THE STABLE COHOMOLOGY OF MODULI SPACES OF SHEAVES ON SURFACES

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Abstract. Let X be a smooth, irreducible, complex projective surface, H a polarization on X. Let $\gamma = (r, c, \Delta)$ be a Chern character. In this paper, we study the cohomology of moduli spaces of Gieseker semistable sheaves $M_{X,H}(\gamma)$. When the rank $r = 1$, the Betti numbers were computed by Göttsche. We conjecture that if we fix the rank $r \geq 1$ and the first Chern class c, then the Betti numbers (and more generally the Hodge numbers) of $M_{X,H}(r, c, \Delta)$ stabilize as the discriminant Δ tends to infinity and that the stable Betti numbers are independent of r and c. In particular, the conjectural stable Betti numbers are determined by Göttsche's calculation. We present evidence for the conjecture. We analyze the validity of the conjecture under blowup and wall-crossing. We prove that when X is a rational surface and $K_X \cdot H < 0$, then the classes $[M_{X,H}(\gamma)]$ stabilize in an appropriate completion of the Grothendieck ring of varieties as Δ tends to ∞. Consequently, the virtual Poincaré and Hodge polynomials stabilize to the conjectural value. In particular, the conjecture holds when X is a rational surface, $H \cdot K_X < 0$ and there are no strictly semistable objects in $M_{X,H}(\gamma)$.

Contents

1. Introduction

Let X be a smooth, irreducible, complex projective surface and let H be an ample divisor on X. We denote the Chern character γ of a sheaf on X by $\gamma = (r, c, d)$ or $\gamma = (r, c, \Delta)$, where

$$r = \mathrm{rk}(\gamma),\ c = \mathrm{ch}_1(\gamma),\ d = \mathrm{ch}_2(\gamma),\ \Delta = \frac{1}{2r^2}(c^2 - 2r\,\mathrm{ch}_2(\gamma)).$$

Let $M_{X,H}(\gamma)$ denote the moduli space parameterizing S-equivalence classes of Gieseker semistable sheaves with Chern character γ. These moduli spaces were constructed by Gieseker [Gie77] and

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Maruyama [Mar78] and play a central role in many areas of mathematics ranging from topology [Don90] to representation theory [Nak99] and mathematical physics [Wit95]. Consequently, it is crucial to understand the cohomology of $M_{X,H}(\gamma)$.

Let $X^{[n]}$ denote the Hilbert scheme of n points on X. When $r = 1$, the moduli space $M_{X,H}(1, c, \Delta)$ is isomorphic to $\mathrm{Pic}^c(X) \times X^{[\Delta]}$. The abelian variety $\mathrm{Pic}^c(X)$ has complex dimension equal to the irregularity $q(X) = h^1(X, \mathcal{O}_X)$. Hence, $H^*(\mathrm{Pic}^c(X), \mathbb{Z}) \cong \bigwedge^{2q(X)} H^1(S^1, \mathbb{Z})^{\oplus 2q}$, where S^1 is the circle. The Betti numbers of $X^{[n]}$ have been computed by Göttsche [Got90]. The Künneth formula then determines the Betti numbers of $M_{X,H}(1, c, \Delta)$. An easy analysis ([Got90, Corollary 2.11], see also [Che96, LQW03] and §3) shows that these Betti numbers stabilize as Δ tends to ∞. Let $b_{i,\mathrm{Stab}}(X)$ denote the ith Betti number of $M_{X,H}(1, c, \Delta)$ for $\Delta \gg 0$.

In contrast, the Betti numbers of $M_{X,H}(r, c, \Delta)$ are generally unknown for $r > 1$. In this paper, we give evidence for the following conjecture.

Conjecture 1.1. *Fix a rank $r > 0$ and a first Chern character c. Then the ith Betti number of $M_{X,H}(r, c, \Delta)$ stabilizes to $b_{i,\mathrm{Stab}}(X)$ as Δ tends to ∞. More precisely, given an integer k, there exists $\Delta_0(k)$ such that for $\Delta \geq \Delta_0(k)$ and $i \leq k$*

$$b_i(M_{X,H}(r, c, \Delta)) = b_{i,\mathrm{Stab}}(X).$$

Furthermore, if H is in a compact subset $\mathcal{C}$ of the ample cone of X, then $\Delta_0(k)$ can be chosen independently of $H \in \mathcal{C}$.

Our main philosophy in this paper is that computing the Betti numbers of $M_{X,H}(\gamma)$ is difficult and leads to complicated formulae, whereas computing the stable Betti numbers is much simpler and leads to beautiful formulae.

Remark 1.2. In particular, if the irregularity $q(X) = 0$, then Conjecture 1.1 implies that the Betti numbers of $M_{X,H}(r, c, \Delta)$ stabilize to the stable Betti numbers of the Hilbert scheme of points $X^{[n]}$.

Remark 1.3. Göttsche and Soergel similarly computed the Hodge numbers of $M_{X,H}(1, c, \Delta)$ [GS93]. These also stabilize as Δ tends to ∞. We expect the Hodge numbers of $M_{X,H}(r, c, \Delta)$ to also stabilize to the stable Hodge numbers of $M_{X,H}(1, c, \Delta)$, at least when $M_{X,H}(r, c, \Delta)$ is smooth.

Remark 1.4. A conjecture of Vakil and Wood [VW15, Conjecture 1.25] would imply that the class of $M_{X,H}(1, c, \Delta)$ stabilizes in an appropriate completion A^- of the Grothendieck ring of varieties (see §3 for details). The stabilization is known when X is a rational surface. More generally, one can ask whether the classes of $M_{X,H}(r, c, \Delta)$ stabilize in A^- to the stable class of $M_{X,H}(1, c, \Delta)$ as Δ tends to ∞, assuming that the classes of $M_{X,H}(1, c, \Delta)$ stabilize in A^-. The proof of our main theorem in this paper will hold motivically and show the stabilization in A^-. The results on Betti and Hodge numbers are then consequences.

Remark 1.5. Some care is necessary in order to have a uniform $\Delta_0(k)$ independent of H. For example, let $\mathbb{F}_e$ be a Hirzebruch surface and let H_m be the polarization $E + mF$, where E is the curve with self-intersection $-e$ and F is the fiber class (see §2). If the rank r does not divide $c \cdot F$, then the moduli space $M_{X,H_m}(r, c, \Delta)$ is empty for any fixed Δ if $m \gg 0$. Hence, it is necessary to impose some restrictions on how H varies. It would be interesting to explore whether one can find uniform bounds on compact subsets of the big and nef cone.

Remark 1.6. A more cautious conjecture would replace the Betti numbers in Conjecture 1.1 with the virtual Betti numbers (see §3). We speculate that as Δ increases, the cohomology of $M_{X,H}(r, c, \Delta)$ becomes pure and Poincaré duality holds in larger and larger ranges. All our results on Betti numbers in this paper will be for smooth moduli spaces. Hence, this is the most speculative aspect of Conjecture 1.1.

Conjecture 1.1 builds on the work of many authors and parts of the statement have been suggested by others in print. We were especially influenced by the work of Göttsche and Yoshioka. Yoshioka informs us that he expected the cohomology to stabilize and this influenced his work on the problem in the 1990s (see [Yos96a]). However, we were unable to find this formulation of the conjecture in the literature. The conjecture fits with the philosophy of Donaldson [Don90], Gieseker and Jun Li [GL94, LiJ93, LiJ94, LiJ97] that the geometry of $M_{X,H}(\gamma)$ becomes better behaved as Δ tends to ∞. For example, O'Grady [O'G96] shows that the moduli space $M_{X,H}(\gamma)$ is irreducible and generically smooth if Δ is sufficiently large. He observes that in particular the zeroth Betti number of $M_{X,H}(\gamma)$ is 1 if Δ is sufficiently large and poses the question whether the cohomology stabilizes. Jun Li [LiJ97] shows the stabilization of the first and second Betti numbers when the rank is two. There is also related work on the stabilization of the cohomology of the moduli spaces of locally-free stable sheaves and the Atiyah-Jones conjecture in the gauge theory literature (see [LiJ97, Tau84, Tau89]).

Yoshioka computes the Betti numbers of moduli spaces of rank 2 sheaves on $\mathbb{P}^2$ and proves the stabilization of the Betti numbers [Yos94, Corollary 6.3]. Yoshioka [Yos95, Yos96b] and Göttsche [Got96] compute the Betti and Hodge numbers of $M_{X,H}(\gamma)$ when $r = 2$ and X is a ruled surface. Yoshioka [Yos95, Yos96a] observes the stabilization of the Betti numbers for rank 2 bundles on ruled surfaces. Göttsche observes that the low degree Hodge numbers are independent of the ample H and gives a nice formula for them [Got96]. Göttsche further extends his results to rank 2 bundles on rational surfaces for polarizations that are K_X-negative in [Got99] (see also [Yos95]). Most of the arguments in this paper were originally developed by Göttsche and Yoshioka when $r = 2$ and the paper owes a great intellectual debt to the two of them. Manschot [Man11, Man14] building on the work of Mozgovoy [Moz13] gives a formula for the Betti numbers of the moduli spaces when $X = \mathbb{P}^2$. The stabilization of the Betti numbers can be observed from the tables provided in these papers.

The conjecture is known for smooth moduli spaces of sheaves on K3 and abelian surfaces. By work of Mukai [Muk84], Huybrechts [Huy03] and Yoshioka [Yos99], smooth moduli spaces of sheaves on a K3 surface X are deformations of the Hilbert scheme of points on X of the same dimension. In particular, they are diffeomorphic to $X^{[n]}$ of the same dimension. Hence, their Betti numbers agree without taking any limits.

Yoshioka [Yos01] obtains similar results on abelian surfaces. A smooth moduli space of sheaves $M_{X,H}(\gamma)$ is deformation equivalent to $X^* \times X^{[n]}$ of the same dimension, where X^* is the dual abelian surface. In this case as well, the cohomology is isomorphic to the cohomology of $X^* \times X^{[n]}$ without the need to take limits.

Let $K_0(\mathrm{var}_{\mathbb{C}})$ denote the Grothendieck ring of varieties over the complex numbers. Let $\mathbb{L}$ denote the class of the affine line $\mathbb{A}^1$. Let $R = K_0(\mathrm{var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. The ring R has a $\mathbb{Z}$-graded decreasing filtration $\mathcal{F}$ generated by

$$[X]\mathbb{L}^a \in \mathcal{F}^i \quad \text{if} \quad \dim(X) + a \leq -i.$$

Let A^- be the completion of R with respect to this filtration (see §3 for further details). The moduli stacks $\mathcal{M}_{X,H}(r, c, \Delta)$ of Gieseker semistable sheaves have well defined classes in A^-. Furthermore, the virtual Poincaré and Hodge polynomials extend to A^- (see §3 for a more detailed discussion). Our main theorem in this paper is the following.

Theorem 1.7. *Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. The classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ of the moduli stacks of Gieseker semistable sheaves stabilize in A^-, and their limit is the same as the limit of $[\mathcal{M}_{X,H}(1, c, \Delta)]$ in A^- as Δ tends to ∞. In particular, the virtual Betti and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize, and the* *generating functions for the stable numbers* $\tilde{b}_{i,\mathrm{Stab}}$ *and* $\tilde{h}^{p,q}_{\mathrm{Stab}}$ *are given by*

$$\sum_{i=0}^{\infty} \tilde{b}_{i,\mathrm{Stab}} t^i = \prod_{i=1}^{\infty} \frac{1}{(1-t^{2i})^{\chi_{\mathrm{top}}(X)}} \quad \text{and} \quad \sum_{p,q=0}^{\infty} \tilde{h}^{p,q}_{\mathrm{Stab}} x^p y^q = \prod_{i=1}^{\infty} \frac{1}{(1-(xy)^i)^{\chi_{\mathrm{top}}(X)}}.$$

Remark 1.8. The canonical class on minimal rational surfaces or del Pezzo surfaces is anti-effective. Hence, on these surfaces the condition $H \cdot K_X < 0$ is satisfied for all polarizations. In general, on any rational surface there are polarizations H with $H \cdot K_X < 0$. However, for more general rational surfaces this is a restriction on the polarization.

We will prove Theorem 1.7 by studying the effect of wall-crossing [Joy08, Moz13] and blowing up [Moz13] on the virtual Poincaré polynomials (and more generally, the classes in A^-) of the moduli stack of stable sheaves. Then using calculations of Mozgovoy [Moz13] and Manschot [Man14] on $\mathbb{F}_1$, we will be able to determine the stable limits.

As a consequence of Theorem 1.7, we recover a result of Yoshioka described in a Remark in [Yos96a, §3.6].

Theorem 1.9. *Let* X *be a smooth rational surface and let* H *be a polarization such that* $H \cdot K_X < 0$. *Assume that the moduli spaces* $M_{X,H}(r,c,\Delta)$ *do not contain any strictly semistable sheaves. Then* $b_i(M_{X,H}(r,c,\Delta))$ *stabilizes to* $b_{i,\mathrm{Stab}}(X)$ *as* Δ *tends to* ∞.

Under the assumptions of Theorem 1.9, the Hodge numbers of $M_{X,H}(r,c,\Delta)$ also stabilize to the stable Hodge numbers of the Hilbert scheme of points. Hence, our results have the following corollary, originally due to Yoshioka by different methods, proving a special case of a conjecture of Jun Li [LiJ94].

Corollary 1.10. [Yos96c] *Let* X *be a smooth, rational surface and let* H *be a polarization such that* $H \cdot K_X < 0$. *Let* γ *be a Chern character such that* $M_{X,H}(\gamma)$ *does not contain any strictly semistable sheaves. If* Δ *is sufficiently large, then the Picard rank* $\rho(M_{X,H}(\gamma)) = \rho(X) + 1$ *and the Donaldson morphism identifies* $\mathrm{Pic}_{\mathbb{Q}}(M_{X,H}(\gamma))$ *with* $\gamma^{\perp}$ *in* $K^0(X)$, *where* $\gamma^{\perp}$ *is the orthogonal complement under the Euler pairing.*

Further problems and questions. Conjecture 1.1 raises several further questions.

First, when stabilization holds, it would be desirable to have explicit bounds for $\Delta_0(k)$ in terms of X and the compact $\mathcal{C} \subset \mathrm{Amp}(X)$.

Question 1.11. *When stabilization holds, can we obtain effective bounds for* Δ *to guarantee the stabilization of Betti numbers of* $M_{X,H}(r,c,\Delta)$*?*

Many of our arguments can be made effective; however, we have not made a systematic effort to do so. Following our methods, S. Mandal has given effective bounds when $X = \mathbb{P}^2$ [Mnd20].

We remark that when Δ is small, the moduli spaces $M_{X,H}(r,c,\Delta)$ can exhibit pathological behavior. For example, for any positive integer k, there exists moduli spaces of sheaves of rank 2 on very general hypersurfaces of degree $d \gg 0$ in $\mathbb{P}^3$ that have at least k components with different dimensions [CH18b, Theorem 5]. This suggests that in general, Δ may need to be large before stabilization occurs. See [HL10, O'G96] for effective bounds that guarantee that the moduli space is irreducible.

Conjecture 1.1 is a conjecture about equality of numbers. More generally, one can ask for a geometric reason for the stabilization. We do not in general know algebraic maps between $M_{X,H}(r,c,\Delta)$ for different Δ. However, one can define correspondences using elementary modifications. It would be desirable to have an algebro-geometric reason for the stabilization.

Question 1.12. *Are there algebro-geometric reasons for the stabilization of the cohomology?*

We remark, however, that Conjecture 1.1 is closely related to the Atiyah-Jones Conjecture (see §8). Taubes [Tau84] has constructed differential geometric maps in this context. When (X, H) is a polarized surface with $K_X = 0$ or $K_X \cdot H < 0$ and the rank and $H \cdot c$ are coprime, Baranovsky [Bar00] constructs an action of an oscillator algebra on

$$\bigoplus_{\Delta} H^*(M_{X,H}(r, c, \Delta)).$$

This action potentially gives another geometric interpretation in this restricted setting.

It is natural to wonder whether the stabilization of Betti numbers holds in more general contexts. First, we expect the ith Betti number of moduli spaces of pure one-dimensional sheaves $M_{X,H}(0, c, d)$ on X to stabilize to $b_{i,\mathrm{Stab}}(X)$ as $c \cdot H$ tends to infinity by increasing c. Similarly, we would expect stabilization to hold for moduli spaces closely related to $M_{X,H}(\gamma)$ such as the Matsuki-Wentworth moduli spaces of twisted-Gieseker-semistable sheaves [MW97]. Since twisted-Gieseker semistability is implied by slope-stability and implies slope-semistability, Corollary 4.9 and Theorem 1.7 imply the following corollary.

