GRASSMANNIANS: THE FIRST EXAMPLE OF A MODULI SPACE

1. What is a moduli problem?

Many objects in algebraic geometry vary in algebraically defined families. For example, a conic in $\mathbb{P}^2$ has an equation of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0. \tag{1}$$

As the coefficients in Equation (1) vary, so do the conics defined in $\mathbb{P}^2$. In fact, two equations of the form (1) define the same conic if and only if they are non-zero multiples of each other. The family of conics in $\mathbb{P}^2$ is parameterized by the space of coefficients $(a, b, \dots, f)$ modulo scalars or the algebraic variety $\mathbb{P}^5$. In general, moduli theory studies the geometry of families of algebraic objects.

Let S be a Noetherian scheme of finite type over a field k. Throughout these notes, unless otherwise specified, schemes will be Noetherian and of finite type over a field or a base scheme.

A *moduli functor* is a contravariant functor from the category of S-schemes to the category of sets that associates to an S-scheme X the equivalence classes of families of geometric objects parameterized by X. Examples of such geometric objects include k-dimensional subspaces of an n-dimensional vector space, smooth curves of genus g, closed subschemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial, or stable vector bundles of degree d and rank r on a curve. Typical equivalence relations that one imposes on geometric objects include the trivial relation (for example, when considering k-dimensional subspaces of an n-dimensional linear space, two subspaces are considered equivalent if and only if they are equal), isomorphism (for example, when considering families of smooth curves of genus g, two families are considered equivalent if they are isomorphic) or projective equivalence (for example, when considering families of smooth curves of degree d and genus g in $\mathbb{P}^3$ up to the action of $\mathbb{P}GL(4)$). To make this more concrete consider the following three functors.

Example 1.1 (The Grassmannian Functor)**.** Let S be a scheme, E a vector bundle on S and r a positive integer less than the rank of E. Let

$$Gr(r, S, E) : \{\text{Schemes}/S\}^o \to \{\text{sets}\}$$

be the contravariant functor that associates to an S-scheme X subvector bundles of $X \times_S E$ of rank r and to a morphism $f : X \to Y$ the pull-back f^*.

Example 1.2 (The Hilbert Functor)**.** Let $X \to S$ be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed Hilbert polynomial. Let

$$Hilb_P(X/S) : \{\text{Schemes}/S\}^o \to \{\text{sets}\}$$

be the contravariant functor that associates to an S scheme Y the subschemes of $X \times_S Y$ which are proper and flat over Y and have the Hilbert polynomial P.

Example 1.3 (Moduli of smooth curves)**.** Let

$$\mathcal{M}_g : \{\text{Schemes}\}^o \to \{\text{sets}\}$$

be the functor that assigns to a scheme Z the set of isomorphism classes of families $X \to Z$ flat over Z whose fibers are smooth curves of genus g and to a morphism between schemes the pull-back families.

Each of the functors in the three examples above poses a moduli problem. We would like to understand these families of geometric objects. The first step in the solution of such a problem is to have a description of the set of families parameterized by the scheme X. Ideally, we would like to endow the set of families with some additional algebraic structure.

Recall that the *functor of points* of a scheme X is the functor $h_X : \{\text{Schemes}\}^o \to \{\text{sets}\}$ that associates to a scheme Y the set of morphisms $\text{Hom}(Y, X)$ and to each morphisms $f : Y \to Z$, the map of sets $h_X(f) : \text{Hom}(Z, X) \to \text{Hom}(Y, X)$ given by composition $g \circ f$. By Yoneda's Lemma, the functor of points h_X determines the scheme X. More precisely, if two functor of points h_X and h_Y are isomorphic as functors, then X and Y are isomorphic as schemes.

Definition 1.4. A contravariant functor $F : \{\text{Schemes}/S\}^o \to \{\text{sets}\}$ is called *representable* if there exists a scheme M such that F is isomorphic to the functor of points h_M of M. When M exists, it is called a *fine moduli space* for F.

To make this slightly more concrete, the functor F is represented by a scheme M over S and an element $U \in F(M)$ if for every S scheme Y, the map

$$\text{Hom}_S(Y, M) \to F(Y)$$

given by $g \to g^*U$ is an isomorphism. If it exists, M is unique up to a unique isomorphism.

The ideal situation is for a functor to be represented by a scheme (as happens in Examples 1.1 and 1.2). However, for many moduli problems (such as in Example 1.3), there does not exist a scheme representing the functor. There are two common ways of dealing with this issue. First, we may relax the requirement of representing the functor in the category of schemes and look instead for an algebraic space, Deligne-Mumford stack or, more generally, an Artin stack representing the functor. Second, we can weaken our notion of representing a functor. The most common alternative we will pursue in these notes will be to seek a scheme that coarsely represents the functor.

Definition 1.5. Given a contravariant functor F from schemes over S to sets, we say that a scheme $X(F)$ over S *coarsely represents* the functor F if there is a natural transformation of functors

$$\Phi : F \to \mathrm{Hom}_S(*, X(F))$$

such that

(1) $\Phi(spec(k)) : F(spec(k)) \to \mathrm{Hom}_S(spec(k), X(F))$ is a bijection for every algebraically closed field k,
(2) For any S-scheme Y and any natural transformation

$$\Psi : F \to \mathrm{Hom}_S(*, Y),$$

there is a unique natural transformation

$$\Pi : \mathrm{Hom}_S(*, X(F)) \to \mathrm{Hom}_S(*, Y)$$

such that $\Psi = \Pi \circ \Phi$.

If $X(F)$ coarsely represents F, then it is called a *coarse moduli space*.

Observe that a coarse moduli space, if it exists, is unique up to isomorphism.

Exercise 1.6. The fine/coarse moduli spaces are endowed with a scheme structure and carry more subtle information than the underlying set parameterizing the geometric objects. To appreciate the distinction, show that $\mathbb{P}^1_{\mathbb{C}}$ is a fine moduli space for one dimensional subspaces of $\mathbb{C}^2$. Let C be the cuspidal cubic curve in $\mathbb{P}^2$ defined by $y^2z = x^3 + x^2z$. Although the map $f : \mathbb{P}^1 \to C$ given by $(u,t) \mapsto (u^2t, u^3, t^3)$ is a bijection between $\mathbb{P}^1$ and C, show that C is not a coarse moduli space parameterizing one dimensional subspaces of $\mathbb{C}^2$.

Finding a moduli space, that is a scheme or stack (finely or coarsely) representing a functor, is only the first step of a moduli problem. Usually the motivation for constructing a moduli space is to understand the objects this space parameterizes. This in turn requires a good knowledge of the geometry of the moduli space itself. The questions about moduli spaces that we will be concerned about in these notes include:

(1) Is the moduli space proper? If not, does it have a modular compactification? Is the moduli space projective?
(2) What is the dimension of the moduli space? Is it connected? Is it irreducible? What are its singularities?
(3) What is the cohomology/Chow ring of the moduli space?
(4) What is the Picard group of the moduli space? Assuming the moduli space is projective, which of the divisors are ample? Which of the divisors are effective?
(5) Can the moduli space be rationally parameterized? What is the Kodaira dimension of the moduli space?
(6) Can one run the Mori program on the moduli space? What are the different birational models of the moduli space? Do they in turn have modular interpretations?

The second step of the moduli problem is answering as many of these questions as possible. The focus of this course will be the second step of the moduli problem. In this course, we will not concentrate on the constructions of the moduli spaces. We will often stop at outlining the main steps of the constructions only in so far as they help us understand the geometry. We will spend most of the time talking about the explicit geometry of these moduli spaces.

We begin our study with the Grassmannian. The Grassmannian is the scheme that represents the functor in Example 1.1. Grassmannians lie at the heart of moduli theory. Their existence is a major step in the proof of the existence of the Hilbert scheme. Many moduli spaces we will discuss in turn can be constructed as quotients of Hilbert schemes. More importantly, almost every construction in moduli theory is inspired by or mimics some aspect of Grassmannian geometry. For example, the cohomology ring of the Grassmannian is generated by the Chern classes of tautological bundles. Similarly, the cohomology of some important moduli spaces, like the Quot scheme on $\mathbb{P}^1$ or the moduli space of stable vector bundles of rank r and degree d with fixed determinant over a curve, can be understood in terms of tautological classes constructed via a universal family or a universal bundle. Even when the Chern classes of tautological bundles are far from generating the cohomology ring, as in the case of the moduli space of curves of genus g, they still generate an important subring of the cohomology. The Grassmannian has a natural stratification given by Schubert cells. Similarly, several stratifications of the moduli spaces we discuss, such as the topological stratification of the moduli space of curves, will pay an important role. Finally, there are fast algorithms such as the Littlewood-Richardson rule for computing intersection products of Schubert cycles in the Grassmannian. We will study similar algorithms for multiplying geometrically defined classes in the Kontsevich moduli spaces and the moduli space of curves of genus g. To study many aspects of moduli theory in a simple setting motivates us to begin our exploration with the Grassmannian.

Additional references: For a more detailed introduction to moduli problems you should read [HM] Chapter 1 Section A, [H] Lecture 21, [EH] Section VI and [K] Section I.1.

2. Preliminaries about the Grassmannian

There are many good references for the geometry of Grassmannians. I especially recommend [H] Lectures 6 and 16, [GH] Chapter I.5, [Ful2] Chapter 14, and the two papers [Kl2] and [KL].

Let V be an n-dimensional vector space. $G(k,n)$ is the Grassmannian that parameterizes k-dimensional linear subspaces of V. $G(k,n)$ naturally carries the structure of a smooth, projective variety of dimension $k(n-k)$. It is often convenient to think of $G(k,n)$ as the parameter space of $(k-1)$-dimensional projective linear spaces in $\mathbb{P}^{n-1}$. When using this point of view, it is customary to denote the Grassmannian by $\mathbb{G}(k-1,n-1)$.

