The stabilization of the cohomology of moduli spaces of sheaves

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UIC

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Preliminaries

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$$P_{H,\mathcal{F}}(m) = \chi(\mathcal{F}(mH)) = a_2 \frac{m^2}{2} + \mathrm{l.o.t.} \quad p_{H,F}(m) = \frac{P_{H,\mathcal{F}}(m)}{a_2}$$

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The slope and the discriminant are defined by the formulae

$$\mu = \frac{\mathrm{ch}_1}{r}, \quad \mu_H = \frac{\mathrm{ch}_1 \cdot H}{r\ H^2}, \quad \Delta = \frac{\mu^2}{2} - \frac{\mathrm{ch}_2}{r}$$

A sheaf $\mathcal{F}$ is called *Gieseker H-semistable* if for every proper subsheaf $\mathcal{E}$, $p_{\mathcal{E},H}(m) \leq p_{\mathcal{F},H}(m)$ for $m \gg 0$.

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A torsion free sheaf $\mathcal{F}$ admits a unique *Harder-Narasimhan filtration* with respect to either Gieseker H-semistability or μ_H-semistability, i.e. a filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{F}$$

such that the quotients $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable with decreasing invariants.

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A semistable sheaf further admits a *Jordan-Hölder* filtration into stable objects. Two semistable sheaves are called *S-equivalent* if they have the same Jordan-Hölder factors.

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The main question today: What is the topology of $M_{X,H}(\gamma)$?

The rank 1 case

Let $X^{[n]}$ denote the Hilbert scheme of n points on X.

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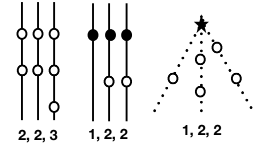
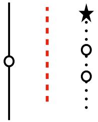
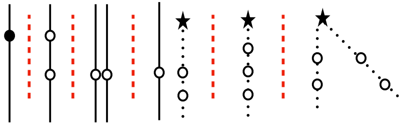
Each basis element depends on 3 partitions, whose total sum is n.



Codimension 1



Codimension 2



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Recall Macdonald's formula

$$\zeta_X(q,t) = \sum_{n=0}^{\infty} P_{X^{(n)}}(t)q^n = \frac{(1+qt)^{b_1(X)}(1+qt^3)^{b_3(X)}}{(1-q)(1-qt^2)^{b_2(X)}(1-qt^4)}.$$

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Göttsche's Formula

$$F(q,t) = \sum_{n=0}^{\infty} P_{X^{[n]}}(t) q^n = \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t).$$

The Betti numbers of $X^{[n]}$ and more generally $M_{X,H}(1, c, \Delta)$ stabilize to $b_{i,\mathrm{Stab}}(X)$ as n or Δ tend to infinity.

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Consider

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The coefficient of t^i is a polynomial in q.

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The coefficient of t^i is a polynomial in q.

In contrast, the Betti numbers of higher rank moduli spaces are generally unknown.

The Main Conjecture

Conjecture (C-Woolf)

Fix a rank $r > 0$ and a first Chern character c. Then the ith Betti number of $M_{X,H}(r, c, \Delta)$ stabilizes as Δ tends to ∞ and the limit is independent of r, c, H.

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Fix a rank $r > 0$ and a first Chern character c. Then the ith Betti number of $M_{X,H}(r, c, \Delta)$ stabilizes as Δ tends to ∞ and the limit is independent of r, c, H.
More precisely, given an integer k, there exists $\Delta_0(k)$ such that for $\Delta \geq \Delta_0(k)$ and $i \leq k$

$$b_i(M_{X,H}(r, c, \Delta)) = b_{i,\mathrm{Stab}}(X).$$

Furthermore, if H is in a compact subset $\mathcal{C}$ of the ample cone of X, then $\Delta_0(k)$ can be chosen independently of $H \in \mathcal{C}$.