Corollary 1.13. *Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. The classes $[\mathcal{M}_{X,H,D}(r, c, \Delta)]$ of the moduli stacks of (H, D)-twisted-Gieseker semistable sheaves stabilize in A^- to the stable limit of $[\mathcal{M}_{X,H}(1, c, \Delta)]$ in A^- as Δ tends to ∞. In particular, the virtual Betti and Hodge numbers of $\mathcal{M}_{X,H,D}(r, c, \Delta)$ stabilize to those of $\mathcal{M}_{X,H}(1, c, \Delta)$.*

More generally, one can ask whether the Betti numbers of moduli spaces of Bridgeland stable objects stabilizes as Δ tends to ∞. For an ample divisor H and an arbitrary $\mathbb{Q}$-divisor D on X, consider the (H, D)-slice of the Bridgeland stability manifold (see [CH18a] for details).

Question 1.14. *Let $\sigma_{s,t}(\Delta)$ be a sequence of stability conditions in the (H, D)-slice (depending on Δ) bounded away from the collapsing wall. Do the Betti numbers of the Bridgeland moduli spaces $M_{X,\sigma_{s,t}(\Delta)}(r, c, \Delta)$ of semistable objects on X stabilize as Δ tends to ∞? Do they stabilize to the stable Betti numbers of $M_{X,H}(1, c, \Delta)$?*

If $D = 0$ and $\sigma_{s,t}(\Delta)$ are a sequence of stability conditions lying above the Gieseker wall, then Question 1.14 reduces to Conjecture 1.1. In general, one has to show some care. Already on $\mathbb{P}^2$, by [CHW17, Theorem 5.7], if we fix a stability condition $\sigma_{s,t}$ and let Δ tend to ∞, then $\sigma_{s,t}$ is below the collapsing wall and the moduli space $M_{\mathbb{P}^2,\sigma_{s,t}}(r, c, \Delta)$ is empty. The Question 1.14 is most interesting when the stability conditions $\sigma_{s,t}(\Delta)$ are between the Gieseker and the collapsing wall. There is a positive answer to Question 1.14 when X is a K3 or abelian surface, the Chern character γ is primitive and the stability conditions avoid walls. For K3 surfaces, the Bridgeland moduli space is a holomorphic symplectic manifold of K3 type [BM14], hence diffeomorphic to a Hilbert scheme of points on X. For abelian surfaces, under certain conditions, these moduli spaces are deformation equivalent to $X^* \times X^{[n]}$ of the same dimension [MYY18]. Similarly, [ABCH13, BC13] provide some evidence that there may be a positive answer to Question 1.14 for rational surfaces.

Organization of the paper. In §2, we review basic facts concerning rational surfaces and moduli spaces of sheaves on surfaces. In §3, we discuss the relation between properties of generating functions and stabilization of Betti numbers. We introduce the appropriate completion of the Grothendieck ring of varieties where our calculations take place. In §4, using Joyce's wall-crossing formula, we analyze the effect of wall-crossing on Betti numbers. We then use the blowup formula of Yoshioka and Mozgovoy to analyze the effect of blowing up. In §5, following calculations of Manschot, we show stabilization for $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_1$. In §6, we prove our main theorem at the level of stacks. In §7 we discuss the relation between the stabilization for the moduli stack and the stabilization for the moduli space. In §8, we discuss the relation between our conjecture and the Atiyah-Jones Conjecture. Finally, in §9, we give applications to the Néron-Severi space of the moduli space.

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2. Preliminaries

In this section, we review the basic definitions and facts required in the rest of the paper.

2.1. **Semistable sheaves.** We begin by reviewing notions of stability. We refer the reader to [HL10] and [CH15] for more detailed discussions.

Let X be a smooth projective surface and H be an ample divisor on X. All sheaves we consider in this paper will be coherent and torsion-free. The Hilbert polynomial and the reduced Hilbert polynomial of a sheaf $\mathcal{F}$ are defined by

$$P_{\mathcal{F},H}(m) = \chi(\mathcal{F}(mH)) = a_2\frac{m^2}{2} + \text{l.o.t.}, \quad p_{\mathcal{F},H} = \frac{P_{\mathcal{F},H}}{a_2},$$

respectively. A sheaf $\mathcal{F}$ is called *Gieseker H-semistable* if for every proper subsheaf $\mathcal{E}$, $p_{\mathcal{E},H}(m) \leq p_{\mathcal{F},H}(m)$ for $m \gg 0$.

Define the H-slope μ_H and the discriminant Δ of a sheaf $\mathcal{F}$ by

$$\mu_H(\mathcal{F}) = \frac{\operatorname{ch}_1(\mathcal{F}) \cdot H}{\operatorname{ch}_0(\mathcal{F})H^2}, \quad \Delta(\mathcal{F}) = \frac{1}{2\operatorname{ch}_0(\mathcal{F})^2}(\operatorname{ch}_1(\mathcal{F})^2 - 2\operatorname{ch}_0(\mathcal{F})\operatorname{ch}_2(\mathcal{F})).$$

A sheaf $\mathcal{F}$ is *μ_H-semistable* if for every proper subsheaf $\mathcal{E}$ of smaller rank, we have $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})$. The sheaf is called *$\mu_H$-stable* if the inequalities are strict.

A torsion free sheaf $\mathcal{F}$ admits a unique *Harder-Narasimhan filtration* with respect to either Gieseker H-semistability or μ_H-semistability, i.e. a filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{F}$$

such that the quotients $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable with decreasing invariants. A semistable sheaf further admits a *Jordan-Hölder* filtration into stable objects. Two semistable sheaves are called *S-equivalent* if they have the same Jordan-Hölder factors.

Let γ be the Chern character of a sheaf. Let $M_{X,H}(\gamma)$ be the moduli space of S-equivalence classes of Gieseker H-semistable sheaves with Chern character γ. These were constructed by Gieseker [Gie77] and Maruyama [Mar78].

Let $M^{\mu,s}_{X,H}(\gamma)$ and $M^{\mu,s,\circ}_{X,H}(\gamma)$ denote the open subsets of $M_{X,H}(\gamma)$ parameterizing μ_H-stable sheaves and locally-free μ_H-stable sheaves, respectively. Let $\mathcal{M}_{X,H}(\gamma)$, $\mathcal{M}^{\mu,s}_{X,H}(\gamma)$ and $\mathcal{M}^{\mu,s,\circ}_{X,H}(\gamma)$ denote the corresponding moduli stacks. Let $\mathcal{M}^{\mu}_{X,H}(\gamma)$ denote the moduli stack of μ_H-semistable sheaves with Chern character γ.

We will use the following well-known proposition several times.

Proposition 2.1. *Let X be a smooth surface and let H be a polarization such that $K_X \cdot H < 0$. If $M_{X,H}(\gamma)$ is nonempty and contains no strictly semistable sheaves, then $M_{X,H}(\gamma)$ is a smooth projective variety of the expected dimension $1 - \chi(\gamma,\gamma)$.*

Proof. Let $V \in M_{X,H}(\gamma)$ be a sheaf. By assumption, V is stable, hence $\hom(V,V) = 1$. Since $K_X \cdot H < 0$, $\operatorname{ext}^2(V,V) = \hom(V, V(K_X)) = 0$ by stability. We conclude that $\operatorname{ext}^1(E,E) = -\chi(\gamma,\gamma) + 1$ and the moduli space is smooth. □

2.2. **Zeta functions.** Let X be a projective variety. Let $X^{(n)}$ denote the nth symmetric product X^n/S_n, where the symmetric group S_n acts on the product X^n by permuting the factors. Given a variety Y, let

$$P_Y(t) = \sum_{i=0}^{2\dim(Y)} b_i(Y)t^i$$

denote the Poincaré polynomial of Y, where $b_i(Y)$ is the ith Betti number of Y.

Following Kapranov [Kap00], it is customary to define the *motivic zeta function* of a variety X as follows

$$Z_X(q) = \sum_{n=0}^{\infty}[X^{(n)}]q^n,$$

where $[X^{(n)}]$ denotes the class of $X^{(n)}$ in the Grothendieck ring of varieties. When X is a smooth projective surface, we will be interested in the Betti realization of the zeta function. In order to distinguish this function from the motivic zeta function, we will denote it by $\zeta_X(q,t)$. The zeta function is given as follows [Mac62]

$$\zeta_X(q,t) = \sum_{n=0}^{\infty} P_{X^{(n)}}(t)q^n = \frac{(1+qt)^{b_1(X)}(1+qt^3)^{b_3(X)}}{(1-q)(1-qt^2)^{b_2(X)}(1-qt^4)}. \tag{1}$$

2.3. **The cohomology of the moduli spaces of rank 1 sheaves.** Let $X^{[n]}$ denote the Hilbert scheme of n points on X. When $r=1$, a stable sheaf $\mathcal{F} \in M_{X,H}(1,c,\Delta)$ is isomorphic to $L \otimes I_Z$, where L is a line bundle with $\mathrm{ch}_1(L) = c$ and I_Z is an ideal sheaf of points on X with $|Z| = \Delta$. There is a natural isomorphism from $\mathrm{Pic}^c(X) \times X^{[\Delta]}$ to $M_{X,H}(1,c,\Delta)$ given by tensor product. The inverse morphism is given by considering the determinant map $M_{X,H}(1,c,\Delta) \to \mathrm{Pic}^c(X)$ and the map $M_{X,H}(1,c,\Delta) \to X^{[\Delta]}$ given by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{F}^*$ to $X^{[\Delta]}$.

Göttsche [Got90] computed the Betti numbers of $X^{[n]}$. It is convenient to form a generating function $F(q,t)$ incorporating the Poincaré polynomials of $X^{[n]}$. Göttsche proves the following formula

$$F(q,t) = \sum_{n=0}^{\infty} P_{X^{[n]}}(t)q^n = \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t).$$

By the Künneth formula, it follows that the Betti numbers of $M_{X,H}(1,c,\Delta)$ are given by

$$G(q,t) = \sum_{\Delta=0}^{\infty} \sum_{i=0}^{4\Delta+b_1(X)} b_i(M_{X,H}(1,c,\Delta))t^i q^{\Delta} = (1+t)^{b_1(X)} \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t).$$

2.4. **Rational surfaces.** In this subsection, we collect some facts on surfaces that will play a role in our discussion. We refer the reader to [Bea83, Cos06a, Cos06b, Har77] for more detailed discussions.

For an integer $e \geq 0$, let $\mathbb{F}_e$ denote the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. The Picard group of $\mathbb{F}_e$ is isomorphic to $\mathbb{Z}E \oplus \mathbb{Z}F$, where E is a section with $E^2 = -e$ and F is the class of a fiber. They satisfy the intersection numbers

$$E^2 = -e, \quad E \cdot F = 1, \quad F^2 = 0.$$

The nef cone of $\mathbb{F}_e$ is spanned by F and $E + eF$. The canonical class of $\mathbb{F}_e$ is $-2E - (e+2)F$. Since $-K_{\mathbb{F}_e}$ is effective, we have that $K_{\mathbb{F}_e} \cdot H < 0$ for every ample divisor H on $\mathbb{F}_e$. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_e$ with $e \neq 1$. The surface $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at a point. Every smooth rational surface can be obtained from one of these surfaces by a sequence of blowups.

Del Pezzo surfaces are surfaces with ample anti-canonical class. They are $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^2$ at no more than 8 general points. Since $-K_X$ is ample, $K_X \cdot H < 0$ for every ample divisor H.

Let $\pi : \hat{X} \to X$ be the blowup of a smooth surface X at a point p. Let E_p denote the exceptional divisor over p. Then $\mathrm{Pic}(\hat{X}) = \mathrm{Pic}(X) \oplus E_p$ and $E_p^2 = -1$ and $E_p \cdot D = 0$ for any divisor D pulled back from X. Furthermore, $K_{\hat{X}} = \pi^* K_X + E_p$. If H is a polarization on X such that $K_X \cdot H < 0$, then $\pi^* H$ is a big and nef class on $\hat{X}$ with $K_{\hat{X}} \cdot \pi^* H < 0$. Since there are ample classes arbitrarily close to $\pi^* H$, one can also find polarizations H' on $\hat{X}$ with $K_{\hat{X}} \cdot H' < 0$. The condition $K_X \cdot H < 0$ is a convex condition on the ample cone. Hence any two polarizations H_1 and H_2 satisfying this condition can be joined by a line segment contained in the K_X-negative part of the ample cone.

2.5. **Moduli Spaces of Vector Bundles on Curves.** In this interlude, we discuss the well-known stabilization of the Betti numbers of moduli spaces of vector bundles on curves as the rank tends to infinity. The purpose of this subsection is to point out that there are contexts in dimensions other than 2 where stabilization of Betti numbers occurs. The stabilization easily follows from Zagier's formula for the Betti numbers of the moduli space of vector bundles of rank r and degree d on a genus g curve.

Theorem 2.2 (Zagier [Zag96])**.** *If r and d are coprime, then the Poincaré polynomial of the moduli space of vector bundles of rank r and degree d on a genus g curve is given by*

$$\sum_{k=1}^{r} \sum_{\substack{r_1+\cdots+r_k=r \\ r_1,\ldots,r_k>0}} \frac{(-1)^{k-1} t^{2M_g(r_1,\ldots,r_k;d/n)}}{(1-t^{2r_1+2r_2})\cdots(1-t^{2r_{k-1}+2r_k})} P_{r_1}\cdots P_{r_k},$$

where

$$P_i = \frac{(1+t)^{2g}(1+t^3)^{2g}\cdots(1+t^{2i-1})^{2g}}{(1-t^2)^2(1-t^4)^2\cdots(1-t^{2i-2})^2(1-t^{2i})}$$

and

$$M_g(r_1,\ldots,r_k;\lambda) = \sum_{j=1}^{k-1}(r_j+r_{j+1})[(r_1+\cdots+r_j)\lambda] + (g-1)\sum_{i<j} r_i r_j,$$

with

$$[x] = 1 + \lfloor x \rfloor - x.$$

Corollary 2.3. *Let C be a smooth, projective curve of genus at least 2. The Betti numbers of the moduli spaces of vector bundles on C with coprime rank and degree stabilize as the rank increases, and the generating function for the stable Betti numbers is given by*

$$\prod_{k=1}^{\infty} \frac{(1+t^{2k-1})^{2g}}{(1-t^{2k})^2}$$

Proof. The main term in Zagier's formula is the term with $k=1$—as r increases, the other terms only contribute to larger Betti numbers. It suffices to show that if $k>1$, given any N, then $M_g(r_1,\ldots,r_k;d/n) > N$ for r sufficiently large. Since

$$M_g \geq (g-1)\sum_{1\leq i<j\leq k} r_i r_j,$$

it suffices to bound the right hand side from below. We can rewrite

$$(g-1)\sum_{1\leq i<j\leq k} r_i r_j = \frac{g-1}{2}(r^2 - \sum_{i=1}^{k} r_i^2) \geq (g-1)(r-1).$$

The last inequality follows from the fact that if $k \geq 2$, the quantity $(r^2 - \sum_{i=1}^k r_i^2)$ is minimized when, up to permuting indices, $k = 2$, $r_1 = r - 1$ and $r_2 = 1$. Since $(g-1)(r-1)$ grows arbitrarily large as r increases, the corollary follows. □

One can ask whether Conjecture 1.1 has any higher dimensional analogues. By work of Macdonald [Mac62] (see also [Che96]), the Betti numbers of symmetric products $X^{(n)}$ stabilize for any smooth projective variety X as n tends to infinity. Note, however, even when X is a surface, the stable Betti numbers of $X^{(n)}$ and $X^{[n]}$ are different. It would be interesting to find analogous stabilizations for moduli spaces of sheaves on X for higher dimensional X.

3. Generating Functions and Stabilization

In this section, we express stabilization of Betti numbers in terms of properties of generating functions. More generally, we will discuss the stabilization of the class of a sequence of varieties in an appropriate completion of the Grothendieck ring of varieties.

Let N be an integer. Let

$$P_d(t) = \sum_{i=0}^{s_d} a_{i,d} t^i$$

be a collection of polynomials of degree s_d in t indexed by integers $d \geq N$. We introduce the shifted polynomials

$$\tilde{P}_d(t) = \sum_{i=0}^{s_d} a_{i,d} t^{i-s_d} = \sum_{j=-s_d}^{0} b_{j,d} t^j.$$

We say that the polynomials $P_d(t)$ *stabilize* if for each j there exists an integer $d_0(j)$ such that $b_{j,d} = b_{j,d+1}$ for $d \geq d_0(j)$. In this case, let $\beta_j = b_{j,d}$ for $d \gg 0$ and let the *stable limit* of $P_d(t)$ be the power series in t^{-1} given by

$$\tilde{P}_\infty(t) = \sum_{j=-\infty}^{0} \beta_j t^j.$$

If the polynomials $P_d(t)$ satisfy Poincaré duality for $d \gg 0$, i.e.

$$t^{s_d} P_d(t^{-1}) = P_d(t),$$

then this notion is equivalent to the more naive notion of stabilization that there exists $d_0(i)$ such that $a_{i,d} = a_{i,d+1}$ for $d \geq d_0(i)$. In this case, let $\alpha_i = a_{i,d}$ for $d \gg 0$. Then the stable limit is

$$\tilde{P}_\infty(t) = \sum_{i=0}^{\infty} \alpha_i t^{-i}.$$

For convenience, we refer to

$$P_\infty(t) = \tilde{P}_\infty(t^{-1}) = \sum_{i=0}^{\infty} \alpha_i t^i$$

as the *generating function of the stable coefficients*.