We will now give $G(k,n)$ the structure of an abstract variety. Given a k-dimensional subspace Ω of V, we can represent it by a $k\times n$ matrix. Choose a basis $v_1,\dots,v_k$ for Ω and form a matrix with $v_1,\dots v_k$ as the row vectors

$$M = \begin{pmatrix} \vec{v_1} \\ \dots \\ \vec{v_k} \end{pmatrix}.$$

The general linear group $GL(k)$ acts on the set of $k \times n$ matrices by left multiplication. This action corresponds to changing the basis of Ω. Therefore, two $k \times n$ matrices represent the same linear space if and only if they are in the same orbit of the action of $GL(k)$. Since the k vectors span Ω, the matrix M has rank k. Hence, M has a non-vanishing $k \times k$ minor. Consider the Zariski open set of matrices that have a fixed non-vanishing $k \times k$ minor. We can normalize M so that this submatrix is the identity matrix. For example, if the $k \times k$ minor consists of the first k columns, the normalized matrix has the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & \cdots & * \\ & & & \cdots & & & \\ 0 & 0 & \cdots & 1 & * & \cdots & * \end{pmatrix}.$$

A normalized matrix gives a unique representation for the vector space Ω. The space of normalized matrices such that a fixed $k \times k$ minor is the identity matrix is affine space $\mathbb{A}^{k(n-k)}$. When a matrix has two non-zero $k \times k$ minors, the transition from one representation to another is clearly given by algebraic functions. We thus endow $G(k,n)$ with the structure of a $k(n-k)$ dimensional abstract variety. When $V = \mathbb{C}^n$, this construction endows $G(k,n)$ with the structure of a complex manifold of dimension $k(n-k)$. Since the unitary group $U(n)$ maps continuously onto $G(k,n)$, we conclude that $G(k,n)$ is a connected, compact complex manifold.

So far we have treated the Grassmannian simply as an abstract variety. However, it is easy to endow it with the structure of a smooth, projective variety. We now describe the Plücker embedding of $G(k,n)$ into $\mathbb{P}(\bigwedge^k V)$. Given a k-plane Ω, choose a basis for it $v_1, \ldots, v_k$. The Plücker map $Pl : G(k,n) \to \mathbb{P}(\bigwedge^k V)$ is defined by sending the k-plane Ω to $v_1 \wedge \cdots \wedge v_k$. If we pick a different basis $w_1, \ldots, w_k$ for Ω, then

$$w_1 \wedge \cdots \wedge w_k = \det(M) v_1 \wedge \cdots \wedge v_k,$$

where M is the matrix giving the change of basis of Ω from $v_1, \ldots, v_k$ to $w_1, \ldots, w_k$. Hence, the map Pl is a well-defined map independent of the chosen basis.

The map Pl is injective since we can recover Ω from its image $p = [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V)$ as the set of all vectors $v \in V$ such that $v \wedge v_1 \wedge \cdots \wedge v_k = 0$. We say that a vector in $\bigwedge^k V$ is *completely decomposable* if it can be expressed as $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for k vectors $v_1, \ldots, v_k \in V$.

Exercise 2.1. When $1 < k < \dim V$, most vectors in $\bigwedge^k V$ are not completely decomposable. Show, for example, that $e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 V$ is not completely decomposable if e_1, e_2, e_3, e_4 is a basis for V.

A point of $\mathbb{P}(\bigwedge^k V)$ is in the image of the map Pl if and only if the representative $\sum p_{i_1,\dots,i_k} e_1 \wedge \cdots \wedge e_{i_k}$ is completely decomposable. It is not hard to characterize the subvariety of $\mathbb{P}(\bigwedge^k V)$ corresponding to completely decomposable elements. Given a vector $u \in V^*$, we can define a contraction

$$u\lrcorner : \bigwedge^k V \to \bigwedge^{k-1} V$$

by setting

$$u\lrcorner(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} u(v_i)\ v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k$$

and extending linearly. The contraction map extends naturally to $u \in \bigwedge^j V^*$. An element $x \in \bigwedge^k V$ is completely decomposable if and only if $(u\lrcorner x) \wedge x = 0$ for every $u \in \bigwedge^{k-1} V^*$. We can express these conditions in coordinates. Choose a basis $e_1, \dots, e_n$ for V and let $u_1, \dots, u_n$ be the dual basis for V^*. Expressing the condition $(u\lrcorner x) \wedge x = 0$ in these coordinates, for every distinct set of $k-1$ indices $i_1, \dots, i_{k-1}$ and a disjoint set of $k+1$ distinct indices $j_1, \dots, j_{k+1}$, we obtain the Plücker relation

$$\sum_{t=1}^{k+1} (-1)^s p_{i_1,\dots,i_{k-1},j_t} p_{j_1,\dots,\hat{j}_t,\dots j_{k+1}} = 0.$$

The set of all Plücker relations generates the ideal of the Grassmannian.

Example 2.2. The simplest and everyone's favorite example is $G(2,4)$. In this case, there is a unique Plücker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

The Plücker map embeds $G(2,4)$ in $\mathbb{P}^5$ as a smooth quadric hypersurface.

Exercise 2.3. Write down all the Plücker relations for $G(2,5)$ and $G(3,6)$. Prove that the Plücker relations generate the ideal of $G(k,n)$.

We can summarize our discussion in the following theorem.

Theorem 2.4. *The Grassmannian $G(k,n)$ is a smooth, irreducible, rational, projective variety of dimension $k(n-k)$.*

The cohomology ring of the complex Grassmannian (and more generally, the Chow ring of the Grassmannian) can be very explicitly described. Fix a flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V,$$

where F_i is an i-dimensional subspace of V. Let λ be a partition with k parts satisfying the conditions

$$n - k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

We will call partitions satisfying these properties *admissible partitions*. Given a flag $F_\bullet$ and an admissible partition λ, we can define a subvariety of the Grassmannian called the *Schubert variety* $\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ of type λ with respect to the flag $F_\bullet$ to be

$$\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet) := \{\ [\Omega] \in G(k,n)\ :\ \dim(\Omega \cap F_{n-k+i-\lambda_i}) \geq i\ \}.$$

The homology and cohomology classes of a Schubert variety depend only on the partition λ and do not depend on the choice of flag. For each partition λ, we get a homology class and a cohomology class (Poincaré dual to the homology class). When writing partitions, it is customary to omit the parts that are equal to zero. We will follow this custom and write, for example, Σ_1 instead of $\Sigma_{1,0}$.

The Schubert classes give an additive basis for the cohomology ring of the Grassmannian. In order to prove this, it is useful to introduce a stratification of $G(k,n)$. Pick an ordered basis $e_1, e_2, \ldots, e_n$ of V and let $F_\bullet$ be the standard flag for V defined by setting $F_i =< e_1, \ldots, e_i >$. The Schubert cell $\Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ is defined as $\{[\Omega] \in G(k,n) \mid$

$$\dim(\Omega \cap F_j) = \left\{ \begin{array}{lc} 0 & \text{for } j < n-k+1-\lambda_1 \\ i & \text{for } n-k+i-\lambda_i \leq j < n-k+i+1-\lambda_{i+1} \\ k & \text{for } n-\lambda_k \leq j \end{array} \right\}.$$

Given a partition λ, define the *weight* of the partition to be

$$|\lambda| = \sum_{i=1}^{k} \lambda_i.$$

The Schubert cell $\Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$. For $\Omega \in \Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ we can uniquely choose a distinguished basis so that the matrix having as rows this basis has the form

$$\begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 \\ & & & & \cdots & & & & & & & \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

where the only non-zero entry in the $(n-k+i-\lambda_i)$-th column is a 1 in row i and all the (i,j) entries are 0 if $j > n-k+i-\lambda_i$. Thus, we see that the Schubert cell is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$. We can express the Grassmannian as a disjoint union of these Schubert cells

$$G(k,n) = \bigsqcup_{\lambda \text{ admissible}} \Sigma^c_\lambda(F_\bullet).$$

Let $\sigma_{\lambda_1,\dots,\lambda_k}$ denote the cohomology class Poincaré dual to the fundamental class of the Schubert variety $\Sigma_{\lambda_1,\dots,\lambda_k}$. Since the Grassmannian has a cellular decomposition where all the cells have even real dimension, we conclude the following theorem.

Theorem 2.5. *The integral cohomology ring $H^*(G(k,n),\mathbb{Z})$ is torsion free. The classes of Schubert varieties σ_λ as λ varies over admissible partitions give an additive basis of $H^*(G(k,n),\mathbb{Z})$.*

Exercise 2.6. Deduce as a corollary of Theorem 2.5 that the Euler characteristic of $G(k,n)$ is $\binom{n}{k}$. Compute the Betti numbers of $G(k,n)$.

Exercise 2.7. Using the fact that $G(k,n)$ has a stratification by affine spaces, prove that the Schubert cycles give an additive basis of the Chow ring of $G(k,n)$. Show that the cycle map from the Chow ring of $G(k,n)$ to the cohomology ring is an isomorphism.

Example 2.8. To make the previous discussion more concrete, let us describe the Schubert varieties in $G(2,4) = \mathbb{G}(1,3)$. For drawing pictures, it is more convenient to use the projective viewpoint of lines in $\mathbb{P}^3$. The possible admissible partitions are $(1),(1,1),(2),(2,1),(2,2)$ and the empty partition. A flag in $\mathbb{P}^3$ corresponds to a choice of point q contained in a line l contained in a plane P contained in $\mathbb{P}^3$.

(1) The codimension 1 Schubert variety Σ_1 parameterizes lines that intersect the line l.
(2) The codimension 2 Schubert variety $\Sigma_{1,1}$ parameterizes lines that are contained in the plane P.
(3) The codimension 2 Schubert variety Σ_2 parameterizes lines that pass through the point p.
(4) The codimension 3 Schubert variety $\Sigma_{2,1}$ parameterizes lines in the plane P and that pass through the point p.
(5) The codimension 4 Schubert variety $\Sigma_{2,2}$ is a point corresponding to the line l.