Numerical evidence

Ellingsrud-Stromme's table for the Betti Numbers of $\mathbb{P}^{2[n]}$

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n											
1	1	1									
2	1	2	3								
3	1	2	5	6							
4	1	2	6	10	13						
5	1	2	6	12	21	24					
6	1	2	6	13	26	39	47				
7	1	2	6	13	28	49	74	83			
8	1	2	6	13	29	54	94	131	150		
9	1	2	6	13	29	56	105	167	232	257	
10	1	2	6	13	29	57	110	189	298	395	440

Yoshioka's table for rank 2, $c_1 = H$ bundles on $\mathbb{P}^2$

c_2													
1	1												
2	1	2	3										
3	1	2	6	9	12								
4	1	2	6	13	24	35	41						
5	1	2	6	13	29	51	85	113	129				
6	1	2	6	13	29	57	106	175	262	337	370		
7	1	2	6	13	29	57	113	200	342	527	746	922	1002
8	1	2	6	13	29	57	113	208	372	625	995	1464	1978

Manschot's table for rank 3, $c_1 = H$ bundles on $\mathbb{P}^2$

c_2													
2	1	1											
3	1	2	5	8	10								
4	1	2	6	12	24	38	54	59					
5	1	2	6	13	28	52	94	149	217	273	298		
6	1	2	6	13	29	56	108	189	322	505	744	992	1200

K3 and abelian surfaces

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- By Mukai, Huybrechts and Yoshioka, smooth moduli spaces of sheaves on a K3 surface X are deformations of $X^{[n]}$ of the same dimension. Hence, they are diffeomorphic to $X^{[n]}$.
- By Yoshioka, a smooth moduli space of sheaves $M_{X,H}(\gamma)$ is deformation equivalent to $X^* \times X^{[n]}$ of the same dimension, where X^* is the dual abelian surface.

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Connection to the Atiyah-Jones Conjecture and work of Taubes

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- Manschot building on the work of Mozgovoy gave a formula for the Betti numbers of the moduli spaces when $X = \mathbb{P}^2$.

Remarks

- Göttsche and Soergel computed the Hodge numbers of $M_{X,H}(1,c,\Delta)$. They stabilize as Δ tends to ∞. We expect the Hodge numbers of $M_{X,H}(r,c,\Delta)$ to also stabilize to the stable Hodge numbers of $M_{X,H}(1,c,\Delta)$, at least when $M_{X,H}(r,c,\Delta)$ is smooth.

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- Are there geometric reasons for the equality of numbers?

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Given smooth projective varieties X_i of dimension d_i, we want a notion of stabilization in A^- that guarantees that the low-degree Betti numbers of X_i stabilize. Consider $\mathbb{L}^{-d_i}[X_i]$. By Poincaré duality,

$$P_{[X_i]}(t) = P_{\mathbb{L}^{-d_i}[X_i]}(t^{-1}), \quad H_{[X_i]}(x,y) = H_{\mathbb{L}^{-d_i}[X_i]}(x^{-1}, y^{-1}).$$

If $\mathbb{L}^{-d_i}[X_i]$ converges in A^-, then the low-degree Betti/Hodge numbers of X_i stabilize.

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One can ask whether the classes of $M_{X,H}(r, c, \Delta)$ also stabilize in A^{-} to the stable class of $M_{X,H}(1, c, \Delta)$ as Δ tends to ∞?

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- Little evidence for the conjecture for general type surfaces.

  The geometry of the moduli space for small Δ can be awful!

Theorem (C-Huizenga)

For any integer k, there is a number $d_k \gg 0$ such that if $d \geq d_k$, then a very general surface $X \subset \mathbb{P}^3$ of degree d has some moduli space $M_X(2, H, s)$ with at least k components.

Main Theorem

Theorem (C-Woolf)

Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. The classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ of the moduli stacks of Gieseker semistable sheaves stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\mathrm{top}}(X)}}$$

as Δ tends to ∞.

Corollary (C-Woolf)

The virtual Betti and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize, and the generating functions for the stable numbers $\tilde{b}_{i,\mathrm{Stab}}$ and $\tilde{h}^{p,q}_{\mathrm{Stab}}$ are given by

$$\sum_{i=0}^{\infty} \tilde{b}_{i,\mathrm{Stab}} t^i = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\mathrm{top}}(X)}}$$

$$\sum_{p,q=0}^{\infty} \tilde{h}^{p,q}_{\mathrm{Stab}} x^p y^q = \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\mathrm{top}}(X)}}.$$

Corollary (C-Woolf–Yoshioka)

Let X be a rational surface and H a polarization such that $K_X \cdot H < 0$. Assume that there are no strictly semistable sheaves of rank r and first Chern class c. Then the Poincaré and Hodge polynomials of $M_{X,H}(r,c,\Delta)$ stabilize as Δ tends to ∞ and the generating functions for the stable Betti and Hodge numbers are given by

$$(1-t^2)\prod_{i=1}^{\infty}\frac{1}{(1-t^{2i})^{\chi_{\mathrm{top}}(X)}}$$

$$(1-xy)\prod_{i=1}^{\infty}\frac{1}{(1-(xy)^i)^{\chi_{\mathrm{top}}(X)}},$$

respectively.