Form the generating function

$$\tilde{F}(q,t) = \sum_{d=N}^{\infty} \tilde{P}_d(t) q^d = \sum_{d=N}^{\infty} \sum_{j=-s_d}^{0} b_{j,d} t^j q^d.$$

Without assuming that the polynomials $P_d(t)$ satisfy Poincaré duality, the following proposition gives a criterion for $P_d(t)$ to stabilize in terms of the generating function $\tilde{F}(q,t)$.

Proposition 3.1. *The polynomials $P_d(t)$ stabilize if and only if for every index i the coefficient of t^i in $(1-q)\tilde{F}(q,t)$ is a Laurent polynomial in q. Moreover, if the coefficients stabilize, the stable coefficients are obtained by evaluating $(1-q)\tilde{F}(q,t)$ at $q=1$.*

Proof. If the polynomials $P_d(t)$ stabilize, then the coefficient of t^i in $\tilde{P}_d(t)$ is β_i for $i \geq d_0(i)$. Without loss of generality, we may assume that $d_0(i) > 0$. For convenience, set $b_{i,d} = 0$ for $d < N$. Hence, the coefficient of t^i in $\tilde{F}(q,t)$ has the form

$$\sum_{d=N}^{d_0(i)-1} b_{i,d} q^d + \sum_{d=d_0(i)}^{\infty} \beta_i q^d.$$

Let ℓ_i be the Laurent polynomial defined by

$$\ell_i := \sum_{d=N}^{-1} b_{i,d} q^d + \sum_{d=0}^{d_0(i)-1} (b_{i,d} - \beta_i) q^d.$$

We can then rewrite the coefficient of t^i as

$$\sum_{d=0}^{\infty} \beta_i q^d + \ell_i = \frac{\beta_i}{1-q} + \ell_i.$$

Hence, the coefficient of t^i in $(1-q)\tilde{F}(q,t)$ is a Laurent polynomial $\beta_i + (1-q)\ell_i$.

Conversely, suppose that the coefficient of t^i in $(1-q)\tilde{F}(q,t)$ is given by a Laurent polynomial $\sum_{d=N}^{M} \gamma_{i,d} q^d$. Then the coefficient of t^i in $\tilde{F}(q,t)$ is given by

$$\left(\sum_{d=N}^{M} \gamma_{i,d} q^d\right) \times \sum_{d=0}^{\infty} q^d.$$

The coefficients of q^d stabilize for $d \geq M - N$ to $\sum_{d=N}^{M} \gamma_{i,d}$. Consequently, the polynomials $P_d(t)$ stabilize.

Now assume that the polynomials $P_d(t)$ stabilize. Since the coefficient of t^i in $(1-q)\tilde{F}(q,t)$ is given by $\beta_i + (1-q)\ell_i$, if we evaluate $(1-q)\tilde{F}(q,t)$ at $q=1$, we get the series $\sum_i \beta_i t^i$. This series encodes the stable limit of the polynomials $P_d(t)$. □

If the polynomials $P_d(t)$ satisfy Poincaré duality for $d \gg 0$, then there is a similar criterion for stabilization. Let

$$F(q,t) = \sum_{d=N}^{\infty} P_d(t) q^d = \sum_{d=N}^{\infty} \sum_{i=0}^{s_d} a_{i,d} t^i q^d.$$

Corollary 3.2. *Assume that the polynomials $P_d(t)$ satisfy Poincaré duality for $d \gg 0$. Then they stabilize if and only if the coefficient of t^i in $(1-q)F(q,t)$ is a Laurent polynomial in q. The stable limit $\tilde{P}_\infty$ is then given by evaluating $(1-q)F(q,t^{-1})$ at $q=1$ and the generating function for the stable coefficients is given by evaluating $(1-q)F(q,t)$ at $q=1$.*

As a corollary we obtain the following well-known proposition (see [Got90, Che96] or [LQW03]).

Proposition 3.3. [Got90, Corollary 2.11] *Let $M_{X,H}(1,c,\Delta)$ be moduli spaces of rank one sheaves with $c_1 = c$ on a smooth, irreducible projective surface X. Then the Betti numbers of $M_{X,H}(1,c,\Delta)$ stabilize as Δ tends to ∞. Moreover, the generating function for the stable Betti numbers $b_{i,\mathrm{Stab}}(M_{X,H}(1,c,\Delta))$ is given by*

$$\sum_{i=0}^{\infty} b_{i,\mathrm{Stab}}(M_{X,H}(1,c,\Delta)) t^i = (1-t^2) \prod_{m=1}^{\infty} \frac{(1+t^{2m-1})^{2b_1}}{(1-t^{2m})^{b_2+2}}, \tag{2}$$

where b_i denotes the ith Betti number of X.

Proof. By Göttsche's formula [Got90], we have that

$$F(q,t) = \sum_{\Delta=0}^{\infty} P_{X^{[\Delta]}}(t)q^{\Delta} = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1}(1+t^{2m+1}q^m)^{b_3}}{(1-t^{2m-2}q^m)^{b_0}(1-t^{2m}q^m)^{b_2}(1-t^{2m+2}q^m)^{b_4}}.$$

By Corollary 3.2, to verify that the Betti numbers of $X^{[\Delta]}$ stabilize, it suffices to show that the coefficients of t^i in $(1-q)F(q,t)$ are Laurent polynomials in q. By Göttsche's formula, we have

$$(1-q)F(q,t) = \prod_{m=2}^{\infty} \frac{1}{(1-t^{2m-2}q^m)^{b_0}} \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1}(1+t^{2m+1}q^m)^{b_3}}{(1-t^{2m}q^m)^{b_2}(1-t^{2m+2}q^m)^{b_4}}.$$

To compute the coefficient of t^i, we can consider the expression modulo t^{i+1}. Then all but finitely many of the terms in the numerator become 1. Every term that occurs in the denominator of $(1-q)F(q,t)$ is of the form $(1-t^lq^m)$ for a positive l. We can express the terms in the denominator by the power series

$$\frac{1}{1-t^lq^m} = \sum_{j=0}^{\infty} t^{lj}q^{mj}.$$

Modulo t^{i+1}, these sums are all finite and all but finitely many of them are equal to 1. Consequently, the coefficient of t^i is a polynomial in q. Therefore, the Betti numbers of $X^{[\Delta]}$ stabilize. By Corollary 3.2, the generating function for the stable cohomology of $X^{[\Delta]}$ is

$$\prod_{m=2}^{\infty} \frac{1}{(1-t^{2m-2})^{b_0}} \prod_{m=1}^{\infty} \frac{(1+t^{2m-1})^{b_1}(1+t^{2m+1})^{b_3}}{(1-t^{2m})^{b_2}(1-t^{2m+2})^{b_4}}.$$

If X is a regular surface, the moduli space of rank 1 sheaves $M_{X,H}(1,c,\Delta)$ is isomorphic to the Hilbert scheme of points $X^{[\Delta]}$. Hence, there is nothing further to show. More generally, combining Göttsche's calculation with the Künneth formula, we obtain that the generating function for the Poincaré polynomials of $M_{X,H}(1,c,\Delta)$ is

$$G(q,t) = (1+t)^{b_1} \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1}(1+t^{2m+1}q^m)^{b_3}}{(1-t^{2m-2}q^m)^{b_0}(1-t^{2m}q^m)^{b_2}(1-t^{2m+2}q^m)^{b_4}}.$$

It is again clear that the coefficient of t^i in $(1-q)G(q,t)$ is a polynomial in q. Hence, the Betti numbers of $M_{X,H}(1,c,\Delta)$ stabilize as Δ tends to ∞. The generating function for the stable Betti numbers is given by

$$(1+t)^{b_1} \prod_{m=2}^{\infty} \frac{1}{(1-t^{2m-2})^{b_0}} \prod_{m=1}^{\infty} \frac{(1+t^{2m-1})^{b_1}(1+t^{2m+1})^{b_3}}{(1-t^{2m})^{b_2}(1-t^{2m+2})^{b_4}}.$$

If X is a smooth, irreducible, projective surface, this expression simplifies to

$$(1-t^2) \prod_{m=1}^{\infty} \frac{(1+t^{2m-1})^{2b_1}}{(1-t^{2m})^{b_2+2}}.$$

□

Proposition 3.4. *Let X be a smooth, irreducible projective surface X. Then the Hodge numbers of $M_{X,H}(1,c,\Delta)$ stabilize as Δ tends to ∞. The generating function for the stable Hodge numbers*

$$G(x,y) = \sum_{a,b\geq 0} h^{a,b}_{\text{Stab}} x^a y^b$$

is equal to

$$(1+x)^{\frac{b_1}{2}}(1+y)^{\frac{b_1}{2}} \prod_{\substack{a,b=0,\\(a,b)\neq(0,0)}}^{2} (1+(-1)^{a+b+1}x^a y^b)^{(-1)^{a+b+1}h^{a,b}(X)} \cdot$$

$$\prod_{m=2}^{\infty}\prod_{a,b=0}^{2}(1+(-1)^{a+b+1}x^{a+m-1}y^{b+m-1})^{(-1)^{a+b+1}h^{a,b}(X)}.$$

Proof. Göttsche, Soergel [GS93] and Cheah [Che96] compute the generating function for the Hodge numbers of $X^{[\Delta]}$.

$$F(x,y,q)=\sum_{\Delta=1}^{\infty}h^{a,b}(X^{[\Delta]})x^a y^b q^{\Delta}=\prod_{m=1}^{\infty}\prod_{a,b=0}^{2}(1+(-1)^{a+b+1}x^{a+m-1}y^{b+m-1}q^m)^{(-1)^{a+b+1}h^{a,b}(X)},$$

where $h^{p,q}(X)$ denotes the (p,q)-Hodge number of X. The denominator depends only on q, precisely when $a=b=0$ and $m=1$. In $(1-q)F(x,y,q)$, the coefficient of every term $x^{\alpha}y^{\beta}$ is a polynomial in q. Hence, the Hodge numbers stabilize and the generating function for the stable Hodge numbers is given by

$$\prod_{\substack{a,b=0,\\(a,b)\neq(0,0)}}^{2} (1+(-1)^{a+b+1}x^a y^b)^{(-1)^{a+b+1}h^{a,b}(X)} \prod_{m=2}^{\infty}\prod_{a,b=0}^{2}(1+(-1)^{a+b+1}x^{a+m-1}y^{b+m-1})^{(-1)^{a+b+1}h^{a,b}(X)}.$$

Finally, by the Künneth formula, we have

$$G(x,y,q)=\sum_{\Delta=1}^{\infty}h^{a,b}(M_{X,H}(1,c,\Delta))x^a y^b q^{\Delta}=(1+x)^{\frac{b_1}{2}}(1+y)^{\frac{b_1}{2}}F(x,y,q).$$

Hence, these Hodge numbers stabilize to

$$(1+x)^{\frac{b_1}{2}}(1+y)^{\frac{b_1}{2}}((1-q)F(x,y,q)|_{q=1}).$$

□

More generally, we are interested in the stabilization of the motive of $M_{X,H}(r,c,\Delta)$. We recall the definition and some basic properties of the Grothendieck ring of varieties. The *Grothendieck ring of varieties over a field k*, $K_0(\mathrm{var}_k)$, is the quotient of the free abelian group on varieties $[X]$ of finite type over k by the scissor relations,

$$[X]=[Y]+[Z]$$

if Y and Z are disjoint locally closed subvarieties of X with $X=Y\cup Z$. Define multiplication by

$$[X]\cdot[Y]=[X\times Y].$$

We denote the class of the affine line $[\mathbb{A}^1]$ by $\mathbb{L}$. The Grothendieck ring of schemes over k is defined similarly, but the scissor relations imply that the class of a scheme is the same as the class of the variety with the induced reduced structure.

Consider $R=K_0(\mathrm{var}_k)[\mathbb{L}^{-1}]$. The ring R has a $\mathbb{Z}$-graded decreasing filtration $\mathcal{F}$ generated by

$$[X]\mathbb{L}^a\in\mathcal{F}^i \quad \text{if} \quad \dim(X)+a\leq -i.$$

Define the ring A^- by

$$A^- := \varprojlim_i \mathcal{F}^0R/\mathcal{F}^iR\otimes_{\mathcal{F}^0R}R.$$

We note that in A^-, the elements $\mathbb{L}$ and $\mathbb{L}^k - 1$ for k a positive integer are invertible, so we have a well-defined map from $\tilde{R} = R[(\mathbb{L}^k - 1)^{-1}]$ to A^-. Consequently, the classes of moduli stacks $[\mathcal{M}_{X,H}(r, c, \Delta)]$ are well-defined in A^- [BD07, Joy07].

Elements of the ring A^- have a natural notion of dimension generalizing the dimension of smooth projective varieties. We have the inequality

$$\dim(X + Y) \leq \dim(X) + \dim(Y).$$

By Hironaka's resolution of singularities [Hir64], if k has characteristic 0, $K_0(\mathrm{var}_k)$ is generated by the classes of smooth projective irreducible varieties. The Poincaré polynomial induces a map

$$P(t) : K_0(\mathrm{var}_k) \to \mathbb{Z}[t]$$

and the resulting map is known as the *virtual Poincaré polynomial*. Similarly, the Hodge polynomial induces a map

$$H(x, y) : K_0(\mathrm{var}_k) \to \mathbb{Z}[x, y]$$

and the resulting map is known as the *virtual Hodge polynomial*. When X is a smooth, complex projective variety, then the virtual Poincaré and the virtual Hodge polynomials coincide with the ordinary Poincaré and Hodge polynomials. More generally, if the cohomology of a variety is pure, then the virtual Poincaré and Hodge polynomials coincide with the ordinary Poincaré and Hodge polynomials. The virtual Poincaré polynomial can be extended to R (resp. A^-), where it takes values in $\mathbb{Z}[t^{\pm 1}]$ (resp. $\mathbb{Z}((t^{-1}))$) [Joy07]. Similarly, the virtual Hodge polynomial can be extended to R (respectively, A^-), were it takes values in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ (resp., $\mathbb{Z}((x^{-1}, y^{-1}))$) [Joy07].

Given a sequence of smooth projective varieties X_i of dimension d_i, we would like to have a notion of stabilization in A^- that guarantees that the low-degree Betti numbers of X_i stabilize. The correct notion of stabilization is *not* the naive one of asking the sequence $[X_i]$ to be convergent in A^-. Instead we consider the sequence $\mathbb{L}^{-d_i}[X_i]$. By Poincaré duality,

$$P_{[X_i]}(t) = P_{\mathbb{L}^{-d_i}[X_i]}(t^{-1}), \quad H_{[X_i]}(x, y) = H_{\mathbb{L}^{-d_i}[X_i]}(x^{-1}, y^{-1}).$$

If the sequence $\mathbb{L}^{-d_i}[X_i]$ converges in A^-, then the low-degree Betti numbers (and Hodge numbers) of the X_i stabilize.

Definition 3.5. We say a sequence of elements $a_i \in A^-$ *stabilizes* to a if the sequence $\mathbb{L}^{-\dim(a_i)} a_i$ converges to a.

It is convenient to think of the sequence $a_i = \mathbb{L}^{-d_i}[X_i]$ as a generating function

$$F(q) = \sum_i a_i q^i,$$

which we can think of as an element of $A^-\{\{q\}\}$, the ring of Puiseux series in q with coefficients in A^-. We will often index our varieties by the discriminant Δ or the second Chern character d. Since Δ and d can take fractional values, it is convenient to allow series in q with fractional exponents with bounded denominator. The following proposition generalizes Proposition 3.1 to the present context.

Proposition 3.6. *Let a_i be a sequence in A^-. Then a_i converges to a in A^- if and only if the series $(1 - q) \sum a_i q^i$ is convergent at $q = 1$ and the sum at $q = 1$ is a.*

Proof. The proof is almost identical to the proof of Proposition 3.1. Set $F(q) = \sum_{i=1}^{\infty} a_i q^i$. We can write $(1 - q)F(q)$ as $\sum_{i=1}^{\infty}(a_i - a_{i-1})q^i$, where $a_0 = 0$. If we evaluate this sum at $q = 1$, the nth partial sum is a_n. The proposition follows. $\square$

Let $\mathcal{P}(\Delta)$ denote the set of partitions of Δ. Let α be the partition $\alpha_1^{i_1}, \cdots, \alpha_j^{i_j}$, where the part α_m is repeated i_m times. Let $|\alpha|$ be the length (equivalently, the number of parts) of the partition. Let $X^{(\alpha)}$ denote the product $X^{(i_1)} \times \cdots \times X^{(i_j)}$. Göttsche [Got01] calculates the motive of the Hilbert scheme of points and finds that

$$[X^{[\Delta]}] = \sum_{\alpha \in \mathcal{P}(\Delta)} [X^{(\alpha)} \times \mathbb{A}^{\Delta - |\alpha|}].$$

This furthermore implies the following equality of generating functions

$$\sum_{\Delta=1}^{\infty} [X^{[\Delta]}] \mathbb{L}^{-2\Delta} q^{\Delta} = \prod_{m=1}^{\infty} Z_X(\mathbb{L}^{-m-1} q^m), \tag{3}$$

where Z_X is the motivic Zeta function of X (see §2.2).