A pictorial representation of these Schubert varieties is given in the next figure.

Exercise 2.9. Following the previous example, work out the explicit geometric description of all the Schubert varieties in $G(2,5)$ and $G(3,6)$.

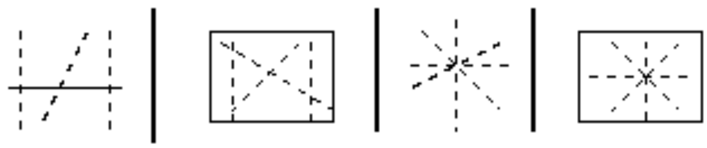

FIGURE 1. Pictorial representations of $\Sigma_1, \Sigma_{1,1}, \Sigma_2$ and $\Sigma_{2,1}$, respectively.

Since the cohomology of Grassmannians is generated by Schubert cycles, given two Schubert cycles σ_λ and σ_μ, their product in the cohomology ring can be expressed as a linear combination of Schubert cycles.

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \; \sigma_\nu$$

The structure constants $c^\nu_{\lambda,\mu}$ of the cohomology ring with respect to the Schubert basis are known as Littlewood - Richardson coefficients.

We will describe several methods for computing the Littlewood-Richardson coefficients. It is crucial to know when two varieties intersect transversely. Kleiman's Transversality Theorem provides a very useful criterion for ascertaining that the intersection of two varieties in a homogeneous space is transverse. Here we will recall the statement and sketch a proof. For a more detailed treatment see [Kl1] or [Ha] Theorem III.10.8.

Theorem 2.10. *(Kleiman) Let k be an algebraically closed field. Let G be an integral algebraic group scheme over k and let X be an integral algebraic scheme with a transitive G action. Let $f : Y \to X$ and $f' : Z \to X$ be two maps of integral algebraic schemes. For each rational element of $g \in G$, denote by gY the X-scheme given by $y \mapsto gf(y)$.*

(1) *Then there exists a dense open subset U of G such that for every rational element $g \in U$, the fiber product $(gY) \times_X Z$ is either empty or equidimensional of the expected dimension*

$$\dim(Y) + \dim(Z) - \dim(X).$$

(2) *If the characteristic of k is zero and Y and Z are regular, then there exists an open, dense subset U' of G such that for $g \in U'$, the fiber product $(gY) \times_X Z$ is regular.*

Proof. The theorem follows from the following lemma.

Lemma 2.11. *Suppose all the schemes in the following diagram are integral over an algebraically closed field k.*

$$\begin{array}{ccccc} & W & & & Z \\ p \swarrow & & q \searrow & r \swarrow & \\ S & & & X & \end{array}$$

If q is flat, then there exists a dense open subset of S such that $p^{-1}(s)\times_X Z$ is empty or equidimensional of dimension

$$\dim(p^{-1}(s)) + \dim(Z) - \dim(X).$$

If in addition, the characteristic of k is zero, Z is regular and q has regular fibers, then $p^{-1}(s) \times_X Z$ is regular for a dense open subset of S.

The theorem follows by taking $S = G$, $W = G \times Y$ and $q : G \times Y \to X$ given by $q(g, y) = gf(y)$. The lemma follows by flatness and generic smoothness. More precisely, since q is flat, the fibers of q are equidimensional of dimension $\dim(W) - \dim(X)$. By base change the induced map $W \times_X Z \to Z$ is also flat, hence the fibers have dimension $\dim(W \times_X Z) - \dim(Z)$. Consequently,

$$\dim(W \times_X Z) = \dim(W) + \dim(Z) - \dim(X).$$

There is an open subset $U_1 \subset S$ over which p is flat, so the fibers are either empty or equidimensional with dimension $\dim(W) - \dim(S)$. Similarly there is an open subset $U_2 \subset S$, where the fibers of $p \circ pr_W : X \times_X Z \to S$ is either empty or equidimensional of dimension $\dim(X \times_X Z) - \dim(S)$. The first part of the lemma follows by taking $U = U_1 \cap U_2$ and combining these dimension statements. The second statement follows by generic smoothness. This is where we use the assumption that the characteristic is zero. $\square$

The Grassmannians $G(k, n)$ are homogeneous under the action of $GL(n)$. Taking $f : Y \to G(k, n)$ and $f' : Z \to G(k, n)$ to be the inclusion of two subvarieties in Kleiman's transversality theorem, we conclude that $gY \cap Z$ is either empty or a proper intersection for a general $g \in GL(n)$. Furthermore, if the characteristic is zero and Y and Z are smooth, then $gY \cap Z$ is smooth. In particular, the intersection is transverse. Hence, Kleiman's Theorem is an extremely powerful tool for computing products in the cohomology.

Example 2.8 continued. Let us work out the Littlewood - Richardson coefficients of $G(2, 4) = \mathbb{G}(1, 3)$. It is simplest to work dually with the intersection of Schubert varieties. Suppose we wanted to calculate $\Sigma_2 \cap \Sigma_2$. Σ_2 is the class of lines that pass through a point. If we take two distinct points, there will be a unique line containing them both. We conclude that $\Sigma_2 \cap \Sigma_2 = \Sigma_{2,2}$. By Kleiman's transversality theorem, we know that the intersection is transverse. Therefore, this equality is a scheme theoretic equality. Similarly, $\Sigma_{1,1} \cap \Sigma_{1,1} = \Sigma_{2,2}$, because there is a unique line contained in two distinct planes in $\mathbb{P}^3$. On the other hand $\Sigma_{1,1} \cap \Sigma_2 = 0$ since there will not be a line contained in a plane and passing through a point not contained in the plane.

The hardest class to compute is $\Sigma_1 \cap \Sigma_1$. Since Schubert classes give an additive basis of the cohomology, we know that $\Sigma_1 \cap \Sigma_1$ is expressible as a linear combination of $\Sigma_{1,1}$ and Σ_2. Suppose

$$\Sigma_1 \cap \Sigma_1 = a\Sigma_{1,1} + b\Sigma_2$$

We just computed that both $\Sigma_{1,1}$ and Σ_2 are self-dual cycles. In order to compute the coefficient we can calculate the triple intersection. $\Sigma_1 \cap \Sigma_1 \cap \Sigma_2$ is the set of lines that meet two lines l_1, l_2 and contain a point q. There is a unique such line given by $\overline{ql_1} \cap \overline{ql_2}$. The other coefficient can be similarly computed to see $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

Exercise 2.12. Work out the multiplicative structure of the cohomology ring of $G(2,4) = \mathbb{G}(1,3)$, $G(2,5) = \mathbb{G}(1,4)$ and $G(3,6) = \mathbb{G}(2,5)$.

In the calculations for $G(2,4)$, it was important to find a dual basis to the Schubert cycles in $H^4(G(2,4), \mathbb{Z})$. Given an admissible partition λ, we define a dual partition λ^* by setting $\lambda_i^* = n-k-\lambda_{k-i+1}$. Pictorially, if the partition λ is represented by a Young diagram inside a $k \times (n-k)$ box, the dual partition λ^* is the partition complementary to λ in the $k \times (n-k)$ box.

Exercise 2.13. Show that the dual of the Schubert cycle $\sigma_{\lambda_1,\dots,\lambda_k}$ is the Schubert cycle $\sigma_{n-k-\lambda_k,\dots,n-k-\lambda_1}$. Conclude that the Littlewood - Richardson coefficient $c_{\lambda,\mu}^{\nu}$ may be computed as the triple product $\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^*}$.

The method of undetermined coefficients we just employed is a powerful technique for calculating the classes of subvarieties of the Grassmannian. Let us do an example to show another use of the technique.

Example 2.14. How many lines are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$? In order to work out this problem we can calculate the class of lines contained in a quadric hypersurface in $\mathbb{P}^4$ and square the class. The dimension of the space of lines on a quadric hypersurface is 3. The classes of dimension 3 in $\mathbb{G}(1,4)$ are given by σ_3 and $\sigma_{2,1}$. We can, therefore, write this class as $a\sigma_3 + b\sigma_{2,1}$. The coefficient of σ_3 is zero because σ_3 is self-dual and corresponds to lines that pass through a point. As long as the quadric hypersurface does not contain the point, the intersection will be zero.

On the other hand, $b = 4$. $\Sigma_{2,1}$ parameterizes lines in $\mathbb{P}^4$ that intersect a $\mathbb{P}^1$ and are contained in a $\mathbb{P}^3$ containing the $\mathbb{P}^1$. The intersection of the quadric hypersurface with the $\mathbb{P}^3$ is a quadric surface. The lines have to be contained in this surface and must pass through the two points of intersection of the $\mathbb{P}^1$ with the quadric surface. There are four such lines. We conclude that there are 16 lines that are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$.

Exercise 2.15. Another way to verify that there are 16 lines in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$ is to observe that such an intersection is a quartic Del Pezzo surface D_4. Such a surface is the blow-up of $\mathbb{P}^2$ at 5 general points embedded by its anti-canonical linear system. Check that the lines in this embedding correspond to the (-1)-curves on the surface and show that the number of (-1)-curves on this surface is 16 (see [Ha] Chapter 5).

Exercise 2.16. Let C be a smooth, complex, irreducible, non-degenerate curve of degree d and genus g in $\mathbb{P}^3$. Compute the class of the variety of lines that are secant to C.

We now give two presentations for the cohomology ring of the Grassmannian. These presentations are useful for theoretical computations. However, we will soon develop Littlewood-Richardson rules, positive combinatorial rules for computing Littlewood-Richardson coefficients, that are much more effective in computing and understanding the structure of the cohomology ring of $G(k, n)$.

A partition λ with $\lambda_2 = \cdots = \lambda_k = 0$ is called a *special partition.* A Schubert cycle defined with respect to a special partition is called a *Pieri cycle.* Pieri's rule is a formula for multiplying an arbitrary Schubert cycle with a Pieri cycle.

