

The stabilization of the cohomology of moduli spaces of sheaves

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Preliminaries

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$$P_{H,\mathcal{F}}(m) = \chi(\mathcal{F}(mH)) = a_2 \frac{m^2}{2} + \text{l.o.t.} \quad \rho_{H,\mathcal{F}}(m) = \frac{P_{H,\mathcal{F}}(m)}{a_2}$$

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The slope and the discriminant are defined by the formulae

$$\mu = \frac{ch_1}{r}, \quad \mu_H = \frac{ch_1 \cdot H}{r H^2}, \quad \Delta = \frac{\mu^2}{2} - \frac{ch_2}{r}$$

A sheaf \mathcal{F} is called *Gieseker H -semistable* if for every proper subsheaf \mathcal{E} , $\rho_{\mathcal{E},H}(m) \leq \rho_{\mathcal{F},H}(m)$ for $m \gg 0$.

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A torsion free sheaf \mathcal{F} admits a unique *Harder-Narasimhan filtration* with respect to either Gieseker H -semistability or μ_H -semistability, i.e. a filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{F}$$

such that the quotients $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable with decreasing invariants.

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A semistable sheaf further admits a *Jordan-Hölder* filtration into stable objects. Two semistable sheaves are called *S -equivalent* if they have the same Jordan-Hölder factors.

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The main question today: What is the topology of $M_{X,H}(\gamma)$?

The rank 1 case

Let $X^{[n]}$ denote the Hilbert scheme of n points on X .

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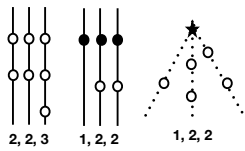
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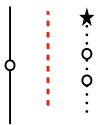
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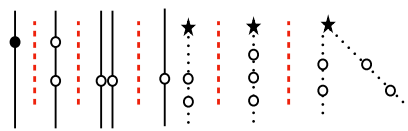
Each basis element depends on 3 partitions, whose total sum is n .



Codimension 1



Codimension 2



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Recall Macdonald's formula

$$\zeta_X(q, t) = \sum_{n=0}^{\infty} P_{X^{(n)}}(t) q^n = \frac{(1 + qt)^{b_1(X)} (1 + qt^3)^{b_3(X)}}{(1 - q)(1 - qt^2)^{b_2(X)} (1 - qt^4)}.$$

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Göttsche's Formula

$$F(q, t) = \sum_{n=0}^{\infty} P_{X^{[n]}}(t) q^n = \prod_{m=1}^{\infty} \zeta_X(t^{2m-2} q^m, t).$$

The Betti numbers of $X^{[n]}$ and more generally $M_{X,H}(1, c, \Delta)$ stabilize to $b_{i, \text{Stab}}(X)$ as n or Δ tend to infinity.

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Consider

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In contrast, the Betti numbers of higher rank moduli spaces are generally unknown.

The Main Conjecture

Conjecture (C-Woolf)

Fix a rank $r > 0$ and a first Chern character c . Then the i th Betti number of $M_{X,H}(r, c, \Delta)$ stabilizes as Δ tends to ∞ and the limit is independent of r, c, H .

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More precisely, given an integer k , there exists $\Delta_0(k)$ such that for $\Delta \geq \Delta_0(k)$ and $i \leq k$

$$b_i(M_{X,H}(r, c, \Delta)) = b_{i, \text{Stab}}(X).$$

Furthermore, if H is in a compact subset \mathcal{C} of the ample cone of X , then $\Delta_0(k)$ can be chosen independently of $H \in \mathcal{C}$.

Numerical evidence

Ellingsrud-Stromme's table for the Betti Numbers of $\mathbb{P}^2[n]$

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n												
1	1	1										
2	1	2	3									
3	1	2	5	6								
4	1	2	6	10	13							
5	1	2	6	12	21	24						
6	1	2	6	13	26	39	47					
7	1	2	6	13	28	49	74	83				
8	1	2	6	13	29	54	94	131	150			
9	1	2	6	13	29	56	105	167	232	257		
10	1	2	6	13	29	57	110	189	298	395	440	

Yoshioka's table for rank 2, $c_1 = H$ bundles on \mathbb{P}^2

c_2														
1	1													
2	1	2	3											
3	1	2	6	9	12									
4	1	2	6	13	24	35	41							
5	1	2	6	13	29	51	85	113	129					
6	1	2	6	13	29	57	106	175	262	337	370			
7	1	2	6	13	29	57	113	200	342	527	746	922	1002	
8	1	2	6	13	29	57	113	208	372	625	995	1464	1978	

Manschot's table for rank 3, $c_1 = H$ bundles on \mathbb{P}^3

c_2													
2	1	1											
3	1	2	5	8	10								
4	1	2	6	12	24	38	54	59					
5	1	2	6	13	28	52	94	149	217	273	298		
6	1	2	6	13	29	56	108	189	322	505	744	992	1200

K3 and abelian surfaces

The conjecture is known for smooth moduli spaces of sheaves on K3 and abelian surfaces.