Vakil and Wood [VW15, Conjecture 1.25] conjecture that the sequence $[X^{(\Delta)}]\mathbb{L}^{-2\Delta}$ converges in A^-. By Göttsche's formula (3), this conjecture also implies that the sequence $[X^{[\Delta]}]\mathbb{L}^{-2\Delta}$ converges. The conjecture is known when X is a rational surface, but is open in general (see [VW15, Proposition 4.2] and [LL04, Theorem 3.9]).

In this paper, we will show that if X is a rational surface and H is an ample line bundle with $K_X \cdot H < 0$, then the sequence $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilizes in A^- and compute the limit explicitly (see Theorem 6.2). The sequences $[X^{(\Delta)}]$ and $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilize in A^- to limits that only differ by a factor of $(1 - \mathbb{L})$. We expect the same theorem to hold for any ample line bundle H. For more general surfaces X, we do not know whether the sequence $[X^{[\Delta]}]$ stabilizes in A^-. It is still worth asking whether the sequence $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilizes in A^-. We expect that when both $[X^{[\Delta]}]$ and $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilize in A^-, they stabilize to the same limit up to a factor of $(1 - \mathbb{L})$. However, we do conjecture that the (virtual) Poincaré or Poincaré-Hodge polynomials of the sequence $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilize.

4. The Wall-crossing and Blowup formulae

In this section, we study the stabilization of the cohomology of the moduli spaces of sheaves under wall-crossing and blowing up.

4.1. **Wall-crossing.** We begin by studying the effect of the change of polarization on the class of the Gieseker moduli stack $\mathcal{M}_{X,H}(\gamma)$ using Joyce's wall-crossing formula. In this subsection, to simplify notation, we will omit the surface X from the notation since it will be fixed throughout the discussion.

Given γ, the ample cone of X admits a chamber decomposition where for ample divisors H in a given chamber the moduli stacks $\mathcal{M}_H(\gamma)$ are isomorphic. When the ample H crosses a wall, certain sheaves in $\mathcal{M}_H(\gamma)$ become destabilized and new sheaves may become semistable. Joyce gives an inductive formula for computing the change in $[\mathcal{M}_H(\gamma)]$ in $\tilde{R}$ in terms of the possible Harder-Narasimhan filtrations of unstable sheaves. Let $\gamma_1, \ldots, \gamma_\ell$ be Chern characters with $\sum_{i=1}^{\ell} \gamma_i = \gamma$. The Chern characters γ_i are the potential characters of the Harder-Narasimhan factors of certain sheaves with character γ. Let r_i, μ_i and Δ_i be the rank, the slope and the discriminant of γ_i.

Definition 4.1. If for all i, we have either

A) $\mu_{H_1}(\gamma_i) > \mu_{H_1}(\gamma_{i+1})$ and $\mu_{H_2}(\sum_{j=1}^{i} \gamma_j) \leq \mu_{H_2}(\sum_{j=i+1}^{\ell} \gamma_j)$, or

B) $\mu_{H_1}(\gamma_i) \leq \mu_{H_1}(\gamma_{i+1})$ and $\mu_{H_2}(\sum_{j=1}^{i} \gamma_j) > \mu_{H_2}(\sum_{j=i+1}^{\ell} \gamma_j)$,

then $S^{\mu}(\gamma_1, \cdots, \gamma_\ell; H_1, H_2)$ is $(-1)^u$, where u is the number of i for which the inequalities of case B hold. Otherwise, $S^{\mu}(\gamma_1, \cdots, \gamma_\ell; H_1, H_2) = 0$. Similarly, we define $S(\gamma_1, \cdots, \gamma_\ell; H_1, H_2)$ with μ_{H_i} replaced with the reduced Hilbert polynomial with respect to H_i.

Joyce proves the following theorem. Joyce assumes that $-K_X$ is nef, but his proof goes through assuming that K_X has negative intersection with H_1 and H_2.

Theorem 4.2. [Joy08, Theorem 6.21] *If H_1 and H_2 are ample line bundles with $K_X \cdot H_i < 0$ for $i \in \{1,2\}$, then the following equation holds in $\tilde{R}$,*

$$[\mathcal{M}_{H_2}(\gamma)] = \sum_{\gamma_1+\cdots+\gamma_\ell=\gamma} \mathrm{S}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)\,\mathbb{L}^{-\sum_{1\le i<j\le\ell}\chi(\gamma_j,\gamma_i)} \prod_{i=1}^{\ell}[\mathcal{M}_{H_1}(\gamma_i)]. \tag{4}$$

Theorem 4.2 holds if we replace all the $\mathcal{M}_{X,H_i}(\gamma)$ with $\mathcal{M}^{\mu}_{X,H_i}(\gamma)$, the moduli stack of slope-H_i-semistable sheaves, and $\mathrm{S}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)$ with $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)$.

As part of the proof of this theorem, Joyce shows that if one of H_1 and H_2 is on a wall and the other is in an adjoining chamber, then there's a simpler description of $\mathrm{S}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)$.

Corollary 4.3. *If H_1 is on a wall and H_2 is in an adjacent chamber, then*

- $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2) = 1$ *if $\mu_{H_1}(\gamma_i) = \mu_{H_1}(\gamma)$ and $\mu_{H_2}(\gamma_i) > \mu_{H_2}(\gamma_{i+1})$ for all i,*
- $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2) = 0$ *otherwise.*

If H_2 is on a wall and H_1 is in an adjacent chamber, then

- $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2) = 1$ *if $\ell = 1$,*
- $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2) = -1$ *if $\ell > 1$, $\mu_{H_2}(\gamma_i) = \mu_{H_2}(\gamma)$ and $\mu_{H_1}(\gamma_i) > \mu_{H_1}(\gamma_{i+1})$ for all i,*
- $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2) = 0$ *otherwise.*

The same statement holds with $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)$ replaced by $\mathrm{S}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)$ and μ_{H_i} replaced by the reduced Hilbert polynomial with respect to H_i.

We want to apply Theorem 4.2 in a slightly more general setting, where the line bundles are no longer ample but nef. Fortunately, Joyce's proof extends to give the following.

Corollary 4.4. *Theorem 4.2 holds even if the H_i are only nef, so long as all the terms on the right hand side of Equation (4) are in A^- and the sum is convergent. In particular, Theorem 4.2 holds if the sum is finite and all the stacks involved are of finite type.*

Definition 4.5. We say that a line bundle L is *admissible* if for all Chern characters γ the terms on the right hand side of Equation (4) are in A^- and the sum is convergent when $L = H_1$ and H_2 is ample and H_1 is ample and $L = H_2$.

Corollary 4.6. *If L is a big and nef line bundle with $K_X \cdot L < 0$, then L is admissible.*

Proof. In [Joy08, Theorem 5.16], Joyce proves that if H_1 and H_2 are ample, then the sum on the right hand side of Equation (4) is finite. His proof only uses the ampleness to show that given two divisor classes ξ_1, ξ_2 such that $H \cdot (\xi_1 - \xi_2) = 0$, with $H = aH_1 + bH_2$ with a, $b \geq 0$, then $(\xi_1 - \xi_2)^2 \leq 0$. This is still true if $H_1 = L$ and H_2 is ample or vice versa. From this it follows that all the moduli stacks of μ_L-stable sheaves are of finite type. These two facts together are enough to show that L is admissible. □

We show that if $\Delta(\gamma)$ is sufficiently large, then the dimensions of the contributions with $\ell > 1$ have arbitrarily negative upper bounds. More precisely, we have the following.

Proposition 4.7. *Fix a positive integer r and a class $c \in \mathrm{Pic}(X)$. Let H_1 be an ample class and let H_2 be a big and nef class. Assume that $H_i \cdot K_X < 0$. Then for all integers v, there is a $\Delta_0(v) > 0$ such that for all $\gamma \in K^0(X)$ with $\mathrm{rk}(\gamma) = r$, $c_1(\gamma) = c$, and $\Delta(\gamma) \geq \Delta_0(v)$, all the terms in the sum with $\ell > 1$ have dimension less than v. Moreover, if $\mathcal{C}$ is a compact convex subset of the K_X-negative part of the big and nef cone and $H_1, H_2 \in \mathcal{C}$, then $\Delta_0(v)$ can be chosen to depend only on $\mathcal{C}$.*

In particular, $[\mathcal{M}_{H_1}(\gamma)]$ stabilizes in A^- as Δ tends to ∞ if and only if $[\mathcal{M}_{H_2}(\gamma)]$ does. In that case, they have the same stabilization.

Proof. By Corollary 4.6, H_2 is admissible and the moduli stacks $\mathcal{M}^{\mu,s}_{X,H_2}(\gamma)$ are of finite-type. In this case, since Joyce proves that there are finitely many walls, we can analyze the wall-crossing one wall at a time. The dimension of $[\mathcal{M}_{H_1}(\gamma_i)]$ is $-\chi(\gamma_i,\gamma_i)$. Hence, the dimension of the term $\mathbb{L}^{-\sum_{1\leq i<j\leq \ell}\chi(\gamma_j,\gamma_i)}\prod_{i=1}^{\ell}[\mathcal{M}_{H_1}(\gamma_i)]$ is given by

$$\sum_{i=1}^{\ell} -\chi(\gamma_i,\gamma_i) - \sum_{1\leq i<j\leq \ell}\chi(\gamma_j,\gamma_i).$$

By the biadditivity of the Euler pairing, this is equal to

$$\sum_{1\leq i<j\leq \ell}\chi(\gamma_i,\gamma_j) - \chi(\gamma,\gamma). \tag{5}$$

Denote the rank, slope and discriminant of γ_i by r_i,μ_i and Δ_i, respectively. The Riemann-Roch Theorem says that

$$\chi(\gamma,\gamma) = r^2(\chi(\mathcal{O}_X) - 2\Delta) \tag{6}$$

$$\chi(\gamma_i,\gamma_j) = r_ir_j\left(\frac{(\mu_i-\mu_j)^2}{2} - \frac{K_X}{2}\cdot(\mu_j-\mu_i) + \chi(\mathcal{O}_X) - \Delta_i - \Delta_j\right) \tag{7}$$

Substituting Equations (6) and (7) into Equation (5), we obtain

$$r^2(2\Delta - \chi(\mathcal{O}_X)) + \sum_{1\leq i<j\leq \ell} r_ir_j\left(\frac{(\mu_j-\mu_i)^2}{2} - \frac{K_X}{2}\cdot(\mu_j-\mu_i) + \chi(\mathcal{O}_X) - \Delta_i - \Delta_j\right).$$

The dimension of the $\ell = 1$ summand is $r^2(2\Delta - \chi(\mathcal{O}_X))$, so the difference between the two dimensions is given by

$$d_{\gamma_1,\dots,\gamma_\ell} = -\sum_{1\leq i<j\leq \ell} r_ir_j\left(\frac{(\mu_j-\mu_i)^2}{2} - \frac{K_X}{2}\cdot(\mu_j-\mu_i) + \chi(\mathcal{O}_X) - \Delta_i - \Delta_j\right).$$

Since $\sum_{i=1}^{\ell}\gamma_i = \gamma$, we can express the slope μ and discriminant Δ of γ in terms of the slopes μ_i and discriminants Δ_i of γ_i as follows

$$\mu = \sum_{i=1}^{\ell}\frac{r_i}{r}\mu_i \quad \text{and} \quad \Delta = \frac{1}{2r^2}\left(\sum_{i=1}^{\ell} r_i\mu_i\right)^2 - \frac{1}{2r}\sum_{i=1}^{\ell} r_i(\mu_i^2 - 2\Delta_i).$$

In particular,

$$\sum_{i=1}^{\ell} r_i\Delta_i = r\Delta - \frac{1}{2r}\left(\sum_{i=1}^{\ell} r_i\mu_i\right)^2 + \frac{1}{2}\sum_{i=1}^{\ell} r_i\mu_i^2.$$

Observe that

$$\begin{aligned}
\frac{1}{2r}\sum_{1\leq i<j\leq \ell} r_ir_j(\mu_j-\mu_i)^2 &= \frac{1}{2r}\sum_{i=1}^{\ell} r_i(r-r_i)\mu_i^2 - \frac{1}{r}\sum_{1\leq i<j\leq \ell} r_ir_j\mu_i\mu_j \\
&= -\frac{1}{2r}\left(\sum_{i=1}^{\ell} r_i\mu_i\right)^2 + \frac{1}{2}\sum_{i=1}^{\ell} r_i\mu_i^2 = \left(\sum_{i=1}^{\ell} r_i\Delta_i\right) - r\Delta.
\end{aligned}$$

Moreover,

$$-\sum_{1\le i<j\le \ell} r_ir_j(-\Delta_i-\Delta_j)=\sum_{i=1}^{\ell} r_i(r-r_i)\Delta_i.$$

Using these two observations and splitting the coefficient of the quadratic term as $\frac{1}{2}=\frac{1}{2r}+\frac{1}{2}\left(1-\frac{1}{r}\right)$, the expression for $d_{\gamma_1,\dots,\gamma_\ell}$ becomes

$$d_{\gamma_1,\dots,\gamma_\ell}=r\Delta-\sum_{1\le i<j\le \ell} r_ir_j\left(\frac{1}{2}\left(1-\frac{1}{r}\right)(\mu_j-\mu_i)^2-\frac{1}{2}K_X\cdot(\mu_j-\mu_i)+\chi(\mathcal{O}_X)\right)+\sum_{i=1}^{\ell} r_i(r-r_i-1)\Delta_i.$$

Therefore, we can write

$$d_{\gamma_1,\dots,\gamma_\ell}=r\Delta+G(\mu_j-\mu_i)+\sum_{i=1}^{\ell}\alpha_i\Delta_i,$$

where $\alpha_i\ge 0$ and G is a quadratic polynomial with the coefficients of the square terms $(\mu_j-\mu_i)^2$ strictly negative. If a μ-semistable sheaf gets destabilized on a wall corresponding to an ample (or big and nef) divisor H, then it becomes strictly semistable on the wall and the Harder-Narasimhan factors all have the same H-slope. Since there is a big and nef H (either H_1 or H_2) such that $H\cdot(\mu_j-\mu_i)=0$, the Hodge index theorem implies that $(\mu_j-\mu_i)^2\le 0$ with equality if and only if $\mu_j-\mu_i=0$. Since the intersection form is negative definite on $H^\perp$, as the μ_i vary, G is bounded below by some constant $C_{r_1,\dots,r_\ell}$. By the Bogomolov inequality, if $\mathcal{M}_{H_1}(\gamma_i)$ is nonempty, then $\Delta_i\ge 0$. Hence, if we fix $\ell\ge 2$ and the ranks r_i, then

$$d_{\gamma_1,\dots,\gamma_\ell}\ge r\Delta+C_{r_1,\dots,r_\ell}.$$

Since there are only finitely many possible choices for ℓ and the r_i, letting C be the minimum over all possible constants $C_{r_1,\dots,r_\ell}$ we get

$$d:=\min d_{\gamma_1,\dots,\gamma_\ell}\ge r\Delta+C.$$

If Δ is sufficiently large, then d can be made arbitrarily large and the difference between $[\mathcal{M}_{X,H_1}(\gamma)]$ and $[\mathcal{M}_{X,H_2}(\gamma)]$ has dimension at least d less than the dimension of $\mathcal{M}_{X,H_1}(\gamma)$. ☐

Taking the virtual Poincaré (resp. Hodge) polynomials, we obtain the following corollary.

Corollary 4.8. *Assume H_1 and H_2 are polarizations on X such that $K_X\cdot H_i<0$. If the virtual Poincaré (resp. Hodge) polynomials of $\mathcal{M}_{X,H_1}(\gamma)$ stabilize as Δ goes to ∞, then the virtual Poincaré (resp. Hodge) polynomials of $\mathcal{M}_{X,H_2}(\gamma)$ stabilize, and they have the same stable limit.*

The next corollary shows that the difference between Gieseker (semi)stability and slope (semi)stability does not effect stabilization.