Theorem 2.17 (Pieri's formula)**.** *Let σ_λ be a Pieri cycle. Suppose σ_μ is any Schubert cycle with parts $\mu_1, \ldots, \mu_k$. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\mu_i \leq \nu_i \leq \mu_{i-1} \\ |\nu| = |\lambda| + |\mu|}} \sigma_\nu \tag{2}$$

Exercise 2.18. Prove Pieri's formula.

Exercise 2.19. Show that the locus where a Plücker coordinate vanishes corresponds to a Schubert variety Σ_1. Observe that the class of Σ_1 generates the second homology of the Grassmannian. In particular, the Picard group is isomorphic to $\mathbb{Z}$. Conclude that $\mathcal{O}_{G(k,n)}(\Sigma_1)$ is the very ample generator of the Picard group and it gives rise to the Plücker embedding.

Exercise 2.20. Compute the degree of the Grassmannian $G(k,n)$ under the Plücker embedding. The answer is provided by $\sigma_1^{k(n-k)}$. When $k=2$, this computation is relatively easy to carry out. By Pieri's formula σ_1 times any cycle in $G(2,n)$ either increases the first index of the cycle or it increases the second index provided that it is less than the first index. This means that the degree of the Grassmannian $G(2,n)$ is the number of ways of walking from one corner of an $(n-2)\times(n-2)$ to the opposite corner without crossing the diagonal. This is well-known to be the Catalan number

$$\frac{(2(n-2))!}{(n-2)!(n-1)!}.$$

The general formula is more involved. The degree of $G(k,n)$ is given by

$$(k(n-k))!\prod_{i=1}^{k}\frac{(i-1)!}{(n-k+i-1)!}.$$

The special Schubert cycles generate the cohomology ring of the Grassmannian. In order to prove this we have to express every Schubert cycle $\sigma_{\lambda_1,\dots,\lambda_k}$ as a linear combination of products of special Schubert cycles. Consider the following example

$$\sigma_{4,3,2} = \sigma_2\cdot\sigma_{4,3} - \sigma_4\cdot\sigma_{4,1} + \sigma_6\cdot\sigma_{2,1}.$$

To check this equality, using Pieri's rule expand the products.

$$\sigma_2\cdot\sigma_{4,3} = \sigma_{4,3,2}+\sigma_{4,4,1}+\sigma_{5,3,1}+\sigma_{5,4}+\sigma_{6,3}$$

$$\sigma_4\cdot\sigma_{4,1} = \sigma_{4,4,1}+\sigma_{5,3,1}+\sigma_{6,2,1}+\sigma_{7,1,1}+\sigma_{5,4}+\sigma_{6,3}+\sigma_{7,2}+\sigma_{8,1}$$

$$\sigma_6\cdot\sigma_{2,1} = \sigma_{7,1,1}+\sigma_{6,2,1}+\sigma_{7,2}+\sigma_{8,1}$$

Note the following features of this calculation. The class $\sigma_{4,3,2}$ only occurs in the first product. All other products occur twice with different signs.

Exercise 2.21. Using Pieri's formula generalize the preceding example to prove the following identity

$$(-1)^k\sigma_{\lambda_1,\dots,\lambda_k} = \sum_{j=1}^{k}(-1)^j\sigma_{\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1}-1,\dots,\lambda_k-1}\cdot\sigma_{\lambda_j+k-j}$$

Theorem 2.22 (Giambelli's formula)**.** *Any Schubert cycle may be expressed as a linear combination of products of special Schubert cycles as follows*

$$\sigma_{\lambda_1,\dots,\lambda_k} = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \sigma_{\lambda_2+k-2} \\ & \cdots & \cdots & \cdots & \\ \sigma_{\lambda_k-k+1} & \sigma_{\lambda_k-k+2} & \sigma_{\lambda_k-k+3} & \cdots & \sigma_{\lambda_k} \end{vmatrix}$$

Exercise 2.23. Expand the determinant by the last column and use the previous exercise to prove Giambelli's formula by induction.

Exercise 2.24. Use Giambelli's formula to express $\sigma_{3,2,1}$ in $G(4,8)$ in terms of special Schubert cycles. Using Pieri's rule find the class of its square.

Pieri's formula and Giambelli's formula together give an algorithm for computing the cup product of any two Schubert cycles. Unfortunately, in practice this algorithm is tedious to use. We will rectify this problem shortly.

Before continuing the discussion, we recall a few basic facts about Chern classes. The reader should refer to [Ful2] 3.2 and 14.4 for more details. Given a vector bundle $E \to X$ of rank r on a smooth, complex projective variety, one associates a collection of classes $c_i(E) \in H^{2i}(X)$ called *Chern classes* and a *total Chern class*

$$c(E) = 1 + c_1(E) + \cdots + c_r(E)$$

satisfying the following properties:

(1) $c_i(E) = 0$ for all $i > 0$.
(2) For any exact sequence
$$0 \to E_1 \to E_2 \to E_3 \to 0$$
of vector bundles on X, the total Chern classes satisfy the Whitney sum formula $c(E_2) = c(E_1)c(E_3)$.
(3) If $E = \mathcal{O}_X(D)$ is a line bundle, then $c_1(E) = [D]$.
(4) If $f : Y \to X$ is a proper morphism, then the following projection formula holds
$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$
for every cycle α on Y and every i.
(5) If $f : Y \to X$ is a flat morphism, then the following pull-back formula holds
$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$
for all cycles α on X and all i.

Exercise 2.25. Using the Euler sequence and the Whitney sum formula, show that the total Chern class of the tangent bundle of $\mathbb{P}^n$ is $(1+h)^{n=1}$ in $\mathbb{Z}[h]/(h^{n+1})$, where h is the hyperplane class.

A very important consequence of the Whitney sum formula and the pull-back formula is the splitting principle, which allows one to compute the Chern classes of any vector bundle obtained from E via linear algebraic operations (for example, $E^*, \bigwedge^s E, E \otimes E$) in terms of the Chern classes of E. One factors the total Chern class

$$c(E) = \prod_{j=1}^{r}(1+\alpha_j)$$

into a product of Chern roots so that $c_i(E)$ is the i-th elementary symmetric polynomial of the Chern roots. For example, $c_1(E) = \sum_{j=1}^{r} \alpha_j$ and $c_2(E) = \sum_{i<j} \alpha_i \alpha_j$. Then the Chern roots of a vector bundle obtained from E via linear algebraic operations can be expressed by applying the linear algebraic operations to the roots.

For example, if the Chern roots of E are α_j, then the Chern roots of the dual E^* are $-\alpha_j$. Hence, by the splitting principle

$$c(E^*) = \prod_{j=1}^{r}(1-\alpha_j).$$

We conclude that $c_i(E^*) = (-1)^i c_i(E)$.

Exercise 2.26. As an other important example, let L be a line bundle with $c_1(L) = \beta$, then

$$c(E \otimes L) = \prod_{j}(1+\beta+\alpha_j)$$

Conclude that

$$c_i(E) = \sum_{j=1}^{i}(-1)^{i-j}\binom{r-j}{i-j} c_1(L)^{j-i} c_j(E \otimes L).$$

Exercise 2.27. Let the Chern roots of E be $\alpha_1, \dots, \alpha_r$ and let the Chern roots of F be $\beta_1, \dots, \beta_s$. Using the splitting principle, calculate that

$$c(\bigwedge^m E) = \prod_{i_1<i_2<\cdots<i_m} (1+\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_m}).$$

Using the splitting principle calculate that

$$c(E \otimes F) = \prod_{i,j}(1+\alpha_i+\beta_j).$$

Geometrically, the Chern class $c_i(E)$ can be interpreted as a degeneracy locus (assuming that E has sufficiently many sections). More precisely, let $s_1, \ldots, s_r$ be r sections such that for any $i \leq r$, the set

$$D_i(E) = \{x \in X \mid s_1(x), \ldots, s_{r-i+1}(x) \text{ are dependent } \}$$

is empty or has pure dimension i. Then $c_i(E)$ is the class of this degeneracy locus D_i. If E does not have enough global sections, we can tensor E with a sufficiently ample line bundle L such that $E \otimes L$ has r global sections with the property that the degeneracy locus $D_i(E \otimes L)$ is empty or of pure dimension i. Using Exercise 2.26 we can obtain the Chern classes of E from the Chern classes of $E \otimes L$ and L.

One extremely useful way of presenting the cohomology of the Grassmannian comes from considering the universal exact sequence of bundles on $G(k,n)$. Let U denote the tautological bundle over $G(k,n)$. Recall that the fiber of U over a point $[\Omega]$ is the vector subspace Ω of V. There is a natural inclusion

$$0 \to U \to \underline{V} \to Q \to 0$$

with quotient bundle Q.

Theorem 2.28. *As a ring the cohomology ring of $G(k,n)$ is isomorphic to*

$$\mathbb{R}[c_1(U), \ldots, c_k(U), c_1(Q), \ldots, c_{n-k}(Q)]/(c(U)c(Q) = 1).$$

Moreover, the chern classes of the Quotient bundle generate the cohomology ring.

The Chern classes of the tautological bundle and the quotient bundle are easy to see in terms of Schubert cycles. As an exercise prove the following proposition:

Proposition 2.29. *The chern classes of the tautological bundle are given as follows:*

$$c_i(U) = (-1)^i \sigma_{1,\ldots,1}$$

where there are i ones. The chern classes of the quotient bundle are given by

$$c_i(Q) = \sigma_i.$$

Example 2.30. We can calculate the number of lines that are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$ yet another way. A quadric hypersurface Q is a section of the vector bundle $\mathrm{Sym}^2(U^*)$ over $G(2,5)$. The locus of lines contained in Q is the locus where the section defined by Q vanishes. Hence, this class can be computed as $c_3(\mathrm{Sym}^2(U^*))$. If the Chern roots of U are α, β, then using the splitting principle, $c_3(\mathrm{Sym}^2(U^*)) = 4\alpha\beta(\alpha+\beta) = 4\sigma_{2,1}$. We recover that there are 16 lines in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$.