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- By Mukai, Huybrechts and Yoshioka, smooth moduli spaces of sheaves on a K3 surface X are deformations of $X^{[n]}$ of the same dimension. Hence, they are diffeomorphic to $X^{[n]}$.
- By Yoshioka, a smooth moduli space of sheaves $M_{X,H}(\gamma)$ is deformation equivalent to $X^* \times X^{[n]}$ of the same dimension, where X^* is the dual abelian surface.

History

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Connection to the Atiyah-Jones Conjecture and work of Taubes

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- Manschot building on the work of Mozgovoy gave a formula for the Betti numbers of the moduli spaces when $X = \mathbb{P}^2$.

Motto:

Computing the Betti numbers are hard
Computing the stable Betti numbers are easy.

Remarks

- Göttsche and Soergel computed the Hodge numbers of $M_{X,H}(1, c, \Delta)$. They stabilize as Δ tends to ∞ . We expect the Hodge numbers of $M_{X,H}(r, c, \Delta)$ to also stabilize to the stable Hodge numbers of $M_{X,H}(1, c, \Delta)$, at least when $M_{X,H}(r, c, \Delta)$ is smooth.

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- Are there geometric reasons for the equality of numbers?

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Complete the resulting with respect to the \mathbb{Z} -graded decreasing filtration

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Given smooth projective varieties X_i of dimension d_i , we want a notion of stabilization in A^- that guarantees that the low-degree Betti numbers of X_i stabilize. Consider $\mathbb{L}^{-d_i}[X_i]$. By Poincaré duality,

$$P_{[X_i]}(t) = P_{\mathbb{L}^{-d_i}[X_i]}(t^{-1}), \quad H_{[X_i]}(x, y) = H_{\mathbb{L}^{-d_i}[X_i]}(x^{-1}, y^{-1}).$$

If $\mathbb{L}^{-d_i}[X_i]$ converges in A^- , then the low-degree Betti/Hodge numbers of X_i stabilize.

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One can ask whether the classes of $M_{X,H}(r, c, \Delta)$ also stabilize in A^- to the stable class of $M_{X,H}(1, c, \Delta)$ as Δ tends to ∞ ?

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The geometry of the moduli space for small Δ can be awful!

Theorem (C-Huizenga)

For any integer k , there is a number $d_k \gg 0$ such that if $d \geq d_k$, then a very general surface $X \subset \mathbb{P}^3$ of degree d has some moduli space $M_X(2, H, s)$ with at least k components.

Main Theorem

Theorem (C-Woolf)

Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. The classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ of the moduli stacks of Gieseker semistable sheaves stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\text{top}}(X)}}$$

as Δ tends to ∞ .

Corollary (C-Woolf)

The virtual Betti and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize, and the generating functions for the stable numbers $\tilde{b}_{i, \text{Stab}}$ and $\tilde{h}_{\text{Stab}}^{p,q}$ are given by

$$\sum_{i=0}^{\infty} \tilde{b}_{i, \text{Stab}} t^i = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\text{top}}(X)}}$$

$$\sum_{p,q=0}^{\infty} \tilde{h}_{\text{Stab}}^{p,q} x^p y^q = \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\text{top}}(X)}}.$$

Corollary (C-Woolf)

Let X be a rational surface and H a polarization such that $K_X \cdot H < 0$. Assume that there are no strictly semistable sheaves of rank r and first Chern class c . Then the Poincaré and Hodge polynomials of $M_{X,H}(r, c, \Delta)$ stabilize as Δ tends to ∞ and the generating functions for the stable Betti and Hodge numbers are given by

$$(1 - t^2) \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\text{top}}(X)}}$$

$$(1 - xy) \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\text{top}}(X)}},$$

respectively.

Wall-Crossing

Joyce's Wall-Crossing formula:

$$[\mathcal{M}_{H_2}(\gamma)] = \sum_{\sum_{i=1}^{\ell} \gamma_i = \gamma} S(\gamma_1, \dots, \gamma_{\ell}) \mathbb{L}^{-\sum_{1 \leq i < j \leq \ell} \chi(\gamma_j, \gamma_i)} \prod_{i=1}^{\ell} [\mathcal{M}_{H_1}(\gamma_i)]$$

Proposition (C-Woolf)

Let H_1 be ample and let H_2 be big and nef. Assume that $H_i \cdot K_X < 0$. Then for all integers v , there is a $\Delta_0(v) > 0$ such that for all $\gamma \in K^0(X)$ with $\text{rk}(\gamma) = r$, $c_1(\gamma) = c$, and $\Delta(\gamma) \geq \Delta_0(v)$, all the terms in the sum with $\ell > 1$ have dimension less than v .

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Moreover, if \mathcal{C} is a compact convex subset of the K_X -negative part of the big and nef cone and $H_1, H_2 \in \mathcal{C}$, then $\Delta_0(v)$ can be chosen to depend only on \mathcal{C} .