Corollary 4.9. *Let H be a big and nef divisor and assume that $K_X\cdot H<0$. Then the following are equivalent:*

(1) *The classes $[\mathcal{M}^{\mu}_{X,H}(r,c,\Delta)]$ stabilize in A^-.*
(2) *The classes $[\mathcal{M}_{X,H}(r,c,\Delta)]$ stabilize in A^-.*
(3) *The classes $[\mathcal{M}^{\mu,s}_{X,H}(r,c,\Delta)]$ stabilize in A^-.*

If any of them stabilize, then they all have the same stable limit. Furthermore, the same holds for the virtual Poincaré (resp. Hodge) polynomials.

Proof. Consider the locus in $\mathcal{M}^{\mu}_{X,H}(r,c,\Delta)$ which are not μ_H-stable. Such sheaves have a Jordan-Hölder filtration of length greater than 1. Suppose F has a Jordan-Hölder filtration of type $F_0\subset F_1\subset\cdots\subset F_\ell$. Let $E_i=F_i/F_{i-1}$ with class γ_i. The space of sheaves having such a filtration has codimension $\sum_{1\le i<j\le\ell}-\chi(\gamma_i,\gamma_j)$. By the proof of Proposition 4.7, this codimension tends to ∞ as Δ tends to ∞. The argument for the Gieseker-semistable locus is identical. ☐

4.2. **Blowup.** We now address blowing up. Let $p : \hat{X} \to X$ be the blowup of X at a point x. Let E be the exceptional divisor in $\hat{X}$. Let H be an ample line bundle on X. Let $\mathcal{M}^{\mu,s}_{X,H}(r,C,d)$ denote the moduli stack of torsion-free μ_H-stable sheaves on X of class (r,C,d). Similarly, let $\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r,C-mE,d)$ denote the moduli stack of torsion-free μ_{p^*H}-stable sheaves on $\hat{X}$ of class $(r,C-mE,d)$. In this subsection, it is more convenient to use ch_2 instead of Δ in order to conform to the literature. Define the following generating functions

$$Y_{X,H}(q) = \sum [\mathcal{M}^{\mu,s}_{X,H}(r,C,d)]q^{-d} \quad \text{and} \quad \hat{Y}_{X,H,m}(q) = \sum [\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r,C-mE,d)]q^{-d}.$$

Then by [Moz13, Proposition 7.3 and Corollary 7.7], the following equality holds:

$$\hat{Y}_{X,H,m}(q) = F_m(q)Y_{X,H}(q),$$

where

$$F_m(q) = \prod_{k\geq 1} \frac{1}{(1-\mathbb{L}^{rk}q^k)^r} \sum_{\sum_{i=1}^r a_i = 0,\, a_i \in \mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}} q^{-\sum_{i<j} a_i a_j}.$$

Lemma 4.10. *The sequence $[M^{\mu,s}_{\hat{X},p^*H}(r,C-mE,d)]$ stabilizes in A^- if and only if the sequence $[M^{\mu,s}_{X,H}(r,C,d)]$ does.*

Proof. We begin by defining auxiliary generating functions

$$Y'_{X,H}(q) = \sum [M^{\mu,s}_{X,H}(r,C,d)]\mathbb{L}^{2rd-C^2+r^2\chi(\mathcal{O}_X)}q^{-d}, \text{ and}$$

$$\hat{Y}'_{X,H,m}(q) = \sum [M^{\mu,s}_{\hat{X},p^*H}(r,C-mE,d)]\mathbb{L}^{2rd-C^2+r^2\chi(\mathcal{O}_X)}q^{-d}.$$

By Proposition 3.6, we need to show that $(1-q)Y'_{X,H}(q)$ is convergent at $q=1$ if and only if $(1-q)\hat{Y}'_{X,H,m}(q)$ is. Since the blowup formula provides the relation $\hat{Y}_{X,H,m}(q) = F_m(q)Y_{X,H}(q)$, it follows that $\hat{Y}'_{X,H,m}(q) = F_m(\mathbb{L}^{-2r}q)Y'_{X,H}(q)$. Hence, it suffices to show that $F_m(\mathbb{L}^{-2r}q)$ is convergent when we evaluate at $q=1$. Since $\prod_{k\geq 1}\frac{1}{(1-\mathbb{L}^{-rk}q^k)^r}$ is convergent in A^- when $q=1$, it suffices to show that

$$\sum_{\sum_{i=1}^r a_i = 0,\, a_i \in \mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}} \mathbb{L}^{-2r\sum_{i<j}a_ia_j} q^{-\sum_{i<j} a_i a_j}$$

converges in A^- when $q=1$. It is enough to show that there are only finitely many $(a_1,\dots,a_r)$ with $\sum_{i=1}^r a_i = 0$ and $a_i \in \mathbb{Z}+\frac{m}{r}$ where the quadratic form $Q(a_1,\dots,a_r) = \sum_{i<j} a_ia_j$ takes a given fixed value C.

Consider the subspace V of $\mathbb{R}^r$ where the coordinates sum to 0. We can restrict Q to a quadratic form on V. Since $\sum a_i = 0$, we have $(\sum a_i)^2 = 0$, so $\sum a_i^2 = -2\sum_{i<j} a_ia_j$. It follows that Q is negative-definite on V. This means that the level sets $Q^{-1}(\{C\})$ are compact. Once we add the requirements that the $a_i \in \mathbb{Z}+\frac{m}{r}$, there are only finitely many possible solutions as desired. □

Proposition 4.11. *Suppose that the $[\mathcal{M}^{\mu,s}_{X,H}(r,C,d)]$ stabilize in A^-. We have the relation*

$$\lim_{q\to 1}\Big((1-q)\hat{Y}'_{X,H,m}(q)\Big) = \prod_{k=1}^{\infty}\frac{1}{1-\mathbb{L}^{-k}} \lim_{q\to 1}\big((1-q)Y'_{X,H}(q)\big)$$

between the generating series for $\mathcal{M}^{\mu,s}_{X,H}(r,C,d)$ and $\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r,C-mE,d)$.

Proof. As in the proof of Lemma 4.10, we have $\hat{Y}'_{X,H,m}(q) = F_m(\mathbb{L}^{-2r}q)Y'_{X,H}(q)$. We want to multiply both sides by $1-q$ and take the limit as q goes to 1. This means that we need to find $\lim_{q\to 1} F_m(\mathbb{L}^{-2r}q)$.

We have

$$F_m(\mathbb{L}^{-2r}q) = \prod_{k\geq 1} \frac{1}{(1-\mathbb{L}^{-rk}q^k)^r} \sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}}\mathbb{L}^{2r\sum_{i<j}a_ia_j}q^{-\sum_{i<j}a_ia_j}.$$

If we take the limit as q goes to 1, we get

$$F_m(\mathbb{L}^{-2r}) = \prod_{k\geq 1} \frac{1}{(1-\mathbb{L}^{-rk})^r} \sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}}\mathbb{L}^{2r\sum_{i<j}a_ia_j}. \tag{8}$$

First, we show that the sum

$$\sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}+2r\sum_{i<j}a_ia_j}$$

is independent of m.

We can rewrite this sum as

$$\sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{-\sum_{i<j}\binom{a_i-a_j}{2}}$$

We will now give a bijection between the following two sets of ordered sequences

$$A=\{(a_i)_{i=1}^r : \sum_{i=1}^r a_i = 0,\ a_i\in\mathbb{Z}+\frac{m}{r}\}$$

and

$$B=\{(b_i)_{i=1}^r : \sum_{i=1}^r b_i = 0,\ b_i\in\mathbb{Z}+\frac{m-1}{r}\}$$

which exchanges

$$\mathbb{L}^{-\sum_{i<j}\binom{a_i-a_j}{2}} \quad \text{with} \quad \mathbb{L}^{-\sum_{i<j}\binom{b_i-b_j}{2}}.$$

Set $b_i = a_{i-1}-\frac{1}{r}$ for $i\geq 2$ and $b_1 = a_r + \frac{r-1}{r}$. This is clearly a bijection between A and B. We can write

$$\sum_{i<j}\binom{b_i-b_j}{2} = \sum_{1<i<j}\binom{b_i-b_j}{2} + \sum_{1<j}\binom{b_1-b_j}{2} = \sum_{1<i<j}\binom{a_{i-1}-a_{j-1}}{2} + \sum_{1<j}\binom{a_r+1-a_{j-1}}{2}$$

$$= \sum_{i<j<r}\binom{a_i-a_j}{2} + \sum_{j<r}\binom{a_j-a_r}{2} = \sum_{i<j}\binom{a_i-a_j}{2}.$$

This shows that the given bijection between A and B preserves the summands, and so the two sums

$$\sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{-\sum_{i<j}\binom{a_i-a_j}{2}} \quad \text{and} \quad \sum_{\sum_{i=1}^r b_i=0,\, b_i\in\mathbb{Z}+\frac{m-1}{r}} \mathbb{L}^{-\sum_{i<j}\binom{b_i-b_j}{2}}$$

are equal. Since these sums are independent of m, we can assume that $m=0$.

We will now use the Macdonald identities for the root system A_{r-1} [Coo97, Mac72] to prove the following identity

$$\sum_{\sum_{i=1}^r a_i=0,\, a_i\in\mathbb{Z}} \mathbb{L}^{-\sum_{i<j}\binom{a_i-a_j}{2}} = \prod_{k=1}^{\infty}\frac{(1-\mathbb{L}^{-rk})^r}{1-\mathbb{L}^{-k}}. \tag{9}$$

Since the expression in Equation (8) is independent of m, substituting Equation (9) into Equation (8), we obtain that

$$F_m(\mathbb{L}^{-2r}) = \prod_{k=1}^{\infty} \frac{1}{1-\mathbb{L}^{-k}}$$

as desired.

To conclude the proof we must prove the identity in Equation (9). Let S_r denote the symmetric group on r letters. For a permutation $\sigma \in S_r$, let $\epsilon(\sigma)$ denote the sign of the permutation. With this notation, the Macdonald identities for the A_{r-1} affine root system read [Mac72, 0.4] (see also [Coo97, Theorem 2.3])

$$\prod_{k=1}^{\infty}(1-t^k)^{r-1} \prod_{1\leq i\neq j\leq r}\left(1-\frac{x_i t^k}{x_j}\right) = \sum_{m_1+\cdots+m_r=0,\, m_i\in\mathbb{Z}} \chi(rm_1,\dots,rm_r)t^{\frac{1}{2}(\sum_{i=1}^r rm_i^2+m_i(r+1-2i))},$$

where

$$\chi(m_1,\dots,m_r) = \frac{\sum_{\sigma\in S_r}\epsilon(\sigma)\prod_{i=1}^r x_i^{m_{\sigma(i)}+r+1-2\sigma(i)}}{\sum_{\sigma\in S_r}\epsilon(\sigma)\prod_{i=1}^r x_i^{r+1-2\sigma(i)}}.$$

The Macdonald identities give an equation in $\mathbb{C}[\frac{x_1}{x_2},\frac{x_2}{x_3},\dots,\frac{x_{r-1}}{x_r}][[t]]$. Hence, we can set $x_i=\omega^i$, where ω is a primitive rth root of unity, and obtain

$$\chi(rm_1,\dots,rm_r)=1$$

for any choice of $m_1,\dots,m_r$. Using the fact that $m_1+\cdots+m_r=0$, we have

$$\frac{1}{2}\sum_{i=1}^{r}\left(rm_i^2 - m_i(r+1-2i)\right) = \sum_{1\leq i<j\leq r}\binom{m_i-m_j}{2}.$$

Since the right hand side of the Macdonald identity is invariant under replacing all the m_i with $-m_i$, we see that the right hand side of the Macdonald identity equals

$$\sum_{m_1+\cdots+m_r=0,\, m_i\in\mathbb{Z}} t^{\sum_{1\leq i<j\leq r}\binom{m_i-m_j}{2}}.$$

On the other hand, the left hand side of the Macdonald identity becomes

$$\prod_{k=1}^{\infty}(1-t^k)^{r-1}\prod_{1\leq i\neq j\leq r}(1-\omega^{i-j}t^k) = \prod_{k=1}^{\infty}(1-t^k)^{r-1}\prod_{i=1}^{r-1}(1-\omega^i t^k)^r.$$

The polynomial $\prod_{i=1}^{r-1}(1-\omega^i x)$ is a polynomial in x whose roots are the inverses of the nontrivial rth roots of unity, each occurring with multiplicity 1. Since the constant term is 1, the polynomial must be $1+x+\cdots+x^{r-1}=\frac{1-x^r}{1-x}$. Hence, we have

$$\prod_{i=1}^{r-1}(1-\omega^i t^k)^r = \frac{(1-t^{kr})^r}{(1-t^k)^r}.$$

This lets us rewrite the left hand side of the Macdonald identities as

$$\prod_{k=1}^{\infty}\frac{(1-t^{rk})^r}{(1-t^k)}.$$

Now setting $t=\mathbb{L}^{-1}$, we obtain Equation (9)

$$\sum_{m_1+\cdots+m_r=0}\mathbb{L}^{-\sum_{1\leq i<j\leq r}\binom{m_i-m_j}{2}} = \prod_{k=1}^{\infty}\frac{(1-\mathbb{L}^{-rk})^r}{(1-\mathbb{L}^{-k})}$$

as desired. $\square$

Corollary 4.12. *The virtual Poincaré (respectively, Hodge) polynomial of $\mathcal{M}^{\mu,s}_{X,H}(r,c,\Delta)$ stabilizes as Δ tends to ∞ if and only if the virtual Poincaré (respectively, Hodge) polynomial of $\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r,c-mE,\Delta)$ does. Furthermore, if these polynomials stabilize, then the ratio of the generating functions of the stable polynomials is given by*

$$\prod_{k=1}^{\infty}\frac{1}{1-t^{2k}} \quad \left(\textit{resp., } \prod_{k=1}^{\infty}\frac{1}{1-x^ky^k}\right).$$

So far in the blowup section we have used the moduli stack of slope-stable sheaves. However, using Corollary 4.9, the same result also holds for moduli stacks of Gieseker semistable sheaves.

Corollary 4.13. *Let X be a smooth surface and let H be a polarization such that $K_X\cdot H<0$. Assume that the classes $[\mathcal{M}_{X,H}(r,c,\Delta)]$ stabilize in A^-. Then $[\mathcal{M}_{\hat{X},p^*H}(r,c-mE,\Delta)]$ stabilize in A^-. The virtual Poincaré (respectively, Hodge) polynomial of $\mathcal{M}_{X,H}(r,c,\Delta)$ stabilizes as Δ tends to ∞ if and only if the virtual Poincaré (respectively, Hodge) polynomial of $\mathcal{M}_{\hat{X},p^*H}(r,c-mE,\Delta)$ does. Furthermore, if these polynomials stabilize, then the ratio of the generating functions of the stable polynomials is given by*

$$\prod_{k=1}^{\infty}\frac{1}{1-t^{2k}} \quad \left(\textit{resp., } \prod_{k=1}^{\infty}\frac{1}{1-x^ky^k}\right).$$

5. The stabilization of cohomology for $\mathbb{F}_1$ and $\mathbb{P}^2$

The previous section studied the behavior of stabilization under wall-crossing and blowing up. In this section, we will show stabilization for $\mathbb{P}^2$ and $\mathbb{F}_1$.

Definition 5.1. Let F be the fiber class on a ruled surface X. A torsion-free sheaf V on X is F-slope-semistable if for all nonzero proper subsheaves $W\subset V$, we have

$$\frac{c_1(W)\cdot F}{\mathrm{rk}(W)}\le\frac{c_1(V)\cdot F}{\mathrm{rk}(V)}.$$

There is an algebraic stack $\mathcal{M}_{X,F}(\gamma)$ of F-slope-semistable sheaves with Chern character γ.

Mozgovoy calculates the classes of these moduli stacks in A^- for any ruled surface X over a curve C. Specifically, when $r\mid c\cdot F$, his [Moz13, Theorem 1.1] gives

$$\sum[\mathcal{M}_{X,F}(r,c_1,\Delta)]q^{r\Delta}=\frac{[\mathrm{Jac}(C)]}{\mathbb{L}-1}\prod_{i=1}^{r-1}Z_C(\mathbb{L}^i)\prod_{k\ge1}\prod_{i=-r}^{r-1}Z_C(\mathbb{L}^{rk+i}q^k),$$

where Z_C is the motivic zeta function of C. These stacks are empty when $r\nmid c\cdot F$.

Proposition 5.2. *Let X be a rational ruled surface. If $r\mid c\cdot F$, then the classes $[\mathcal{M}_{X,F}(r,c,\Delta)]$ stabilize in A^- to*

$$\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-i})^4}.$$

Proof. We want to show that the $[\mathcal{M}_{X,F}(r,c,\Delta)]$ stabilize. Consider the power series

$$\tilde{G}(q)=\sum_{\Delta\ge0}\mathbb{L}^{-r^2(2\Delta-1)}[\mathcal{M}_{X,F}(r,c,\Delta)]q^{r\Delta}.$$

By Proposition 3.6, it is enough to show that the series $(1-q)\tilde{G}(q)$ is convergent at $q=1$.

