to $K_X + (1+\epsilon)\Delta$ for some small ϵ, we get that $K_X + (1+\epsilon)\Delta$ is $\mathbf{Q}$-linearly equivalent to the pullback of an ample $\mathbf{Q}$-divisor A on some contraction of X'. Then the pullback of A is a semiample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor class on X' which spans M_1. □

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