Exercise 2.31. Calculate the number of lines on a general cubic hypersurface in $\mathbb{P}^3$. More generally, calculate the class of the variety of lines contained in a general cubic hypersurface in $\mathbb{P}^n$.

Exercise 2.32. Calculate the number of lines on a general quintic threefold.

Exercise 2.33. Calculate the number of lines contained in a general pencil of quartic surfaces in $\mathbb{P}^3$. Carry out the same calculation for a general pencil of sextic hypersurfaces in $\mathbb{P}^4$.

The local structure of the Grassmannian. The tangent bundle of the Grassmannian has a simple intrinsic description in terms of the tautological bundle U and the quotient bundle Q. There is a natural identification of the tangent bundle of the Grassmannian with homomorphisms from U to Q, in other words

$$TG(k,n) = \mathrm{Hom}(U,Q).$$

In particular, the tangent space to the Grassmannian at a point $[\Omega]$ is given by $\mathrm{Hom}(\Omega, V/\Omega)$. One way to realize this identification is to note that the Grassmannian is a homogeneous space for $GL(n)$. The tangent space at a point may be naturally identified with quotient of the Lie algebra of $GL(n)$ by the Lie algebra of the stabilizer. The Lie algebra of $GL(n)$ is the endomorphisms of V. Those that stabilize Ω are those homomorphisms $\phi : V \to V$ such that $\phi(\Omega) \subset \Omega$. These homomorphisms are precisely homomorphisms $\mathrm{Hom}(\Omega, V/\Omega)$.

Exercise 2.34. Use the above description to obtain a description of the tangent space to the Schubert variety $\Sigma_{\lambda_1,\dots,\lambda_k}$ at a smooth point $[\Omega]$ of the variety.

We can use the description of the tangent space to check that the intersection of Schubert cycles in previous calculations were indeed transverse. For example, suppose we take the intersection of two Schubert varieties Σ_1 in $\mathbb{G}(1,3)$ defined with respect to two skew-lines. Then the intersection is a smooth variety. In vector space notation, we can assume that the conditions are imposed by two non-intersecting two-dimensional vector spaces V_1 and V_2. Suppose a 2-dimensional vector space Ω meets each in dimension 1. The tangent space to Ω at the intersection is given by

$$\phi \in \mathrm{Hom}(\Omega, V/\Omega) \ \text{ such that } \ \phi(\Omega \cap V_i) \subset [V_i] \in V/\Omega.$$

As long as V_1 and V_2 do not intersect, Ω has exactly a one-dimensional intersection with each of V_i and these span Ω. On the other hand, the quotient of V_i in V/Ω is one-dimensional. We conclude that the dimension of such homomorphisms is 2. Since this is equal to the dimension of the variety, we deduce that the variety is smooth.

Exercise 2.35. Carry out a similar analysis for the other examples we did above.

Using the description of the tangent bundle, we can calculate the canonical class of $G(k,n)$. We use the splitting principle for Chern classes. Let $\alpha_1, \dots, \alpha_k$ be the Chern roots of S^*. We then have the equation

$$c(S^*) = \prod_{i=1}^{k}(1+\alpha_i) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,1,\dots,1}.$$

Similarly, let $\beta_1, \dots, \beta_{n-k}$ be the Chern roots of Q. We then have the equation

$$c(Q) = \prod_{j=1}^{n-k}(1+\beta_j) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k}.$$

The Chern classes of the tangent bundle can be expressed as

$$c(TG(k,n)) = c(S^* \otimes Q) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1+\alpha_i+\beta_j).$$

In particular, the first Chern class is equal to $n\sigma_1$. Since this class is n times the ample generator of the Picard group, we conclude the following theorem.

Theorem 2.36. *The canonical class of $G(k,n)$ is equal to $-n\sigma_1$. $G(k,n)$ is a Fano variety of Picard number one and index n.*

Definition 2.37. Let S be a scheme, E a vector bundle on S and k a natural number less than or equal to the rank of E. The functor

$$Gr(k,E) : \{\text{schemes over} S\} \to \{\text{sets}\}$$

associates to every S scheme X the set of rank k subvector bundles of $E \times_S X$.

Theorem 2.38. *The functor $Gr(k,E)$ is represented by a scheme $G_S(k,E)$ and a subvector bundle $U \subset E \times_S G_S(k,E)$ of rank k.*

3. The Geometric Littlewood-Richardson rule

Positive combinatorial rules for determining Littlewood-Richardson coefficients are known as Littlewood-Richardson rules. As an introduction to the degeneration techniques that we will employ through out this course, we give a Littlewood-Richardson rule for the Grassmannian.

There are many Littlewood-Richardson rules for the Grassmannian. You can find other Littlewood-Richardson rules in [Ful1], [V1], [KT]. The rule we will develop here is a geometric Littlewood-Richardson rule. These rules have many applications in geometry. For some examples of applications to positive characteristic, Schubert calculus over $\mathbb{R}$ and monodromy groups see [V2].

Notation 3.1. Let $e_1, \dots, e_n$ be an ordered basis for the vector space V. Let $[e_i, e_j]$ denote the vector space spanned by the consecutive basis elements $e_i, e_{i+1}, \dots, e_{j-1}, e_j$. Given a vector space W spanned by a subset of the basis elements, let $l(W)$ and $r(W)$ denote the basis elements with the smallest, respectively, largest index in W. Given a basis element e_i, let $\#e_i = i$ denote the index of the basis element.

The fundamental example. Consider calculating σ_1^2 in $G(2,4)$. Geometrically we would like to calculate the class of two dimensional linear spaces that meet two general two dimensional linear spaces in a four dimensional vector space. Projectivizing this question is equivalent to asking for the class of lines in $\mathbb{P}^3$ intersecting two general lines.

The idea underlying the approach to answering this question is classical. While it is hard to see the Schubert cycles that constitute this intersection when the two lines that define the two Schubert cycles are general, the result becomes easier if the lines are in special position.

To put the lines l_1 and l_2 in a special position fix a plane containing l_1 and rotate it about a point on it, so that it intersects l_2. As long as l_1 and l_2 do not intersect, they are in general position since the automorphism group of $\mathbb{P}^3$ acts transitively on pairs of skew lines. However, when l_1 and l_2 intersect, then they are no longer in general position.

We can ask the following fundamental question: What is the limiting position of the lines that intersect both l_1 and l_2? Since intersecting the lines are closed conditions, any limit line has to continue to intersect l_1 and l_2. There are two ways that a line can intersect two intersecting lines in $\mathbb{P}^3$. Either the line passes through their intersection point, or if it does not pass through its intersection point then it must lie in the plane spanned by the two lines. Note that these are both Schubert cycles. Since their dimensions are equal to the dimension of the original variety, the class of the original variety has to be the sum of multiples of these two Schubert cycles.

We can determine that the multiplicities are one by a local calculation. It suffices to check that the two cycles in special position intersect generically transversely. Suppose that in the special position the two lines l_1 and l_2 to be the projectivization of the vector spaces $[e_2, e_3]$ and $[e_3, e_4]$, respectively. Let m_1 and m_2 be the two lines that are the projectivization of the vector spaces spanned by $[e_1, e_3]$ and $[e_2, e_4]$, respectively. m_1 and m_2 are contained in the two Schubert cycles of lines intersecting l_i for $i = 1, 2$. In fact, m_1 lies in the component of intersection corresponding to lines that contain e_3 and m_2 lines in the component of the intersection corresponding to lines that are contained in the plane spanned by $[e_2, e_3, e_4]$. The intersection of the tangent spaces to the two Schubert varieties at m_1 is expressed as

$$\{\phi \in \mathrm{Hom}([e_1, e_3], \frac{V}{[e_1, e_3]}) \mid \phi(e_3) \in \frac{[e_2, e_3]}{[e_1, e_3]} \text{ and } \phi(e_3) \in \frac{[e_3, e_4]}{[e_1, e_3]}\}.$$

Since this is clearly a two dimensional space, we conclude that the two Schubert varieties intersect transversely at m_1. Similarly, the intersection of the tangent spaces to the two Schubert varieties at m_2 can be expressed as

$$\{\phi \in \mathrm{Hom}([e_2, e_4], \frac{V}{[e_2, e_4]}) \mid \phi(e_2) \in \frac{[e_2, e_3]}{[e_2, e_4]} \text{ and } \phi(e_4) \in \frac{[e_3, e_4]}{[e_2, e_4]}\}.$$

Since this is also a two dimensional subspace, we conclude that the two Schubert varieties intersect generically transversely and that the multiplicities are equal to one.

We now generalize this example to arbitrary Grassmannians and the product of arbitrary Schubert classes. In fact, we will give a rule for computing the classes of a much larger class of varieties.

Definition 3.2. A *Mondrian tableau* M for $G(k, n)$ consists of a set of k vector spaces $M = \{W_1, \dots, W_k\}$ spanned by consecutive basis elements of the basis $e_1, \dots, e_n$ such that for every i, j

$$\dim(\overline{W_i W_j}) \geq \dim W_i + \#\{W_k \in M \mid W_k \subset \overline{W_i W_j} \text{ and } W_k \not\subset W_i\}. \quad (3)$$

Notation 3.3. We depict a Mondrian tableau as a set of k squares in an $n \times n$ grid. We imagine placing the ordered basis along the diagonal unit squares of the $n \times n$ grid in increasing order from southwest to northeast. We then depict the vector space $[e_i, e_j]$ with the square whose diagonal consists of the unit squares labeled $e_i, \dots, e_j$. In order not to clutter the pictures, we omit the labels of the diagonal unit squares. See Figure 4 for some examples of Mondrian tableaux.