We can write $\tilde{G}(q)$ as

$$\sum_{\Delta\geq 0}\mathbb{L}^{-r^2(2\Delta-1)}[\mathcal{M}_{X,F}(r,c,\Delta)]q^{r\Delta}=\mathbb{L}^{r^2}\sum_{\Delta\geq 0}[\mathcal{M}_{X,F}(r,c,\Delta)](\mathbb{L}^{-2r}q)^{r\Delta}.$$

Substituting $\mathbb{L}^{-2r}q$ for q, we get

$$\tilde{G}(q)=\mathbb{L}^{r^2}\frac{1}{\mathbb{L}-1}\prod_{i=1}^{r-1}Z_{\mathbb{P}^1}(\mathbb{L}^i)\prod_{k\geq 1}\prod_{i=-r}^{r-1}Z_{\mathbb{P}^1}(\mathbb{L}^{-rk+i}q^k).$$

Using the fact that

$$Z_{\mathbb{P}^1}(x)=\frac{1}{(1-x)(1-\mathbb{L}x)},$$

this expression becomes

$$\mathbb{L}^{r^2}\frac{1}{\mathbb{L}-1}\prod_{i=1}^{r-1}\frac{1}{(1-\mathbb{L}^i)(1-\mathbb{L}^{i+1})}\prod_{k\geq 1}\prod_{i=-r}^{r-1}\frac{1}{(1-\mathbb{L}^{-rk+i}q^k)(1-\mathbb{L}^{-rk+i+1}q^k)}.$$

This is a product of terms of the form $\frac{1}{1-\mathbb{L}^a q^b}$. The only term with $a=0$ occurs when $k=1$ and $i=r-1$, in which case we get $\frac{1}{1-q}$. For a given power of a, there are only finitely many such terms, so if we set $q=1$, the series $(1-q)\tilde{G}(q)$ is convergent. The limit of this infinite series is

$$\lim_{q\to 1}(1-q)\tilde{G}(q)=\mathbb{L}^{r^2}\frac{1}{\mathbb{L}-1}\prod_{i=1}^{r-1}\frac{1}{(1-\mathbb{L}^i)(1-\mathbb{L}^{i+1})}\prod_{\substack{k\geq 1,-r\leq i<r\\(k,i)\neq(1,r-1)}}\frac{1}{(1-\mathbb{L}^{-rk+i})(1-\mathbb{L}^{-rk+i+1})}$$

The reader can check that this expression simplifies to

$$\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-i})^4}$$

concluding the proof of the proposition.

□

Corollary 5.3. *If X is a ruled surface and F is the fiber class, then F is admissible in the sense of Definition 4.5.*

Proof. By Mozgovoy's calculation [Moz13] of $[M^{\mu}_{X,F}(\gamma)]$, this class is in A^-. It suffices to show that the sum on the right hand side of Equation (4) is convergent in A^-. This will follow if we can show that for any D, there are only finitely many terms with dimension greater than D. We can assume that H_2 is close to F, in which case we know that the factor $\mathrm{S}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)\neq 0$ only if $\mu_F(\gamma_i)=\mu_F(\gamma)$ for all i, and $\mu_{H_2}(\gamma_i)>\mu_{H_2}(\gamma_{i+1})$ for all i. In particular, $(\mu_i-\mu_j)^2=0$ since these differ by multiples of F. The dimension of a given term is

$$d(\gamma_1,\ldots,\gamma_\ell):=r^2(2\Delta-\chi(\mathcal{O}_X))+\sum_{1\leq i<j\leq\ell}r_ir_j\left(\frac{(\mu_j-\mu_i)^2}{2}-\frac{K_X}{2}\cdot(\mu_j-\mu_i)+\chi(\mathcal{O}_X)-\Delta_i-\Delta_j\right).$$

Hence, we get

$$d(\gamma_1,\ldots,\gamma_\ell)\leq r^2(2\Delta)+\sum_{1\leq i<j\leq\ell}r_ir_j\left(-\frac{K_X}{2}\cdot(\mu_j-\mu_i)+\chi(\mathcal{O}_X)\right).$$

We can write $\mu_j-\mu_i=a_{ij}F$ with $a_{ij}<0$. It follows that $-\frac{K_X}{2}\cdot(\mu_j-\mu_i)=a_{ij}$.

It suffices to show that for any integer $N>0$, there are only finitely many terms with

$$-\frac{K_X}{2}\cdot(\mu_1-\mu_\ell)<N.$$

There are only finitely many positive integers r_i that add up to r. Since $\Delta_i \geq 0$ and their denominators are bounded, there are only finitely many Δ_i with $r\Delta = \sum r_i \Delta_i$. Finally, there are only finitely many μ_i since we can write $\mu_i = aE + b_i F$ with b_i a rational number with bounded denominator in a bounded interval centered at $\mu \cdot \frac{K_X}{2}$. This concludes the proof. $\square$

Theorem 5.4. *Assume $r \mid c \cdot F$. Then the classes $[\mathcal{M}_{\mathbb{F}_1, E+F}(r, c, \Delta)]$ stabilize in A^- to*

$$\prod_{k=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

Proof. The statement is true for F-semistable sheaves by Proposition 5.2. We will apply Corollary 4.4 with $H_1 = F$ and $H_2 = E + F$, to compute $[\mathcal{M}^{\mu}_{\mathbb{F}_1, E+F}(\gamma)]$. We want to show that in Equation (4) in Theorem 4.2 the dimension of any summand on the right hand side with $\ell > 1$ is much less than the dimension of the left hand side. The dimension of the left hand side is $-\chi(\gamma, \gamma)$. The dimension of each summand on the right hand side is $-\sum \chi(\gamma_i, \gamma_i) - \sum_{i<j} \chi(\gamma_j, \gamma_i)$. By the biadditivity of Euler characteristic, the difference between the left hand side and the right hand side is $-\sum_{i<j} \chi(\gamma_i, \gamma_j)$.

Express

$$\chi(\gamma, \gamma) = r^2(\chi(\mathcal{O}_{\mathbb{F}_1}) - 2\Delta), \quad \chi(\gamma_i, \gamma_i) = r_i^2(\chi(\mathcal{O}_{\mathbb{F}_1}) - 2\Delta_i).$$

By the additivity of the Euler characteristic, we have

$$-2 \sum_{1 \leq i < j \leq \ell} \chi(\gamma_i, \gamma_j) = -\chi(\gamma, \gamma) + \sum_{i=1}^{\ell} \chi(\gamma_i, \gamma_i) + \sum_{1 \leq i < j \leq \ell} (\chi(\gamma_j, \gamma_i) - \chi(\gamma_i, \gamma_j)). \tag{10}$$

We would like to estimate the quantity $-2\sum_{1 \leq i < j \leq \ell} \chi(\gamma_i, \gamma_j)$. It is cleanest to take the terms separately.

By Riemann-Roch, we have

$$\sum_{1 \leq i < j \leq \ell} (\chi(\gamma_j, \gamma_i) - \chi(\gamma_i, \gamma_j)) = \sum_{1 \leq i < j \leq \ell} r_i r_j K_{\mathbb{F}_1} \cdot (\mu_j - \mu_i). \tag{11}$$

Let $c_1(\gamma) = raE + bF$ and let $c_1(\gamma_i) = r_i a_i E + b_i F$. We are assuming that a is an integer. We may also assume that all the a_i are integers, otherwise the moduli stack $\mathcal{M}_{\mathbb{F}_1, F}(\gamma_i)$ is empty. Substituting, we obtain

$$r_i r_j(\mu_j - \mu_i) = -r_i r_j a_i E - r_j b_i F + r_i r_j a_j E + r_i b_j F = r_i r_j (a_j - a_i)E + (r_i b_j - r_j b_i)F.$$

Since $K_{\mathbb{F}_1} = -2E - 3F$, we obtain

$$\sum_{1 \leq i < j \leq \ell} r_i r_j K_{\mathbb{F}_1} \cdot (\mu_j - \mu_i) = \sum_{1 \leq i < j \leq \ell} \left(r_i r_j (a_i - a_j) - 2(r_i b_j - r_j b_i) \right). \tag{12}$$

Using the relations

$$b = \sum_{i=1}^{\ell} b_i \quad \text{and} \quad r = \sum_{i=1}^{\ell} r_i,$$

we can express

$$\begin{aligned} \sum_{1 \leq i < j \leq \ell} (r_j b_i - r_i b_j) &= -\sum_{i=2}^{\ell} \sum_{j=i+1}^{\ell} (r_i b_j - r_j b_i) - \sum_{j=2}^{\ell} \left(r_1 b_j - r_j b + \sum_{m=2}^{\ell} r_j b_m \right) \\ &= -\sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} r_i b_j - \sum_{i=2}^{\ell} \sum_{j=2}^{i} r_j b_i - (r_1 - r)b \end{aligned}$$

Following Manschot [Man14], it is convenient to make the following change of variables

$$b_\ell = s_\ell, \quad b_i = s_i - s_{i+1} \text{ for } 2 \leq i < \ell.$$

For convenience, set $s_{\ell+1} = 0$. We obtain

$$\sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} r_i b_j + \sum_{i=2}^{\ell} \sum_{j=2}^{i} r_j b_i = \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} r_i (s_j - s_{j+1}) + \sum_{i=2}^{\ell} \sum_{j=2}^{i} r_j (s_i - s_{i+1}) = \sum_{i=2}^{\ell} (r_i + r_{i-1}) s_i.$$

Substituting these expressions back into Equation (12), we obtain that

$$\sum_{1 \leq i < j \leq \ell} r_i r_j K_{\mathbb{F}_1} \cdot (\mu_j - \mu_i) = \sum_{1 \leq i < j \leq \ell} r_i r_j (a_i - a_j) - 2 \sum_{i=2}^{\ell} (r_{i-1} + r_i) s_i + 2(r - r_1) b. \tag{13}$$

Suppose that the Harder-Narasimhan filtration is given by $0 \subset F_1 \subset \cdots \subset F_\ell$. We next use Yoshioka's relation for discriminants [Yos96b, Equation 2.1]

$$r\Delta = \sum_{i=1}^{\ell} r_i \Delta_i - \sum_{i=2}^{\ell} \frac{r(F_i) r(F_{i-1})}{2r_i} (\mu(F_i) - \mu(F_{i-1}))^2 \tag{14}$$

Using the facts that $r(F_i) = \sum_{j=1}^{i} r_j$ and $c_1(F_i) = \sum_{j=1}^{i} c_1(\gamma_j)$, we can rewrite the expression

$$\sum_{i=2}^{\ell} \frac{r(F_i) r(F_{i-1})}{2r_i} (\mu(F_i) - \mu(F_{i-1}))^2$$

in Equation (14) as follows

$$\sum_{i=2}^{\ell} \frac{1}{2r_i r(F_i) r(F_{i-1})} \left(\sum_{j=1}^{i-1} r_j \left(\sum_{j=1}^{i} (r_j a_j E + b_j F) \right) - \sum_{j=1}^{i} r_j \left(\sum_{j=1}^{i-1} (r_j a_j E + b_j F) \right) \right)^2$$

$$= \sum_{i=2}^{\ell} \frac{1}{2r_i r(F_i) r(F_{i-1})} \left(\sum_{j=1}^{i-1} (r_i r_j (a_i - a_j) E + (r_j b_i - r_i b_j) F) \right)^2$$

$$= \sum_{i=2}^{\ell} \frac{1}{2r_i r(F_i) r(F_{i-1})} \left(- \left(\sum_{j=1}^{i-1} r_i r_j (a_i - a_j) \right)^2 + 2 \left(\sum_{j=1}^{i-1} r_i r_j (a_i - a_j) \right) \left(\sum_{j=1}^{i-1} (r_j b_i - r_i b_j) \right) \right) \tag{15}$$

We now rewrite the term

$$\sum_{i=2}^{\ell} \frac{1}{r(F_i) r(F_{i-1})} \left(\sum_{j=1}^{i-1} r_j (a_i - a_j) \right) \left(\sum_{j=1}^{i-1} (r_j b_i - r_i b_j) \right)$$

using the fact that $b = \sum_{i=1}^{\ell} b_i$, $b_\ell = s_\ell$ and $b_i = s_i - s_{i+1}$. It is not too hard to check that we obtain

$$\sum_{i=2}^{\ell} \frac{1}{r(F_i) r(F_{i-1})} \sum_{j=1}^{i-1} r_j (a_i - a_j) \left(\sum_{j=1}^{i-1} r_j b_i + \sum_{j=i}^{\ell} r_i b_j - r_i b \right)$$

$$= \sum_{i=2}^{\ell} (a_i - a_{i-1}) s_i - b \sum_{i=2}^{\ell} \sum_{j=1}^{i-1} \frac{r_i r_j (a_i - a_j)}{r(F_i) r(F_{i-1})}.$$

Substituting this expression back into the expression in Equation (15), Equation (14) becomes
(16)

$$r\Delta = \sum_{i=1}^{\ell} r_i\Delta_i + \sum_{i=2}^{\ell} \frac{1}{2r_i r(F_i) r(F_{i-1})}\left(\sum_{j=1}^{i} r_i r_j (a_i - a_j)\right)^2 - \sum_{i=2}^{\ell}(a_i - a_{i-1})s_i + b\sum_{i=2}^{\ell}\sum_{j=1}^{i-1}\frac{r_i r_j (a_i - a_j)}{r(F_i) r(F_{i-1})}$$

We can use Riemann-Roch (see Equations (6, 7)) and substitute Equation (11) into Equation (10) to obtain

$$-2\sum_{1\leq i<j\leq \ell}\chi(\gamma_j,\gamma_i) = -r^2(1-2\Delta) + \sum_{i=1}^{\ell} r_i^2(1-2\Delta_i) + \sum_{1\leq i<j\leq\ell} K_{\mathbb{F}_1}\cdot(r_i c_1(\gamma_j) - r_j c_1(\gamma_i)).$$

We next introduce a new variable ϵ to split the term $-r^2(1-2\Delta)$, to rewrite this expression as

$$= -r^2 + \sum_{i=1}^{\ell} r_i^2 + 2r\epsilon\Delta + 2r(r-\epsilon)\Delta - \sum_{i=1}^{\ell} 2r_i^2\Delta_i + \sum_{1\leq i<j\leq\ell} K_{\mathbb{F}_1}\cdot(r_i c_1(\gamma_j) - r_j c_1(\gamma_i)).$$

Using Equations (13) and (16), this becomes

$$\begin{aligned}(17)\quad & -r^2 + \sum_{i=1}^{\ell} r_i^2 + 2r\epsilon\Delta + 2\sum_i r_i(r-\epsilon-r_i)\Delta_i + \\ & (r-\epsilon)\left[\sum_{i=2}^{\ell}\frac{r_i}{r(F_i)r(F_{i-1})}\left(\sum_{j=1}^{i-1} r_j(a_i-a_j)\right)^2 - 2\sum_{i=2}^{\ell}(a_i-a_{i-1})s_i + 2b\sum_{i=2}^{\ell}\sum_{j=1}^{i-1}\frac{r_i r_j(a_i-a_j)}{r(F_i)r(F_{i-1})}\right] \\ & \qquad + \sum_{1\leq i<j\leq\ell} r_i r_j(a_i-a_j) - 2\sum_{i=2}^{\ell}(r_{i-1}+r_i)s_i + 2(r-r_1)b\end{aligned}$$

First, suppose all the a_i are fixed integers. We need to analyze the terms when $\mathrm{S}^{\mu}(\gamma_1,\cdots,\gamma_\ell;\mathrm{H}_1,\mathrm{H}_2)\neq 0$.

For Case A in Definition 4.1, we have

$$a_i - a_{i-1} < 0 \quad\text{and}\quad \frac{\sum_{j=1}^{i-1} b_j}{\sum_{j=1}^{i-1} r_j} \leq \frac{\sum_{j=i}^{\ell} b_j}{\sum_{j=i}^{\ell} r_j}.$$

Using the facts that $b = \sum_{j=1}^{\ell} b_j$ and using the fact that $s_i = \sum_{j=i}^{\ell} b_j$, the second inequality becomes

$$s_i \geq \frac{b}{r}\left(\sum_{i\leq j\leq\ell} r_j\right).$$

Similarly, the inequalities in Case B are

$$a_i - a_{i-1} \geq 0 \quad\text{and}\quad s_i < \frac{b}{r}\left(\sum_{i\leq j\leq\ell} r_j\right).$$

The coefficient of s_i in the expression (17) is given by

$$-2(r-\epsilon)(a_i-a_{i-1}) - 2(r_{i-1}+r_i).$$

If ϵ is sufficiently small, in Case B, this coefficient is negative and s_i is bounded above. Hence, the terms involving s_i in the expression (17) are bounded below. Similarly, in Case A, this coefficient is positive, unless $\ell = 2$ and $a_2 - a_1 = -1$. Since s_i are bounded below, the terms involving s_i in the expression (17) are bounded below. Note that if $\ell = 2$ and $a_2 - a_1 = -1$, then

$$c_1(\gamma) = r_1 a_1 E + b_1 F + r_2(a_1 - 1)E + b_2 F = (ra_1 - r_2)E + b_2 F,$$

so $r \nmid c_1(\gamma) \cdot F$, which contradicts our assumptions. Hence, in all cases the terms involving s_i are bounded below. This is the only place we use the assumption $r \mid c_1(\gamma) \cdot F$.

