Effective divisors on moduli spaces of sheaves on the plane

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(joint work with Jack Huizenga, Matthew Woolf)

Let ξ be the Chern character of a stable sheaf F on $\mathbb{P}^2$. In joint work with Jack Huizenga and Matthew Woolf, we determine the effective cone of the moduli spaces $M(\xi)$ of Gieseker semi-stable sheaves on $\mathbb{P}^2$ with Chern character ξ. For simplicity, assume that the rank r of ξ is positive. It is then convenient to record ξ in terms of the slope $\mu = \frac{c_1}{r}$ and the discriminant $\Delta = \frac{\mu^2}{2} - \frac{\mathrm{ch}_2}{r}$.

A sheaf E *satisfies interpolation* with respect to a coherent sheaf F on $\mathbb{P}^2$ if $h^i(E \otimes F) = 0$ for every i (in particular, $\chi(E \otimes F) = 0$). The stable base locus decomposition of $M(\xi)$ is closely tied to the higher rank interpolation problem.

Problem 1 (Higher rank interpolation)**.** *Given $F \in M(\xi)$ determine the minimal slope $\mu \in \mathbb{Q}$ with $\mu + \mu(\xi) \geq 0$ for which there exists a vector bundle E of slope μ satisfying interpolation with respect to F.*

If E satisfies interpolation with respect to F, then the Brill-Noether divisor

$$D_E := \{G \in M(\xi) | h^1(E \otimes G) \neq 0\}$$

is an effective divisor that does not contain F in its base locus. The interpolation problem in general is very hard, but has been solved in the following cases:

(1) $F = I_Z$, where Z is a complete intersection, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(2) $F = I_Z$, where Z is a monomial, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(3) $F = I_Z$, where Z is a general, zero-dimensional scheme in $\mathbb{P}^2$ [H].
(4) $F \in M(\xi)$ is a general stable sheaf [CHW].

These theorems depend on finding a good resolution of F. If F were unstable, then the maximal destabilizing object would yield an exact sequence

$$0 \to A \to F \to B \to 0.$$

The idea is to destabilize F via Bridgeland stability and use the exact sequence arising from the Harder-Narasimhan filtration just past the wall where F is destabilized. If a bundle E satisfies interpolation with respect to A and B, then E satisfies interpolation for F by the long exact sequence for cohomology. One may hope that for an interpolating bundle E with minimal slope, E satisfies interpolation for F because it does so for both A and B. Because the Harder-Narasimhan filtrations of A and B are "simpler", we can try to prove interpolation inductively. This strategy works in all 4 cases listed above.

We first explain the case of monomial schemes. The Grothendieck K group of $\mathbb{P}^2$ is a free abelian group of rank 3. Let $\chi(\xi, \zeta) = \sum_{i=0}^{2} \mathrm{ext}^i(F, E)$, where F and E are sheaves with Chern characters ξ and ζ. Then there is a natural pairing on $K(\mathbb{P}^2)$ given by $(\xi, \zeta) = \chi(\xi^*, \zeta)$. If we scale by the rank, every Chern character ξ (of positive rank) determines a parabola Q_ξ of orthogonal invariants in the (μ, Δ)-plane defined by $P(\mu(\xi) + \mu) - \Delta(\xi) = \Delta$, where $P(m) = \frac{1}{2}(m^2 + 3m + 2)$ is the

Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^2}$. The invariants of any sheaf satisfying interpolation with respect to F lie on the parabola Q_ξ.

A monomial scheme Z can be represented by a box diagram D_Z recording the monomials that are nonzero in $\mathbb{C}[x,y]/I_Z$. Let h_i be the number of boxes in the ith row counting from the bottom and let v_i be the number of boxes in the ith column counting from the left. Define

$$\mu_j = -1 + \frac{1}{j}\sum_{i=1}^{j}(h_i + i - 1), \quad \nu_k = -1 + \frac{1}{k}\sum_{i=1}^{k}(v_i + i - 1), \quad \mu_Z = \max_{j,k}(\mu_j, \nu_k).$$

Assume that the maximum is achieved by μ_h, (i.e., $\mu_Z = \mu_h$). Let D_U be the portion of D_Z lying above the hth horizontal line and let D_V be the potion of D_Z lying below this line. The diagrams D_U and D_V correspond to monomial zero-dimensional schemes U, V. We then have the following theorem.