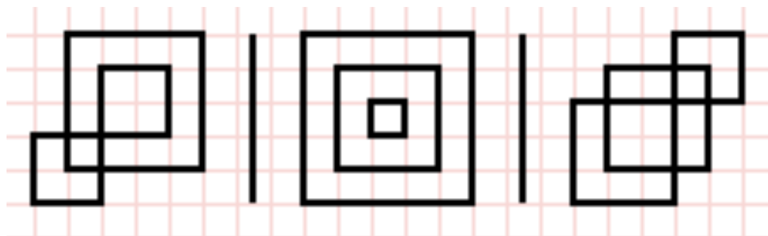

FIGURE 2. Three Mondrian tableaux for $G(3,5)$. Proceeding from left to right, the Mondrian tableaux depicted are $M_1 = \{[e_1, e_2], [e_2, e_5], [e_3, e_4]\}$, $M_2 = \{[e_1, e_5], [e_2, e_4], [e_3]\}$, $M_3 = \{[e_1, e_3], [e_2, e_4], [e_4, e_5]\}$.

Remark 3.4. Mondrian tableaux are named in honor of the Dutch painter Piet Mondrian who painted numerous canvases reminiscent of Mondrian diagrams.

To a Mondrian tableau M we associate a subvariety of the Grassmannian $X(M) \subset G(k,n)$.

Definition 3.5. Let M be a Mondrian tableau for $G(k,n)$. The Mondrian variety $X(M)$ associated to M is the Zariski closure of the locus of k-planes that admit a basis $v_1, \ldots, v_k$ such that $v_i \in W_i$.

Exercise 3.6. (1) Show that the variety associated to a Mondrian tableau with one vector space $[e_i, e_j]$ is isomorphic to $\mathbb{P}^{j-i}$.
(2) Show that the variety associated to a Mondrian tableau with two vector spaces $[e_i, e_j] \subset [e_k, e_l]$ is isomorphic to a Schubert variety in $G(2, l-k+1)$.
(3) Show that the variety associated to a Mondrian tableau with two vector spaces $[e_i, e_j], [e_k, e_l]$ with $j < k$ is isomorphic to a product of projective spaces $\mathbb{P}^{j-i} \times \mathbb{P}^{l-k}$.

Exercise 3.7. Show that the variety $X(M)$ associated to M can alternatively be described as the locus of k-planes Ω such that the dimension of intersection of Ω with any vector space W spanned by a set of basis elements is greater than or equal to the number of subspaces of M contained in W

$$\dim(\Omega \cap W) \geq \#\{W_i \in M \mid W_i \subseteq W\}.$$

Theorem 3.8. *The variety $X(M)$ associated to a Mondrian tableau M is irreducible of dimension*

$$\dim(X(M)) = \sum_{i=1}^{k} \dim(W_i) - \sum_{i=1}^{k} \#\{W_j \in M \mid W_j \subseteq W_i\}.$$

Proof. This theorem is proved by induction on the number of vector spaces constituting M. If M consists of one vector space $[e_i, e_j]$, then $X(M) \cong \mathbb{P}^{j-i}$ and the theorem holds. Let W_i be the largest dimensional vector space in M with minimal $\#l(W)$. Let M' be the Mondrian tableau obtained by omitting W_i. Then there is a dominant morphism from a Zariski open dense subset of $X(M)$ to $X(M')$. Consider the Zariski open subset U of $X(M)$ consisting of k-planes Ω that have a basis $v_1, \ldots, v_k$ with $v_j \in W_j$ such that v_i is not contained in the span of the intersections of Ω with W_j for $j \neq i$. Note that this set is non-empty by the definition of a Mondrian tableau. Mapping Ω to the span of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$, we obtain a morphism ϕ from U to $X(M')$. The map ϕ is dominant. We can assume by induction that $X(M')$ is irreducible of dimension

$$\sum_{j=1, j\neq i}^{k} \dim(W_i) - \sum_{j=1, j\neq i}^{k} \#\{W_l \in M' \mid W_l \subseteq W_j\}.$$

The fiber of ϕ over a point Ω' in the image of ϕ consists of k-planes that contain Ω' and are spanned by Ω' and a vector $v_i \in W_i$. The k-plane then is determined by the choice of v_i which is allowed to vary in an open subset of a projective space of dimension

$$\dim(W_i) - \#\{W_j \in M \mid W_j \subseteq W_i\}.$$

The irreducibility and dimension formula follow from these considerations. $\square$

Exercise 3.9. Show that if the vector spaces constituting a Mondrian tableau M are totally ordered by inclusion $W_1 \subset \cdots \subset W_k$, then $X(M)$ is a Schubert variety. In this case, verify the dimension formula in Theorem 3.8 directly.

Definition 3.10. We will call any Mondrian tableau M with the property that the vector spaces $W_i \in M$ are totally ordered by inclusion $W_1 \subset \cdots \subset W_k$ and $\dim(W_i) = n - k + i - \lambda_i$ a *Mondrian tableau associated to the Schubert cycle* $\sigma_{\lambda_1,\ldots,\lambda_k}$.

The intersection of two Schubert varieties in general position can also be represented by a Mondrian tableau. Let λ and μ be two partitions such that $\lambda_i + \mu_{k-i+1} \leq n-k$ for every i. The product of the two Schubert cycles σ_λ and σ_μ is non-zero if and only if this condition is satisfied. Represent the cycle σ_λ by a totally ordered Mondrian tableau $\Lambda_1 \subset \cdots \subset \Lambda_k$ such that $l(\Lambda_i) = e_1$ for every i. Represent σ_μ by the totally ordered Mondrian tableau $\Gamma_1 \subset \cdots \subset \Gamma_k$ such that $r(\Gamma_i) = e_n$. Let the *Mondrian tableau* $M(\lambda, \mu)$ *associated to the product* $\sigma_\lambda \cdot \sigma_\mu$ be the Mondrian tableau with vector spaces $W_i = \Lambda_i \cap \Gamma_{k-i+1}$. See Figure 3 for an example.

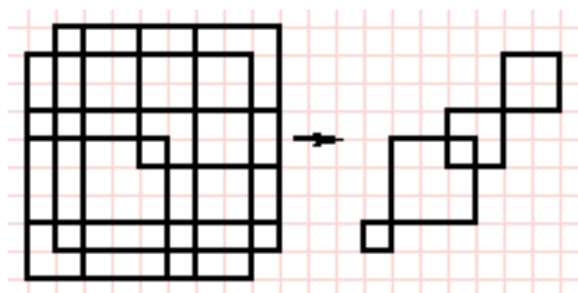

FIGURE 3. The Mondrian tableau representing the product $\sigma_{4,2,2,1} \cdot \sigma_{3,2,1,1}$ in $G(4,9)$. The panel on the left depicts the Mondrian tableaux depicting the two Schubert varieties superimposed on the same $n \times n$ grid.

Exercise 3.11. Show that $X(M(\lambda,\mu))$ is equal to the intersection $\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(G_\bullet)$, where $F_i = [e_1, e_i]$ and $G_j = [e_n, e_{n-j+1}]$.

Exercise 3.12. Show that if M is a Mondrian tableau representing the product of two Schubert cycles in $G(k,n)$, then $W_i \not\subset W_j$ for any $i \neq j$. Conversely, show that any Mondrian tableau satisfying this property represents the product of two Schubert cycles in a Grassmannian.

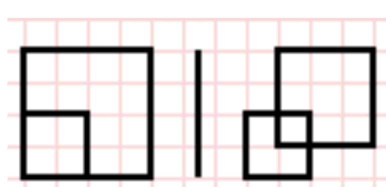

FIGURE 4. Two Mondrian tableaux M_1, M_2 with $X(M_1) = X(M_2)$.

The same variety can be represented by different Mondrian tableaux. For example, consider the two Mondrian tableaux $M_1 = \{[e_1, e_2], [e_1, e_4]\}$ and $M_2 = \{[e_1, e_2], [e_2, e_4]\}$ for $G(2,4)$ depicted in Figure **??**. Requiring a two dimensional subspace to be contained in $[e_1, e_4]$ or to intersect

$[e_2, e_4]$ are both vacuous conditions, so both of these Mondrian tableaux define the Schubert variety Σ_1.

Definition 3.13. A Mondrian tableau is called *normalized* if the following two conditions hold.
(1) $l(W_i) \neq l(W_j)$ for $i \neq j$.
(2) If $W_i \subset W_j$ and $r(W_i) = r(W_j)$, then $W_i \subset W_k$ for any W_k with $\#l(W_k) \leq \#l(W_j)$.

Algorithm 3.14 (Normalization Algorithm)**.** Given a Mondrian tableau M satisfying Condition (2) in Definition 3.13, we can replace it with a normalized Mondrian tableau by running the following algorithm.

- As long as M is not normalized, let e_a be the minimal index vector for which there exists $W_i, W_j \in M$ with $i \neq j$ such that $l(W_i) = l(W_j) = e_a$. Let $W_i \subsetneq W_j$ be the two vector spaces in M of minimal dimension such that $l(W_i) = l(W_j) = e_a$. Replace W_j with $[e_{a+1}, r(W_j)]$.

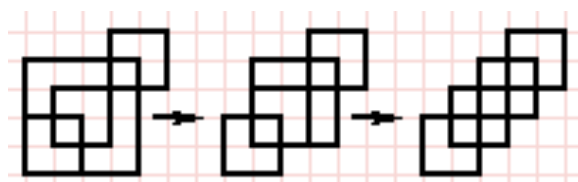

FIGURE 5. Running two steps of the Normalization Algorithm on a Mondrian tableau.

Exercise 3.15. If M' is obtained from M by applying Algorithm 3.14, then M' is a Mondrian tableau and $X(M') = X(M)$.

Exercise 3.16. Show that every Mondrian variety can be represented by a Mondrian tableau with the property that $l(W_i) \neq l(W_j)$ and $r(W_i) \neq r(W_j)$ for $i \neq j$.