Now, we want to show that for any C, there are only finitely many values of a_i with $\sum_{i=1}^{\ell} r_i a_i = ra$ such that

$$\sum_{i=2}^{\ell} \frac{r_i}{r(F_i)r(F_{i-1})} \left(\sum_{j=1}^{i-1} r_j (a_i - a_j) \right)^2 \leq C. \tag{18}$$

We will first prove that for each i, $|a_i - a_{i-1}| \leq \sqrt{C} E_i$, where E_i is a constant depending only on the r_i. The proof is by induction on i. Since all the summands in the expression (18) are nonnegative, by taking the $i = 2$ term, we see that

$$r_1 |a_2 - a_1| \leq \sqrt{\frac{Cr(F_2)r(F_1)}{r_2}},$$

which is the base case $i = 2$ of the induction. By the triangle inequality, we have

$$\left| \sum_{j=1}^{i-1} r_j a_i - \sum_{j=1}^{i-1} r_j a_j \right| + \left| \sum_{j=1}^{i-2} r_j a_j - \sum_{j=1}^{i-2} r_j a_{i-1} \right| \geq \left| \sum_{j=1}^{i-1} r_j a_i - \sum_{j=1}^{i-1} r_j a_{i-1} \right| = \left(\sum_{j=1}^{i-1} r_j \right) \cdot |a_i - a_{i-1}|.$$

Taking only the ith term in (18), we have the inequality

$$\left| \sum_{j=1}^{i-1} r_j a_i - \sum_{j=1}^{i-1} r_j a_j \right| \leq \sqrt{\frac{Cr(F_i)r(F_{i-1})}{r_i}},$$

and by the induction hypothesis, we have

$$\left| \sum_{j=1}^{i-2} r_j a_j - \sum_{j=1}^{i-2} r_j a_{i-1} \right| \leq \sqrt{C} E_i'$$

for some constant E_i' depending only on the ranks $r_1, \dots, r_\ell$. This concludes the induction step.

By the triangle inequality, it follows that all the a_j with $j > 1$ lie in an interval centered around a_1 of radius depending on C, say $D(C)$. If $a_1 < a - D(C)$, then $\sum r_i a_i < \sum r_i a = ra$, and similarly, if $a_1 > a + D(C)$, then $\sum r_i a_i > \sum r_i a = ra$. Since $\sum_{i=1}^{\ell} r_i a_i = ra$, we conclude that a_1 must be an integer within $D(C)$ of a. Hence, there are only finitely many choices for a_1, and consequently only finitely many choices for all the a_i, as desired.

From this, we see that

(19)
$$(r - \epsilon) \left[\sum_{i=2}^{\ell} \frac{r_i}{r(F_i)r(F_{i-1})} \left(\sum_{j=1}^{i-1} r_j (a_i - a_j) \right)^2 - 2 \sum_{i=2}^{\ell} (a_i - a_{i-1}) s_i + 2b \sum_{i=2}^{l} \sum_{j=1}^{i-1} \frac{r_i r_j (a_i - a_j)}{r(F_i)r(F_{i-1})} \right] +$$
$$\sum_{1 \leq i < j \leq \ell} r_i r_j (a_i - a_j) - 2 \sum_{i=2}^{\ell} (r_{i-1} + r_i) s_i + 2(r - r_1) b$$

is bounded below for any fixed choice of a_i and s_i with $\mathrm{S}^{\mu}(\gamma_1, \cdots, \gamma_\ell; \mathrm{H}_1, \mathrm{H}_2) \neq 0$. By the Bogomolov inequality, we have that $\Delta_i \geq 0$. Hence, using the expression (17) for $-2\sum_{i<j}\chi(\gamma_i, \gamma_j)$, if we pick $\epsilon < \min r - r_i$, we get

$$-\sum_{i<j}\chi(\gamma_i, \gamma_j) \geq r\epsilon\Delta + C'$$

for some constant C'. Hence, the dimension of the correction terms are arbitrarily small as Δ tends to ∞. We conclude that $[\mathcal{M}^{\mu}_{\mathbb{F}_1, E+F}(\gamma)]$ stabilize in A^- and have the same stable limit as $[\mathcal{M}^{\mu}_{\mathbb{F}_1,F}(\gamma)]$. The theorem follows by Proposition 5.2 and Corollary 4.9. $\square$

Corollary 5.5. *The classes $[\mathcal{M}_{\mathbb{P}^2}(r, c, \Delta)]$ stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^3}.$$

Proof. The surface $\mathbb{F}_1$ is the blowup $p : \mathbb{F}_1 \to \mathbb{P}^2$ at a point $x \in \mathbb{P}^2$. Moreover, we have that $p^*L = E+F$, where L is the class of $\mathcal{O}_{\mathbb{P}^2}(1)$. Suppose that $c = LH$ and using the division algorithm express $d = qr + m$ with $0 \leq m < r$. The corollary follows by Theorem 5.4 and Proposition 4.11 applied to the moduli spaces $\mathcal{M}_{\mathbb{P}^2,L}(r, c, \Delta)$ and $\mathcal{M}_{\mathbb{F}_1,p*L}(r, c - mE, \Delta)$. $\square$

We can now remove the restriction $r|c_1 \cdot F$ in Theorem 5.4 by using the blowup formula and Corollary 5.5.

Corollary 5.6. *Let H be a big and nef divisor on $\mathbb{F}_1$. Then the classes $[\mathcal{M}_{\mathbb{F}_1,H}(r, c, \Delta)]$ stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

Proof. Every divisor class c on $\mathbb{F}_1$ is a linear combination of the form $aE + bF$. Since tensoring all the sheaves in a moduli space by the line bundle $\mathcal{O}_{\mathbb{F}_1}(F)$ gives an isomorphism between moduli spaces and changes the first Chern class by adding rF, we may assume $b \geq a$. Hence, $c = bp^*L - (b - a)E$. Applying Corollary 5.5 and Proposition 4.11 with $m = b - a$, we see that the classes of $[\mathcal{M}_{\mathbb{F}_1,E+F}(r, c, \Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

By Corollary 4.6, $E + F$ is admissible. Consequently, by Proposition 4.7, the same holds for $[\mathcal{M}_{\mathbb{F}_1,H}(r, c, \Delta)]$ for any ample class H on $\mathbb{F}_1$. $\square$

6. Rational surfaces more generally

In this section, we extend the previous discussion from $\mathbb{P}^2$ and $\mathbb{F}_1$ to more general rational surfaces.

Theorem 6.1. *Let $\mathbb{F}_e$ be a Hirzebruch surface. Let H be an ample on $\mathbb{F}_e$. Then the classes $[\mathcal{M}_{\mathbb{F}_e,H}(r, c, \Delta)]$ stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

Proof. By Corollary 5.6, the theorem is true for the Hirzebruch surface $\mathbb{F}_1$. Recall from §2 that E is the self-intersection $-e$ section of $\mathbb{F}_e$. If we blowup a point x on the exceptional curve $E \subset \mathbb{F}_e$ and blowdown the proper transform of the fiber through x, we obtain the surface $\mathbb{F}_{e+1}$. Conversely, if we blowup a general point x in $\mathbb{F}_e$ and blowdown the proper transform of the fiber through x, we obtain the surface $\mathbb{F}_{e-1}$ [Bea83, §III]. Suppose we know the theorem for $\mathbb{F}_e$, we will show that the theorem also holds for $\mathbb{F}_{e+1}$ and $\mathbb{F}_{e-1}$. Since we know the theorem for $\mathbb{F}_1$, it follows that we know the theorem for all Hirzebruch surfaces $\mathbb{F}_e$. The argument is symmetric for $\mathbb{F}_{e+1}$ and $\mathbb{F}_{e-1}$. Let H_e and H_{e+1} be ample divisors on $\mathbb{F}_e$ and $\mathbb{F}_{e+1}$, respectively. Let X be the common blowup of $\mathbb{F}_e$ and $\mathbb{F}_{e+1}$ with maps $p_e : X \to \mathbb{F}_e$ and $p_{e+1} : X \to \mathbb{F}_{e+1}$. Then $p_e^* H_e$ and $p_{e+1}^* H_{e+1}$ are both K_X-negative, big and nef divisors on X. In particular, by Corollary 4.6, they are admissible. Furthermore, $ap_e^* H_e + bp_{e+1}^* H_{e+1}$ is ample for any $a, b > 0$. By Proposition 4.11 and our inductive hypothesis on $\mathbb{F}_e$, we know that $[\mathcal{M}_{X,p_e^* H_e}(r, c, \Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^5}.$$

By Proposition 4.7, we conclude that $[\mathcal{M}_{X,p_{e+1}^* H_{e+1}}(r, c, \Delta)]$ also stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^5}.$$

By Proposition 4.11, we conclude that $[\mathcal{M}_{\mathbb{F}_{e+1},H_{e+1}}(r, c, \Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

This concludes the inductive step and the proof of the theorem. □

Theorem 6.2. *Let X be a rational surface and let H be a polarization such that $H \cdot K_X < 0$. Then the classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\mathrm{top}}(X)}}.$$

Proof. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_e$ with $e \neq 1$. Every smooth rational surface can be obtained by a sequence of blowups from one of these minimal surfaces. By Theorem 6.1 and Corollary 5.5, the theorem is known for minimal rational surfaces. Assume that the theorem is true for any rational surface X_n obtained by a sequence of n blowups of a minimal rational surface at smooth points. By induction, we prove the theorem for a blowup $p : X_{n+1} \to X_n$ at a smooth point x. We have that

$$\chi_{\mathrm{top}}(X_{n+1}) = \chi_{\mathrm{top}}(X_n) + 1.$$

Let H be an ample divisor on X_n such that $H \cdot K_{X_n} < 0$. Then $p^* H \cdot K_{X_{n+1}} < 0$ and $p^* H$ is admissible by Corollary 4.6. Hence, by Proposition 4.11, the theorem holds for the classes $[\mathcal{M}_{X_{n+1},p^* H}(r, c - mE, \Delta)]$, where E is the exceptional divisor over x. Notice that we can get any Chern class on X_{n+1} as a Chern class of the form $(r, c - mE, \Delta)$ for a suitable m. If $\hat{H}$ is any ample divisor on X_{n+1} such that $\hat{H} \cdot K_{X_{n+1}} < 0$, then we can use Proposition 4.7 to see that the theorem holds for the classes $[\mathcal{M}_{X_{n+1},\hat{H}}(r, c - mE, \Delta)]$. This concludes the inductive step. □

Taking the virtual Poincaré (resp. Hodge) polynomials, we obtain the following corollary.

Corollary 6.3. *Let X be a rational surface and let H be a polarization such that $K_X \cdot H < 0$. Then the virtual Poincaré and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize as Δ tends to ∞ and the generating functions for the stable numbers are given by*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\mathrm{top}}(X)}}, \quad \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\mathrm{top}}(X)}}.$$

Observe that Theorem 6.2 and Corollary 6.3 yield Theorem 1.7 from the Introduction.

7. From the Moduli Stack to the Moduli Space

The results in §4, 5, 6 concern the virtual Poincaré polynomial of the moduli stack of sheaves. In practice, we are often interested in the moduli space of S-equivalence classes of sheaves. In this section, we will show that the virtual Poincaré polynomials of the two spaces have a close relation.

In this section, we will work in a quotient of the Grothendieck ring.

Definition 7.1. Define A to be the quotient of A^- by the relations $[P] = [X][G]$ whenever $G = \mathrm{PGL}_n$ for some n and $P \to X$ is an étale G-torsor.

Note that the classes of PGL_n are invertible in A. The key fact about this is the following, which is essentially [BD07, Theorem A.9].

Theorem 7.2. *The virtual Poincaré and Hodge polynomials descend to well-defined maps on A.*

This theorem lets us easily deduce the following fact.

Proposition 7.3. *The virtual Poincaré and Hodge polynomials of $\mathcal{M}^s_{X,H}(\gamma)$ (the locus of Gieseker-stable sheaves in the moduli stack) and $M^s_{X,H}(\gamma)$ are related as follows:*

$$P_t(M^s_{X,H}(\gamma)) = (t^2-1)P_t(\mathcal{M}^s_{X,H}(\gamma)),$$
$$P_{xy}(M^s_{X,H}(\gamma)) = (xy-1)P_{xy}(\mathcal{M}^s_{X,H}(\gamma))$$

Proof. Let Q be the locus of Gieseker-stable sheaves in a component of $\mathrm{Quot}(X)$ which dominates $\mathcal{M}^s_{X,H}(\gamma)$. There is a natural map $Q \to \mathcal{M}^s_{X,H}(\gamma)$ which is a GL_n-torsor, and $Q \to M^s_{X,H}(\gamma)$ which is a PGL_n-torsor. We then have the following equality in A:

$$[M^s_{X,H}(\gamma)] = [Q]/[\mathrm{GL}_n] = [Q]/[\mathrm{PGL}_n] * [\mathrm{GL}_n]/[\mathrm{PGL}_n] = [\mathcal{M}^s_{X,H}(\gamma)] * [\mathrm{GL}_n]/[\mathrm{PGL}_n].$$

We know that $[\mathrm{GL}_n]/[\mathrm{PGL}_n] = \mathbb{L} - 1$, which finishes the proof. □

This proposition deals with the stable locus, but since the semistable locus is relatively small, we get a similar result asymptotically. To prove this rigorously, we first need the following lemma.

Lemma 7.4. *Let $\mathcal{F}$ be a Gieseker-semistable sheaf of rank r. Then $\dim \mathrm{Aut}(\mathcal{F}) \le r^2$.*

Proof. Since $\mathrm{Aut}(\mathcal{F}) \subset \mathrm{End}(\mathcal{F})$, it suffices to show this for $\mathrm{End}(\mathcal{F})$.

We know that $\mathcal{F}$ has a Jordan-Hölder filtration where the subquotients are stable. There is an isotrivial degeneration of $\mathcal{F}$ into the direct sum of the subquotients of this filtration. Since the dimension of the space of endomorphisms is upper semicontinuous, we can assume that $\mathcal{F} = \bigoplus E_i$ with the E_i stable.

We then have

$$\mathrm{End}(\mathcal{F}) = \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j).$$

We know that any map between stable sheaves of the same reduced Gieseker polynomial is either scalar multiplication or 0, so each $\mathrm{Hom}(E_i, E_j)$ has dimension at most 1. Since there are at most r^2 terms in the direct sum, the lemma follows. □

Proposition 7.5. *Let H be a polarization such that $K_X \cdot H < 0$. Suppose that the virtual Poincaré (resp. Hodge) polynomials of $\mathcal{M}_{X,H}(\gamma)$ stabilize to $F(t)$ (resp. $F(x,y)$) as $\Delta \to \infty$. Then the Poincaré (resp. Hodge) polynomials of $M_{X,H}(\gamma)$ stabilize to $(t^2-1)F(t)$ (resp. $(xy-1)F(x,y)$).*

Proof. We know that the codimension of the strictly semistable locus in $\mathcal{M}_{X,H}(\gamma)$ grows with Δ by the proof of Corollary 4.9. This means that the stable Betti (resp. Hodge) polynomials of $\mathcal{M}_{X,H}(\gamma)$ and $\mathcal{M}^s_{X,H}(\gamma)$ are the same.

The previous proposition then implies the result for $M^s_{X,H}(\gamma)$. To finish the proof, we have to show that the codimension of the strictly semistable locus in $M_{X,H}(\gamma)$ grows with Δ.

First, denote by c the codimension of the strictly semistable locus in $\mathcal{M}_{X,H}(\gamma)$. Next, consider the map $\mathcal{M}_{X,H}(\gamma) \to M_{X,H}(\gamma)$. The fiber over a point $[E]$ in the right hand side consists of the space of all semistable sheaves S-equivalent to E modulo automorphism. In particular, the dimension of the fiber is at least $-r^2$ by the previous lemma.

The codimension of the strictly semistable locus in $M_{X,H}(\gamma)$ is at least $c+1-r^2$. Since c grows with Δ and r is fixed, it follows that this codimension also grows with Δ. $\square$

Corollary 7.6. *Let X be a rational surface and H a polarization such that $K_X \cdot H < 0$. Then the virtual Poincaré and virtual Hodge polynomials of $M_{X,H}(r,c,\Delta)$ stabilize as Δ tends to ∞. The generating functions for the stable Betti and Hodge numbers are given by*

$$(1-t^2)\prod_{i=1}^{\infty}\frac{1}{(1-t^{2i})^{\chi_{\mathrm{top}}(X)}}, \quad (1-xy)\prod_{i=1}^{\infty}\frac{1}{(1-(xy)^i)^{\chi_{\mathrm{top}}(X)}},$$

respectively.