Theorem 2. [CH] *Let Z be a zero-dimensional monomial scheme with Chern character ξ. There exists a vector bundle E of slope $\mu \in \mathbb{Q}$ satisfying interpolation for I_Z if and only if $\mu \geq \mu_Z$. We may take E to be prioritary. Furthermore, if there exists stable bundles of slope μ along Q_ξ, we may take E to be stable.*

The Bridgeland destabilizing sequence is given by

$$0 \to I_U(-h) \to I_Z \to I_{V\subset hL} \to 0,$$

where L is the line defined by $y = 0$. One proves the theorem by inducting on the complexity of Z. In fact, one computes the entire Harder-Narasimhan filtration of I_Z for different Bridgeland stability conditions, inductively decomposing the box diagram of the monomial scheme into pieces until each piece is a rectangle. As a corollary, one determines when monomial schemes are in the stable base loci of linear systems on the Hilbert schemes of points.

We now describe the effective cone of $M(\xi)$ in general. The possible invariants of stable vector bundles on $\mathbb{P}^2$ have been classified by Drézet and Le Potier [DLP], [LP]. First, there are *exceptional bundles* which are stable bundles E such that $\mathrm{Ext}^1(E,E) = 0$. The moduli space of an exceptional bundle is a single isolated point. The slope α of an exceptional bundle E_α is called an *exceptional slope.* The exceptional slopes exhibit remarkable number theoretic properties. For example, the even length continued fraction expansions of exceptional slopes between 0 and $\frac{1}{2}$ are palindromes consisting of 1s and 2s [H].

There is an explicit fractal curve δ in the (μ, Δ)-plane made of pieces of parabolas. For each exceptional slope α, there is an interval $I_\alpha = [\alpha - x_\alpha, \alpha + x_\alpha]$ where over that interval the curve δ is $Q_{-\alpha}$ on $[\alpha - x_\alpha, \alpha]$ and $Q_{-\alpha-3}$ on $[\alpha, \alpha + x_\alpha]$. The complement of these intervals is a Cantor set C with the following property (which plays an essential role in the geometry).

Theorem 3. [CHW] *A point of C is either an end point of an I_α (hence a quadratic irrational) or transcendental.*

Drézet and Le Potier prove that there exists a positive dimensional moduli space of stable bundles with invariants (r, μ, Δ) if and only if $r\mu, r(P(\mu) - \Delta) \in \mathbb{Z}$ and

$\Delta \geq \delta(\mu)$ [DLP]. In this case, the moduli space is normal, projective, $\mathbb{Q}$-factorial of dimension $r^2(2\Delta - 1) + 1$ [LP]. The stable bundles with $\Delta = \delta(\mu)$ are called *height zero* bundles and their moduli spaces have Picard rank 1. For moduli spaces $M(\xi)$ with $\Delta > \delta(\mu)$, the Picard group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ naturally identified with $\xi^\perp$ in $K(\mathbb{P}^2)$ with respect to the Euler pairing introduced above.

Since the intersection of Q_ξ with $\Delta = \frac{1}{2}$ is a quadratic irrational, by Theorem 3, Q_ξ intersects $\Delta = \frac{1}{2}$ along some I_α and determines an exceptional bundle E_α. This bundle controls the effective cone of $M(\xi)$. Let ξ_α be the Chern character of E_α. Our main theorem is in terms of the following invariants:

(1) If $\chi(\xi, \xi_\alpha) > 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha}$.
(2) If $\chi(\xi, \xi_\alpha) = 0$, let $(\mu^+, \Delta^+) = (\alpha, \Delta_\alpha)$.
(3) If $\chi(\xi, \xi_\alpha) < 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha-3}$.

Theorem 4. [CHW] *Let F be a general point of $M(\xi)$ and let r^+ be sufficiently large and divisible. Let ζ be the Chern character with rank r^+, slope μ^+ and discriminant Δ^+. Then the general point E of $M(\zeta)$ satisfies interpolation with respect to F. Furthermore, the Brill-Noether divisor D_E spans an extremal ray of the effective cone of $M(\xi)$. If $\chi(\xi, \xi_\alpha) \neq 0$, then D_E also spans an extremal ray of the movable cone.*

The Beilinson spectral sequence for an exceptional collection explicitly determined by α [Dr] yields a canonical two term complex. One thus obtains a good resolution of F that allows one to compute cohomology. (This resolution also coincides with the destabilizing sequence in the sense of Bridgeland.) In particular, one obtains a rational map to a moduli space of Kronecker modules. When $\chi(\xi, \xi_\alpha) \neq 0$, the divisor D_E is the pullback of the ample generator via this map. Using the fact that there are complete curves in the fibers, we deduce that D_E is extremal. When $\chi(\xi, \xi_\alpha) = 0$, the rational map is birational and contracts the divisor D_E.

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