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In particular, $[\mathcal{M}_{H_1}(\gamma)]$ stabilizes in A^- as Δ tends to ∞ if and only if $[\mathcal{M}_{H_2}(\gamma)]$ does; and they have the same stabilization.

Blowing up

$$Y_{X,H}(q) = \sum [\mathcal{M}_{X,H}^{\mu,s}(r, C, d)] q^{-d}$$

$$\hat{Y}_{X,H,m}(q) = \sum [\mathcal{M}_{\hat{X},p^*H}^{\mu,s}(r, C - mE, d)] q^{-d}.$$

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$$\hat{Y}_{X,H,m}(q) = \sum [\mathcal{M}_{\hat{X},\rho^*H}^{\mu,s}(r, C - mE, d)] q^{-d}.$$

Blowup formula of Yoshioka and Mozgovoy

$$\hat{Y}_{X,H,m}(q) = F_m(q) Y_{X,H}(q),$$

$$F_m(q) = \prod_{k \geq 1} \frac{1}{(1 - \mathbb{L}^{rk} q^k)^r} \sum_{\substack{\sum_{i=1}^r a_i = 0, \\ a_i \in \mathbb{Z} + \frac{m}{r}}} \mathbb{L}^{\sum_{i < j} \binom{a_j - a_i}{2}} q^{-\sum_{i < j} a_i a_j}.$$

$$Y'_{X,H}(q) = \sum [M_{X,H}^{\mu,s}(r, C, d)] \mathbb{L}^{2rd - C^2 + r^2 \chi(\mathcal{O}_X)} q^{-d}$$

$$\hat{Y}'_{X,H,m}(q) = \sum [M_{\hat{X},p^*H}^{\mu,s}(r, C - mE, d)] \mathbb{L}^{2rd - C^2 + r^2 \chi(\mathcal{O}_X)} q^{-d}.$$

$$\hat{Y}'_{X,H,m}(q) = F_m(\mathbb{L}^{-2r} q) Y'_{X,H}(q)$$

Lemma

The sequence $[M_{\hat{X}, p^*H}^{\mu, s}(r, C - mE, d)]$ stabilizes in A^- if and only if the sequence $[M_{X, H}^{\mu, s}(r, C, d)]$ does.

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Proposition

Suppose that the $[M_{X, H}^{\mu, s}(r, C, d)]$ stabilize in A^- . We have the relation

$$\lim_{q \rightarrow 1} \left((1 - q) \hat{Y}'_{X, H, m}(q) \right) = \prod_{k=1}^{\infty} \frac{1}{1 - \mathbb{L}^{-k}} \lim_{q \rightarrow 1} \left((1 - q) Y'_{X, H}(q) \right)$$

between the generating series for $M_{X, H}^{\mu, s}(r, C, d)$ and $M_{\hat{X}, p^*H}^{\mu, s}(r, C - mE, d)$.

The sum

$$\sum_{\sum_{i=1}^r a_i=0, a_i \in \mathbb{Z} + \frac{m}{r}} \mathbb{L}^{\sum_{i < j} \binom{a_j - a_i}{2}} + 2r \sum_{i < j} a_i a_j$$

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$$F_m(\mathbb{L}^{-2r}) = \prod_{k=1}^{\infty} \frac{1}{1 - \mathbb{L}^{-k}}$$

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We first understand the moduli spaces on \mathbb{F}_1 .

Theorem (C-Woelf)

Assume $r \mid c \cdot F$. Then the classes $[\mathcal{M}_{\mathbb{F}_1, E+F}(r, c, \Delta)]$ stabilize in A^- to

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Joyce's wall-crossing starting with Mozgovoy's calculation of $[\mathcal{M}_{\mathbb{F}_1, F}(r, c, \Delta)]$. In fact, Mozgovoy calculates $[\mathcal{M}_{X, F}(r, c, \Delta)]$, for any ruled surface X .

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- 1 If $r \nmid c \cdot F$, then the stack is empty.
- 2 If $r \mid c \cdot F$ and X is a rational ruled surface, then the classes $[\mathcal{M}_{X, F}(r, c, \Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

- ① Using the blowup formula, we conclude that the classes $[\mathcal{M}_{\mathbb{P}^2}(r, c, \Delta)]$ stabilize in A^- to

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- ② Using the blowup formula and wall-crossing, we get that the classes $[\mathcal{M}_{\mathbb{F}_1, H}(r, c, \Delta)]$ stabilize in A^- to

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- ③ \mathbb{F}_e is obtained from \mathbb{F}_{e-1} by an elementary modification. Using the blowup formula twice and induction, the same holds for \mathbb{F}_e .

Finally, by induction on the number of blowups, the blowup formula and wall-crossing, we obtain:

Theorem (C-Woolf)

Let X be a rational surface and let H be a polarization such that $H \cdot K_X < 0$. Then the classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\text{top}}(X)}}.$$

Thank you!