For the “open problem session” in the conference in honor of Joe Harris

Barry Mazur

August 19, 2011

A great thing when you work together with Joe is that you find yourself in the midst of loads of inspiring problems, thanks to his deep engagement with, and intense curiosity about, all aspects of his subject. He's a master of formulating problems on so many levels that it’s already something of an open problem simply to choose just one or two of them in his honor.

Sometimes Joe introduces a problem very broadly and somewhat obliquely, as when he once asked “How many curves are there defined over $\mathbb{Q}$?” Of course, this was an invitation to discuss the dimension—as a function of the genus $g$ —of the Zariski closure of the set of $\mathbb{Q}$-rational points in $M_g$ the moduli space of curves of genus $g$. Hyperelliptic curves already gives you $2/3$ of the dimension of that moduli space (mod $O(1)$, and as $g \to \infty$) but can you get, say, a better fraction than that? This question, of course, immediately connects, via celebrated conjectures of Lang, to questions regarding the algebraic geometric structure of $M_g$.

And sometimes, when you come to Joe with a general question—even a vague unformed one—he will clue you in to a rich tradition of very concrete instantiations of it. This happened recently when John Tate, Mark Kisin, and I asked him a certain general question. In the remaining minutes of my presentation today I’ll hint at that question and mention briefly at least one specific instance of it that Joe and Mike Roth and Jason Starr had investigated (also Joe’s student Arnav Tripathy) opening up, it seems to me, a wide area of excellent algebro-geometric structure.

The general title of the problem could be: **descending cohomology, geometrically.** Take, for example, a smooth projective threefold $X$ over $\mathbb{Q}$ with the property that it possesses no holomorphic everywhere regular differential $3$-forms; i.e., $h^{0,3} = h^{3,0} = 0$. For any prime number $\ell$ let

$$Q_\ell(-1) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T\mu_\ell^{-1},$$

that is, we’ve “Tate- twisted” the constant sheaf down one peg.

Now let

$$H_\ell := H^3_{et}(X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}; Q_\ell(-1))$$

denote the three-dimensional $\ell$-adic étale cohomology group of $X$ viewed as $G_{\overline{\mathbb{Q}}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation, but “descended” in the sense that it is twisted down as indicated.
The reason why it is reasonable to do this is that for all primes \( p \) of good reduction for \( X \), the eigenvalues of Frobenius in the original untwisted representation are algebraic integers divisible by \( p \) (since \( h^{0,3} = 0 \): this is a consequence, for example, of the Katz conjecture, now a theorem) and therefore the downward twisted representation has the property that these eigenvalues remain algebraic integers, and—in the usual sense—“Weyl numbers”.

Even better, the \( \ell \)-adic Hodge numbers of the downward twisted representation are zero except for \( h^{0,1} = h^{1,0} \). I.e., given these numerical invariants, the downward twisted representation could conceivably be the Galois representation attached to an isogeny class of abelian varieties over \( \mathbb{Q} \). By Faltings’ Theorem, if there is such an isogeny class it is unique (also independent of \( \ell \)). Call it the **phantom** isogeny class of abelian varieties over \( \mathbb{Q} \) related to this \( X \). Moreover, for every prime \( p \) of good reduction, there really does exist a unique isogeny class of abelian varieties over \( \mathbb{F}_p \) that would play the role of the reduction mod \( p \) of the **phantom** if the latter were to exist. This follows from theorems of Tate and Honda.

So our questions, in general, were:

1. For which \( X \) does the **phantom abelian variety** actually exist?
2. When it does exist, how can we construct it by “algebro-geometric means”?

A natural place to look for these phantom abelian varieties, at least for threefolds \( X \) that are uniruled \(^1\) is as Albanese varieties attached to, say, appropriate Kontsevich spaces of parametrized genus zero curves in \( X \).

When we asked Joe about this, he led us—as he usually does—to some terrific results in this direction. Here is one of them (I’m quoting from an email of his, with his permission):

Mike Roth, Jason Starr and I looked at an example of this, the cubic threefold \( X \) in \( P^4 \): in this case, it’s known that the Albanese variety of the Fano scheme of lines on \( X \) is what you want; we wanted to see if the same was true for the space of rational curves of any degree. We were able to verify this up to degree 5, but not in general.

This already suggests an interesting stability type question regarding the Albanese varieties attached to Kontsevich spaces \( K := K(X, \beta) \) where \( X \) is a threefold and \( \beta \in H_2(X, \mathbb{Z}) \) a chosen homology class (i.e., the fixed fundamental class of the rational curves in \( X \) that \( K \) parametrizes)\(^2\)—i.e., what can be said about dependence on \( \beta \)? As is often the case with Joe’s work, this type of result opens up for us a treasure chest of interesting, and probably important, problems.

---

\(^1\) **Uniruled** means that for every point \( x \) of \( X \) there exists a rational curve lying in \( X \) passing through \( x \). The condition \( h^{3,0} \) for a threefold is not enough to guarantee that it is uniruled; there is a conjecture that if no positive power of the canonical bundle of \( X \) has sections, then \( X \) is uniruled. This is conjectured in all dimensions, and true for curves and surfaces.

\(^2\) As Joe explained to me, the corresponding “stability question” for fourfolds would involve the MRC quotients of the corresponding Kontsevich spaces and would have a very different look.