For a smooth, projective variety, the virtual Poincaré (resp. Hodge) polynomials agree with the ordinary Poincaré (resp. Hodge) polynomials. Using Proposition 2.1, we obtain the following corollary.

Corollary 7.7. *Let X be a rational surface and H a polarization such that $K_X \cdot H < 0$. Assume that there are no strictly semistable sheaves of rank r and first Chern class c. Then the Poincaré and Hodge polynomials of $M_{X,H}(r,c,\Delta)$ stabilize as Δ tends to ∞ and the generating functions for the stable Betti and Hodge numbers are given by*

$$(1-t^2)\prod_{i=1}^{\infty}\frac{1}{(1-t^{2i})^{\chi_{\mathrm{top}}(X)}}, \quad (1-xy)\prod_{i=1}^{\infty}\frac{1}{(1-(xy)^i)^{\chi_{\mathrm{top}}(X)}},$$

respectively.

Observe that Corollary 7.7 implies Theorem 1.9 from the Introduction.

8. Relation to the generalized Atiyah-Jones conjecture

In this section, we discuss the relation of Conjecture 1.1 and the Atiyah-Jones conjecture.

In [AJ78] Atiyah and Jones made a number of conjectures concerning the homology and homotopy type of the moduli space of instantons on the 4-dimensional sphere S^4. The conjecture was later generalized to other four-manifolds including to polarized surfaces (X,H). In this context, for fixed rank r and first Chern class c, Taubes constructed a differential geometric map between the moduli spaces of locally free μ_H-stable sheaves $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ and $M^{\mu,s,\circ}_{X,H}(r,c,\Delta+\frac{1}{r})$ [Tau84]. The generalized Atiyah-Jones conjecture predicts that these maps are rational homology and homotopy equivalences in degrees that tend to infinity as the discriminant Δ tends to infinity. Taubes [Tau84] shows that if the generalized Atiyah-Jones Conjecture holds, then the Betti numbers of $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ stabilize and the stable Betti numbers can be recovered from the Betti numbers of the space $\mathrm{Map}(X,BPU(r))$ of maps from X to the classifying space of the projective unitary group $PU(r)$. We will call this consequence the *weak generalized Atiyah-Jones conjecture*. Let $\mathrm{Map}^f(X,BPU(r))$ denote the maps in $\mathrm{Map}(X,BPU(r))$ such that the pullback of the universal $PU(r)$ bundle to X defines the underlying projective unitary bundle of the bundles parameterized by $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$. The locus $\mathrm{Map}^f(X,BPU(r))$ is a path component of $\mathrm{Map}(X,BPU(r))$.

Conjecture 8.1 (Weak Generalized Atiyah-Jones Conjecture)**.** *The Betti numbers of $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ stabilize to the Betti numbers of the space* $\mathrm{Pic}(X) \times \mathrm{Map}^f(X, BPU(r))$ *as Δ tends to ∞.*

In this section, we show that the stabilization of the Betti numbers in Conjecture 1.1 is equivalent to the weak generalized Atiyah-Jones conjecture assuming the cohomology of $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ equals to its cohomology with compact support in increasing degrees as Δ tends to infinity. The original Atiyah-Jones Conjecture was proved by Boyer, Hurtubise, Mann and Milgram [BHMM93]. The Atiyah-Jones conjecture for $r = 2$ was proved for ruled surfaces by Hurtubise and Milgram [HM95] and for rational surfaces and certain polarizations by Gasparim [Gas08].

Theorem 8.2. *Assume that the cohomology of $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ with compact supports is pure and equal to the cohomology of $M^{\mu,s,\circ}_{X,H}(r,c,\Delta)$ in increasing degrees as Δ tends to ∞. Then the weak generalized Atiyah-Jones conjecture is equivalent to the stabilization of Betti numbers in Conjecture 1.1.*

Proof. Given any torsion-free sheaf E on a surface X, its double-dual E^{**} is locally free, and the quotient is zero-dimensional. We can stratify the moduli space $M^{\mu,s}_{X,H}(r,c,\Delta)$ based on the length of this quotient.

Let $P_X(t)$ denote the virtual Poincaré polynomial of X. Building on the work of Yoshioka on computing the virtual Poincaré polynomial of the space of length k-quotients of a locally free rank r sheaf [Yos94], Göttsche obtains the following formula [Got99, Proposition 3.1]

$$\sum_{\Delta \geq 0} P_{M^{\mu,s}_{X,H}(r,c,\Delta)}(t) q^{r\Delta} = \left(\prod_{a=1}^{\infty} \prod_{b=1}^{r} \zeta_X(t^{2ra-2b} q^a, t) \right) \sum_{\Delta \geq 0} P_{M^{\mu,s,\circ}_{X,H}(r,c,\Delta)}(t) q^{r\Delta}.$$

Now we compute the shifted series, to obtain

$$\sum_{\Delta \geq 0} P_{M^{\mu,s}_{X,H}(r,c,\Delta)}(t) t^{-2r^2(2\Delta - \chi(\mathcal{O}_X))-2} q^{r\Delta} \tag{20}$$

$$= \left(\prod_{a=1}^{\infty} \prod_{b=1}^{r} \zeta_X(t^{-2ra-2b} q^a, t) \right) \sum_{\Delta \geq 0} P_{M^{\mu,s,\circ}_{X,H}(r,c,\Delta)}(t) t^{-2r^2(2\Delta - \chi(\mathcal{O}_X))-2} q^{r\Delta}.$$

To prove the theorem, we will multiply both sides of Equation (20) by $(1-q)$ and compute the limit as q tends to 1.

To compute

$$\lim_{q \to 1} (1-q) \sum_{\Delta \geq 0} P_{M^{\mu,s}_{X,H}(r,c,\Delta)}(t) t^{-2r^2(2\Delta - \chi(\mathcal{O}_X))-2} q^{r\Delta},$$

we use Conjecture 1.1, Corollary 4.9 and Göttsche's formula (2), to obtain

$$(1-t^{-2}) \prod_{m=1}^{\infty} \frac{(1+t^{-2m+1})^{2b_1}}{(1-t^{-2m})^{b_2+2}},$$

which, using the definition of ζ_X (see Equation (1)), can be rewritten as

$$(1+t^{-1})^{b_1} \frac{(1+t^{-3})^{b_1}(1+t^{-1})^{b_3}}{(1-t^{-4})(1-t^{-2})^{b_2}} \prod_{a=3}^{\infty} \zeta_X(t^{-2a}, t).$$

To compute the right hand side

$$\lim_{q \to 1} (1-q) \left(\prod_{a=1}^{\infty} \prod_{b=1}^{r} \zeta_X(t^{-2ra-2b} q^a, t) \right) \sum_{\Delta \geq 0} P_{M^{\mu,s,\circ}_{X,H}(r,c,\Delta)}(t) t^{-2r^2(2\Delta - \chi(\mathcal{O}_X))-2} q^{r\Delta}, \tag{21}$$

we use the weak generalized Atiyah-Jones Conjecture. By [AB83, §2], the Poincaré polynomial of $\mathrm{Map}^f(X, BPU(r))$ is given by

$$\frac{(1+t^3)^{b_1}(1+t)^{b_3}}{(1-t^4)(1-t^2)^{b_2}} \prod_{a=3}^{r} \frac{(1+t^{2a-1})^{b_1}(1+t^{2a-3})^{b_3}}{(1-t^{2a})(1-t^{2a-2})^{b_2}(1-t^{2a-4})}. \tag{22}$$

If the weak generalized Atiyah-Jones conjecture is true and the virtual Poincaré polynomial equals the actual Poincaré polynomial for the space $M_{X,H}^{\mu,s,\circ}(r,c,\Delta)$ in increasing degrees as Δ tends to infinity, then using the expression (22) $P_{M_{X,H}^{\mu,s,\circ}(r,c,\Delta)}(t)t^{-2r^2(2\Delta-1)-2}$ converge to

$$(1+t^{-1})^{b_1}\frac{(1+t^{-3})^{b_1}(1+t^{-1})^{b_3}}{(1-t^{-4})(1-t^{-2})^{b_2}} \prod_{a=3}^{r} \zeta_X(t^{-2a}, t).$$

Consequently, the expression (21) is equal to

$$\left(\prod_{a=1}^{\infty}\prod_{b=1}^{r} \zeta_X(t^{-2ra-2b}, t)\right)(1+t^{-1})^{b_1}\frac{(1+t^{-3})^{b_1}(1+t^{-1})^{b_3}}{(1-t^{-4})(1-t^{-2})^{b_2}} \prod_{a=3}^{r} \zeta_X(t^{-2a}, t),$$

which can be rewritten as

$$(1+t^{-1})^{b_1}\frac{(1+t^{-3})^{b_1}(1+t^{-1})^{b_3}}{(1-t^{-4})(1-t^{-2})^{b_2}} \prod_{a=3}^{\infty} \zeta_X(t^{-2a}, t).$$

We conclude that the right hand side equals the left hand side. Hence, the stabilization of Betti numbers in Conjecture 1.1 is equivalent to the weak generalized Atiyah-Jones conjecture assuming that the cohomology of $M_{X,H}^{\mu,s,\circ}(r,c,\Delta)$ with compact supports is pure. □

9. Picard Group of the Moduli Space

In this section, we show that if X is a regular surface with geometric genus 0, then stabilization of the Betti numbers allows us to understand the Néron-Severi space of $M_{X,H}(r,c,\Delta)$ when Δ is sufficiently large.

Let X be a smooth, irreducible projective surface whose irregularity $q(X) = h^1(X, \mathcal{O}_X)$ and geometric genus $p_g(X) = h^2(X, \mathcal{O}_X)$ are zero. Let γ be a Chern character such that the moduli space $M_{X,H}(\gamma)$ does not contain any strictly semistable sheaves. Let $\gamma^\perp \subset K^0(X)$ denote the orthogonal complement of γ with respect to the Euler pairing. Let $\mathcal{F}$ be the universal sheaf on $X \times \mathcal{M}_{X,H}(\gamma)$. Given a vector bundle E on X, we get a line bundle on $\mathcal{M}_{X,H}(\gamma)$ by taking

$$\det(\mathbb{R}\pi_{2*}(\mathcal{F} \otimes^{\mathbb{L}} \mathbb{L}\pi_1^* E)).$$

If $\chi(E \otimes^{\mathbb{L}} \gamma) = 0$, then this line bundle descends to a line bundle on $M_{X,H}(\gamma)$, and we get a well-defined map

$$\lambda : \gamma^\perp \to \mathrm{Pic}(M_{X,H}(\gamma)),$$

which we call the *Donaldson morphism* (see [HL10, §8]). If Δ is sufficiently large, one can see that the map induced from $\mathrm{NS}_{\mathbb{Q}}(X) \oplus \mathbb{Q} \to \mathrm{NS}_{\mathbb{Q}}(M_{X,H}(\gamma))$ is injective. We give a quick argument for the reader's convenience (see also [HL10, Example 8.1.7]).

Proposition 9.1. *Let X and γ satisfy the assumptions in this section. Let $K^0_{\mathrm{coh}}(X)$ be the image of the cycle class map in $H^*(X, \mathbb{C})$. The map*

$$\lambda : K^0_{\mathrm{coh}}(X) \supset \gamma^\perp \to \mathrm{NS}(M_{X,H}(\gamma))$$

is injective if $\Delta(\gamma)$ is sufficiently large.

Proof. We proceed by giving curves in $M_{X,H}(\gamma)$ and calculating the intersection numbers of these curves with a basis for the image of the Donaldson morphism.

Let C_i with $1 \leq i \leq n = \rho(X)$ be effective curves which form a basis of $N_1(X)$. Let E be a slope-stable vector bundle on X with $r(E) = r(\gamma)$, $c_1(E) = c_1(\gamma)$, and $\mathrm{ch}_2(E) = \mathrm{ch}_2(\gamma) + r$. By O'Grady's Theorem [O'G96], [HL10, §9], if Δ is sufficiently large, such an E exists.

If we consider the family of sheaves $E \otimes \mathcal{I}_p$, this gives us an embedding of X into $M_{X,H}(\gamma)$. In particular, each of the C_i embed in $M_{X,H}(\gamma)$. We will now produce one more curve in $M_{X,H}(\gamma)$.

Let E' be a slope-stable vector bundle on X with $r(E') = r(\gamma)$, $c_1(E') = c_1(\gamma)$, and $\mathrm{ch}_2(E') = \mathrm{ch}_2(\gamma) + 2r$. As before, if Δ is sufficiently large, such an E' is guaranteed to exist. Now fix a point $p \in X$. The space of all length-two subschemes Z of X supported at p form a $\mathbb{P}^1$. The sheaves $E' \otimes I_Z$ form a curve in $M_{X,H}(\gamma)$ which we will denote by D.

We will now give a basis of $\gamma^{\perp}$. Write $\gamma = (r, c, d)$. For each i, the class

$$\left(0, C_i, \left(\frac{-c}{r} + \frac{K_X}{2}\right) \cdot C_i\right) \in \gamma^{\perp}.$$

The same is true for

$$\left(1, 0, \frac{-d}{r} + \frac{K_X \cdot c}{2r} - \chi(\mathcal{O}_X)\right).$$

These classes form a basis for $\gamma^{\perp} \subset K^0_{\mathrm{coh}}(X)$. Let

$$\mathcal{L}_i = \lambda\left(0, C_i, \left(\frac{-c}{r} + \frac{K_X}{2}\right) \cdot C_i\right) \quad \text{and} \quad \mathcal{L}_0 = \lambda\left(1, 0, \frac{-d}{r} + \frac{K_X \cdot c}{2r} - \chi(\mathcal{O}_X)\right).$$

The curves C_i and D have intersection number 0 with line bundles pulled back from $\mathrm{Pic}^0(X)$, so it makes sense to ask for the intersection number with classes in the image of the Donaldson morphism. It is straightforward to calculate these intersection numbers using Grothendieck-Riemann-Roch, and we get the intersection matrix:

$\cdot$	$\mathcal{L}_1$	$\cdots$	$\mathcal{L}_n$	$\mathcal{L}_0$
C_1	$r(C_1 \cdot C_1)$	$\cdots$	$r(C_1 \cdot C_n)$	$C_1 \cdot c$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
C_n	$r(C_n \cdot C_1)$	$\cdots$	$r(C_n \cdot C_n)$	$C_n \cdot c$
D	0	$\cdots$	0	-4r

It is enough to show that this matrix has full rank. The submatrix formed by omitting the last row and the last column has full rank, since up to a nonzero multiple, it's the intersection matrix for $\mathrm{NS}(X)$. It is clear that the last row is independent from the other rows, so the intersection matrix has full rank as desired.

$\square$

Note that in the previous proposition, the only reason we needed Δ to be large was to ensure that the vector bundles E and E' existed.

Theorem 9.2. *Let X be a smooth, projective irreducible surface with $q(X) = p_g(X) = 0$. Assume that the Betti numbers of the moduli spaces stabilize to those of the Hilbert scheme of points. If $\gamma \in K^0(X)$ is a class such that $M_{X,H}(\gamma)$ does not contain any strictly semistable sheaves and $\Delta(\gamma)$ is sufficiently large, then the map*

$$\lambda : K^{\text{-}0}_{\mathrm{coh}}(X) \otimes \mathbb{Q} \supset \gamma^{\perp} \to \mathrm{NS}(M_{X,H}(\gamma)) \otimes \mathbb{Q}$$

is an isomorphism.

Proof. By the previous proposition, we know that if Δ is sufficiently large, this map is injective. Since the spaces in question are rational vector spaces, it suffices to show that they have the same dimension.

Since X is a surface, the dimension of $K^0_{\mathrm{coh}}(X) \otimes \mathbb{Q}$ is $2 + \rho(X)$. Since X has geometric genus 0, we have $\rho(X) = b_2(X)$. This means that the dimension of $\gamma^\perp$ is $b_2(X) + 1$.

We have

$$\dim \mathrm{NS}(M_{X,H}(\gamma)) \otimes \mathbb{Q} \leq b_2(M_{X,H}(\gamma)).$$

Looking at the formula for the stable Betti numbers of the moduli spaces, we see that if $\Delta(\gamma)$ is sufficiently large, then $b_2(M_{X,H}(\gamma)) = b_2(X) + 1$. This means that λ is an injective map from a vector space of dimension $b_2(X) + 1$ to a vector space of dimension at most $b_2(X) + 1$, and hence it must be an isomorphism. $\square$

For example, Theorem 9.2 applies to rational surfaces. The Picard groups of moduli spaces of sheaves on rational surfaces, and more generally on ruled surfaces, for $\Delta \gg 0$ were computed by Yoshioka in [Yos96c].

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