Exercise 3.17. Let G be an algebraic group and let P be a parabolic subgroup. A *Richardson variety* $R(u, v)$ in the homogeneous variety G/P is in the scheme theoretic intersection of a Schubert variety X_u with a general translate of a Schubert variety gX_v with $g \in G$. It is known that Richardson varieties are reduced and irreducible. Show that if $R(u, v)$ is a Richardson variety in a partial flag variety $F(k_1, \dots, k_r; n)$, then its natural projection to $G(k_r, n)$ is a Mondrian variety. Conversely, show that every Mondrian variety arises this way.

Our goal is to compute the cohomology class of $X(M)$. As in the fundamental example worked out above, we will do this by degenerating one of the vector spaces W_i encoded by M. We begin by describing which vector space to degenerate.

Definition 3.18. A vector space W_i in a Mondrian tableau M is *nested* if the following two conditions hold:

- For every $W_j \in M$ either $W_j \subset W_i$ or $W_i \subset W_j$.
- The set of vector spaces $W_j \in M$ containing W_i is totally ordered by inclusion.

We call a vector space which is not nested *unnested*.

Definition 3.19. Let M be a Mondrian tableau whose vector spaces are not totally ordered by inclusion. The *degenerating vector space* $W_i \in M$ be the vector space with minimal $\#l(W_i)$ among the vector spaces of M that are unnested and the set of vector spaces of M contained in W_i is totally ordered by inclusion.

Definition 3.20. A *neighbor* of the degenerating vector space $W_i = [e_a, e_b]$ is any vector space W_j of M satisfying the following properties.

(1) $\#l(W_i) < \#l(W_j)$
(2) $e_{b+1} \in W_j$
(3) If $\#l(W_i) < \#l(W_k) < \#l(W_j)$, either $W_k \subset W_i$ or $W_j \subset W_k$.

Notation 3.21. Let M be a normalized Mondrian tableau for $G(k, n)$. Assume that the vector spaces in M are not totally ordered by inclusion. Let $W_i = [e_a, e_b]$ be the degenerating vector space of M. Let $N_1, \dots, N_r$ be the neighbors of W_i in M ordered from smallest dimension to largest. Let $W_i' = [e_{a+1}, e_{b+1}]$. Let M_j, $1 \leq j \leq r$ be the Mondrian tableaux obtained by replacing W_i and N_j in M with $W_i' \cap N_j$ and $\overline{W_i N_j}$

$$M_j = (M - \{W_i, N_j\}) \cup \{W_i' \cap N_j, \overline{W_i N_j}\}.$$

If $N_1 \not\subset W_i'$, let M_0 be the Mondrian tableau obtained by normalizing the Mondrian tableau obtained by replacing W_i with W_i'.

Algorithm 3.22 (The Geometric Littlewood-Richardson Rule)**.** Let M be a normalized Mondrian tableau for $G(k, n)$. If the vector spaces in M are totally ordered by inclusion, the Algorithm terminates. Otherwise, replace M by the following Mondrian tableaux.

- If $N_1 \subset W_i'$, then replace M by M_1.
- If $N_1 \not\subset W_i'$ and equality holds in Equation 3 for W_i and N_r, then replace M by M_j for $1 \leq j \leq r$.
- Otherwise, replace M by M_j for $0 \leq j \leq r$.

Example 3.23. The example in Figure 6 applies Algorithm 3.22 to the Mondrian tableau

$$M = \{[e_1, e_4], [e_3, e_8], [e_4, e_6], [e_5, e_7]\}.$$

The degenerating vector space is $[e_1, e_4]$. Algorithm 3.22 replaces M with the following three Mondrian tableaux depicted in Figure 6 in clockwise order.

$$M_0 = \{[e_2, e_5], [e_3, e_8], [e_4, e_6], [e_5, e_7]\}.$$

$$M_1 = \{[e_1, e_6], [e_3, e_8], [e_4, e_5], [e_5, e_7]\}.$$

$$M_2 = \{[e_1, e_8], [e_3, e_5], [e_4, e_6], [e_5, e_7]\}.$$

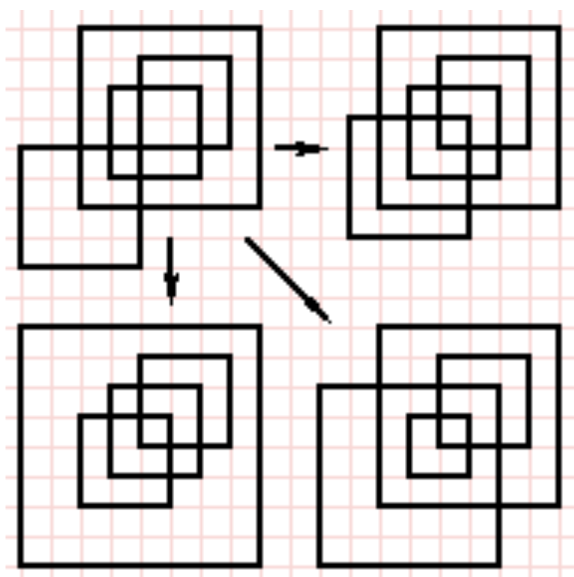

FIGURE 6. An example of Algorithm 3.22

Definition 3.24. A *degeneration path* for M consists of a sequence of Mondrian tableaux $M_1, M_2, \ldots, M_r$ such that $M = M_1$, M_r is a Mondrian tableau associated to a Schubert variety and M_j is one of the Mondrian tableau replacing M_{j-1} in Algorithm 3.22.

Theorem 3.25. *Let M be a normalized Mondrian tableau. Then the cohomology class $[X(M)] = \sum_\lambda c_\lambda \sigma_\lambda$, where c_λ is the number of degeneration paths starting with M and ending with a Mondrian tableau associated to σ_λ.*

Corollary 3.26. *The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is the number of degeneration paths starting with the Mondrian tableau $M(\lambda,\mu)$ and ending with a Mondrian tableau associated to the Schubert cycle σ_ν.*

The example in 7 shows the product $\sigma_{2,1}\cdot\sigma_{2,1}$ in $G(3,6)$.

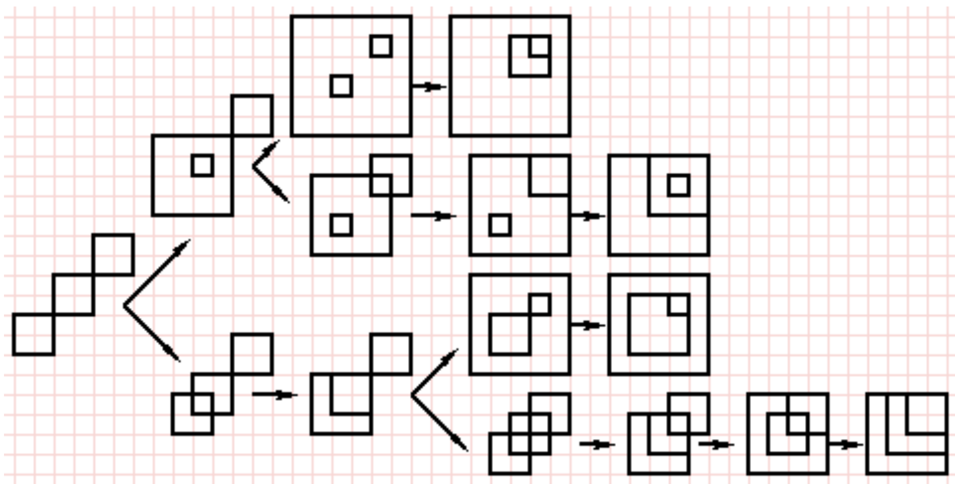

FIGURE 7. The calculation $\sigma_{2,1}\cdot\sigma_{2,1} = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}$ in $G(3,6)$.

Exercise 3.27. When one of the Schubert cycles is a Pieri cycle, show that the theorem implies Pieri's rule.

Exercise 3.28. Using the Geometric Littlewood-Richardson rule calculate $\sigma_{3,2,1}\cdot\sigma_{3,2,1}$ and $\sigma_{2,1}\cdot\sigma_{3,2,1}$ in $G(4,8)$. More generally, calculate all the Littlewood-Richardson coefficients in small Grassmannians such as $G(2,5), G(3,6)$ and $G(3,7)$.

Theorem 3.25 can be proved by interpreting Algorithm 3.22 as describing the flat limits of a degeneration. Let $W_i = [e_a, e_b]$ be the degenerating vector space of M. We consider a one parameter family of ordered bases where we keep all the basis elements except for e_a the same and we replace e_a by $e_a(t) = te_a + (1-t)e_{b+1}$. As long as $t \neq 0$ the one parameter family of varieties $X(M(t))$ for $t \neq 0$, where $M(t)$ differs from M only in that every occurrence of e_a is replaced by $e_a(t)$. The fibers of the family $X(M(t))$ for $t \in \mathbb{A}^1 - 0$ are all projectively equivalent. Hence, $X(M(t))$ forms a flat family. Algorithm 3.22 describes the flat limit at $t = 0$. Theorem 3.25 is an immediate consequence of the following geometric theorem.

Theorem 3.29. *The support of the flat limit of the family $X(M(t))$ at $t = 0$ is equal to the union of $X(M_i)$, where M_i are the Mondrian tableaux replacing M in Algorithm 3.22. Furthermore, the flat limit is generically reduced along every $X(M_i)$.*

Proof. The proof has two components. We first have to identify the limits and then we have to calculate the multiplicity of the limit along the generic point of each component.

For $t \neq 0$, let $B(t)$ denote the basis $e_1, \ldots, e_{a-1}, e_a(t), e_{a+1}, \ldots, e_n$, where $e_a(t) = te_a + (1-t)e_{b+1}$. Let $W(t)$ be a vector space spanned by elements of the basis $B(t)$. Then the flat limit at $t = 0$ of $W(t)$ is easy to describe. If $e_a(t) \notin W(t)$ or if both $e_a(t)$ and e_{b+1} are in $W(t)$, then $W(t)$ is a constant vector space and $W(t) = W(1) = W(0)$. The only interesting case is when $e_a(t) \in W(t)$ but $e_{b+1} \notin W(t)$. In this case, if $W(1)$ is spanned by a set of basis elements S, then $W(0)$ is spanned by $(S - \{e_a\}) \cup \{e_{b+1}\}$. The following is the basic observation.

Observation 3.30. Let $\Omega(t)$ be a vector space parameterized by $X(M(t))$. Then by the definition of a Mondrian variety

$$\dim(\Omega(t) \cap W(t)) \geq \{W_j \in M(1) \mid W_j \subset W(1)\}.$$

Since this is a closed condition, the following inequality must hold for every vector space $\Omega(0)$ parameterized by the flat limit $X(M(0))$:

$$\dim(\Omega(0) \cap W(0)) \geq \{W_j \in M(1) \mid W_j \subset W(1)\},$$

where $W(0)$ is the flat limit of the vector spaces $W(t)$.

In fact, Observation 3.30 suffices to determine the support of the flat limit. Every subvariety of the Grassmannian is contained in a Mondrian variety since for instance the Grassmannian itself is a Mondrian variety. Given a component Y of the support of the flat limit, we would like to describe the minimal dimensional Mondrian varieties containing it. Observation 3.30 allows us to restrict our attention to very special tableaux, namely to those that satisfy $\dim(\Omega(0) \cap W(0)) \geq \#W(M) = \#\{W_j \in M(1) \mid W_j \subset W(1)\}$. We will see that the varieties associated to such Mondrian tableaux have dimension bounded above by the dimension of $X(M)$ and those that have the same dimension are the ones described in Algorithm 3.22. Since flat limits preserve dimension, we can conclude from these observations that the support of the flat limit is contained in the union of the varieties $X(M_i)$, where M_i are the tableaux described in Algorithm 3.22.

Exercise 3.31. Using Theorem 3.8, check that $\dim(X(M_i)) = \dim(X(M))$ for every M_i replacing M in Algorithm 3.22.

Suppose that the degenerating vector space is $S = [e_a, e_b]$. Then $S(0) = [a_{a+1}, e_{b+1}]$. If $S(0)$ contains the smallest neighbor N_1 of S, then $X(M) = X(M_1)$. In this case, Algorithm 3.22 simply the reverses the normalization process. We can interpret this case as replacing M by a tableau that represents the same variety. This agrees with Step 1 of Algorithm 3.22. From now on we can assume that $S(0)$ does not contain N_1.

Let Y be an irreducible component of the flat limit $X(M(0))$ of dimension equal to the dimension of $X(M)$. By the Observation 3.30 the k-planes parameterized by Y have to intersect the vector spaces $W_i(0)$ for $W_i \in M$ in dimension at least $\#W_i(M)$. Let Λ be a general k-plane parameterized by Y. Suppose that the dimension $\dim(\Lambda \cap W_i(0)) = \#W_i(M)$ and Λ is spanned by its intersections with $W_i(0)$. It follows that the support of Y has to be contained in $X(M_0)$. Since they are both irreducible varieties of the same dimension, we conclude that $Y = X(M_0)$.

Exercise 3.32. Show that if equality holds in Equation (3) for S and the largest neighbor of S in M, then $\dim(X(M_0)) < \dim(X(M))$. If the smallest neighbor of S is not contained in $S(0)$ and there is strict inequality in Equation (3) for S and the largest neighbor of S in M, then $\dim(X(M_0)) = \dim(X(M))$.

We can now assume that for a general point $\Lambda \in Y$, the subspaces of Λ contained in $W_i(0)$ do not span Λ. Hence, the subspaces contained in two of the vector spaces $W_i(0)$ and $W_j(0)$ must intersect in a subspace of dimension greater than $\#(W_i(0) \cap W_j(0)$. Let T be the smallest dimensional linear space that is the span of consecutive basis elements with the property that the intersection of Λ with T has dimension larger than the number of vector spaces of M_0 contained in T. If there is more than one vector space with the same dimension, let T be the one that contains a basis element with least index. List the minimal vector spaces of M_0 with respect to inclusion $W_{i_1}(0), \dots, W_{i_m}(0)$ that contain T in order of their lower-left most corners. The fact that Λ intersects T leads the conditions imposed by $W_{i_1}(0), \dots, W_{i_m}(0)$ to be automatically satisfied. However, by Observation 3.30 any k-plane in the flat limit must still intersect the limit of the spans $\overline{W_{i_h} W_{i_h+1}}(0)$ in dimension equal to $\#\overline{W_{i_h} W_{i_h+1}}(M)$Hence, we can replace the tableau M_0, by the tableau where we remove the vector spaces $W_{i_1}(0), \dots, W_{i_m}(0)$ and include the vector spaces $T, \overline{W_{i_1} W_{i_2}}(0), \dots, \overline{W_{i_{m-1}} W_{i_m}}(0)$. Call the resulting tableau U_1. If Λ is spanned by its intersection with these new set of vector spaces, then, by Observation 3.30, Y has to be contained in $X(U_1)$. Otherwise, we repeat the process. Since this cannot go on indefinitely, we obtain a tableau U_a such that Y is contained in $X(U(a))$.

The first step will be complete if we can show that the variety associated to every tableau other than $M_1, \dots, M_r$ obtained by this process has dimension strictly smaller than $\dim(\Sigma_M)$.

First observe that the sum of the dimensions of the vector spaces of U_r is at most one more than the sum of the vector spaces of M. Let $W_1, \dots, W_j$ be a collection of vector spaces ordered by $l(W_i)$ that have a non-empty intersection. Let $\overline{W_h W_{h+1}}$ denote the span of the consecutive vector spaces. Then we have the easy observation that

$$\sum_{h=1}^{j} \dim(W_h) = \dim(W_1 \cap W_2 \cap \cdots \cap W_j) + \sum_{h=1}^{j-1} \dim(\overline{W_h W_{h+1}}).$$

Hence the procedure preserves or decreases the total sum of the dimensions of the vector spaces, unless W_1 is the vector space being degenerated and W_2 contains e_{j+1} but not e_i. In the latte case, the sum of the dimensions increases by one. Since the degenerating vector space occurs at most once during the process, the total sum of the dimensions is at most one larger.

Second we observe that the process increases the total number of containment relations among the vector spaces by at least $j-1$. Initially, $W_h \not\subseteq W_l$ for $h \neq l$. On the other hand, the intersection T is a subspace of all $\overline{W_h W_{h+1}}$. Let $W \in M$ be a vector space other than $W_1, \dots, W_j$, then the process does not change the number of vector spaces contained in W. The process also does not change the number of vector spaces containing W unless the $\#l(W_h) < \#l(W) < \#l(W_{h+1})$ and $r(W) = r(W_{h+1})$ for some h. In this case $W \subset \overline{W_h W_{h+1}}$.

If $j = 1$, then the sum of the dimensions of the vector spaces decreases by at least one since this case corresponds to simply replace one vector space by a smaller one. Hence to preserve dimension, we must have $j \geq 2$ for each run of the procedure. However, the sum of the dimensions of the vector spaces can increase by at most one. We conclude that in order to obtain a variety with dimension equal to $\dim(X(M))$ we can run the procedure at most once. Furthermore, in this case $j = 2$, W_1 is the vector space being degenerated and $e_{j+1} \in W_2$. The vector space T has to be the full intersection of W_1 and W_2 and if there is a vector space W with $\#l(W_1) < \#l(W) < \#l(W_2)$, then W must either contain W_2 or be contained in W_1. We conclude that W_2 has to be a neighbor N_i of the vector space being degenerated. It follows that

$Y = X(M_i)$ for one of the Mondrian tableau replacing M in Algorithm 3.22.

We thus conclude that the support of $\Sigma_M(0)$ is contained in the union of the varieties associated to the tableaux described by Step 2 of the Grassmannian Algorithm. Furthermore, it is easy to write explicit families of k-planes that show that the support of $\Sigma_M(0)$ is the union of the varieties associated to the tableaux described by Step 2 of the Grassmannian Algorithm.

There remains to show that the flat limit $X(M(0))$ is generically reduced along each of the components $X(M_i)$. It suffices to check the multiplicity at the generic point of each irreducible component. Without loss of generality, we may assume that the Mondrian tableau M consists only of the degenerating vector space S and the neighbors $N_1, \dots, N_r$. Let $\mathcal{M} \to \mathbb{P}^1$ denote the total space of the family of varieties associated to Mondrian tableaux. For $t \neq 0$, let $V(t)$ be the vector space which is the span of $S(t)$ and $N_r(t)$. Let $V(0)$ denote the limit of $V(t)$. Suppose there are p vector spaces in M contained in the span of the vector spaces S and N_r. Over a dense Zariski-open subset $\mathcal{U}$ of $\mathcal{M}$ intersecting every component of $X(M(0))$, the morphism obtained by restricting the k-planes to their p-dimensional subspaces contained in $V(t)$ is a smooth morphism. Let T be a vector space contained in S or N_r, but not equal to S or any of the neighbors. After possibly shrinking $\mathcal{U}$ to another Zariski open intersecting every component of $X(M(0))$, the morphism quotienting out the p-planes by their subspaces contained in T is a smooth morphism. It follows that to determine the multiplicities it suffices to treat the case when the tableau consists only of S and the neighbors $N_1, \dots, N_r$.

By induction on the number of neighbors, this reduces to a computation in the Grassmannian of lines. Suppose S has only one neighbor. Then the multiplicity of each tableau is one and the variety associated to each tableau occurs as a component of $X(M(0))$. This easily follows either by the Pieri rule for the Grassmannian of lines or by an easy calculation almost identical to the calculation in the fundamental example above. Now suppose $r > 1$. For $t \neq 0$, let $V(t)$ denote the span of $S(t)$ and $N_{r-1}(t)$. Let $V(0)$ denote the limit of $V(t)$. We can restrict the $(r+1)$-planes to the r-plane contained in $V(t)$. By induction it follows that each component occurs with multiplicity one. This concludes the proof of the theorem.

□

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