Wall-Crossing

Joyce's Wall-Crossing formula:

$$[\mathcal{M}_{H_2}(\gamma)] = \sum_{\sum_{i=1}^{\ell} \gamma_i = \gamma} S(\gamma_1, \ldots, \gamma_\ell) \mathbb{L}^{-\sum_{1 \le i < j \le \ell} \chi(\gamma_j, \gamma_i)} \prod_{i=1}^{\ell} [\mathcal{M}_{H_1}(\gamma_i)]$$

Proposition

Let H_1 *be ample and let* H_2 *be big and nef. Assume that* $H_i \cdot K_X < 0$*. Then for all integers* v*, there is a* $\Delta_0(v) > 0$ *such that for all* $\gamma \in K^0(X)$ *with* $\mathrm{rk}(\gamma) = r$*,* $c_1(\gamma) = c$*, and* $\Delta(\gamma) \geq \Delta_0(v)$*, all the terms in the sum with* $\ell > 1$ *have dimension less than* v*.*

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Moreover, if $\mathcal{C}$ is a compact convex subset of the K_X-negative part of the big and nef cone and $H_1, H_2 \in \mathcal{C}$, then $\Delta_0(v)$ can be chosen to depend only on $\mathcal{C}$.

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Moreover, if $\mathcal{C}$ is a compact convex subset of the K_X-negative part of the big and nef cone and $H_1, H_2 \in \mathcal{C}$, then $\Delta_0(v)$ can be chosen to depend only on $\mathcal{C}$.
In particular, $[\mathcal{M}_{H_1}(\gamma)]$ stabilizes in A^- as Δ tends to ∞ if and only if $[\mathcal{M}_{H_2}(\gamma)]$ does; and they have the same stabilization.

Blowing up

$$Y_{X,H}(q) = \sum [\mathcal{M}^{\mu,s}_{X,H}(r, C, d)] q^{-d}$$

$$\hat{Y}_{X,H,m}(q) = \sum [\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r, C - mE, d)] q^{-d}.$$

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Blowup formula of Yoshioka and Mozgovoy

$$\hat{Y}_{X,H,m}(q) = F_m(q) Y_{X,H}(q),$$

$$F_m(q) = \prod_{k \geq 1} \frac{1}{(1 - \mathbb{L}^{rk} q^k)^r} \sum_{\sum_{i=1}^r a_i = 0,\ a_i \in \mathbb{Z} + \frac{m}{r}} \mathbb{L}^{\sum_{i<j} \binom{a_j - a_i}{2}} q^{-\sum_{i<j} a_i a_j}.$$

$$Y'_{X,H}(q) = \sum [M^{\mu,s}_{X,H}(r, C, d)]\mathbb{L}^{2rd - C^2 + r^2\chi(\mathcal{O}_X)} q^{-d}$$

$$\hat{Y}'_{X,H,m}(q) = \sum [M^{\mu,s}_{\hat{X},p^*H}(r, C - mE, d)]\mathbb{L}^{2rd - C^2 + r^2\chi(\mathcal{O}_X)} q^{-d}.$$

$$\hat{Y}'_{X,H,m}(q) = F_m(\mathbb{L}^{-2r} q) Y'_{X,H}(q)$$

Lemma

The sequence $[M^{\mu,s}_{\hat{X},p^*H}(r, C - mE, d)]$ *stabilizes in* A^- *if and only if the sequence* $[M^{\mu,s}_{X,H}(r, C, d)]$ *does.*

Lemma

The sequence $[M^{\mu,s}_{\hat{X},p^*H}(r, C - mE, d)]$ *stabilizes in* A^- *if and only if the sequence* $[M^{\mu,s}_{X,H}(r, C, d)]$ *does.*

Proposition

Suppose that the $[\mathcal{M}^{\mu,s}_{X,H}(r, C, d)]$ *stabilize in* A^-. *We have the relation*

$$\lim_{q\to 1}\Big((1-q)\hat{Y}'_{X,H,m}(q)\Big) = \prod_{k=1}^{\infty}\frac{1}{1-\mathbb{L}^{-k}}\lim_{q\to 1}\left((1-q)Y'_{X,H}(q)\right)$$

between the generating series for $\mathcal{M}^{\mu,s}_{X,H}(r, C, d)$ *and* $\mathcal{M}^{\mu,s}_{\hat{X},p^*H}(r, C - mE, d)$.

The sum

$$\sum_{\sum_{i=1}^{r} a_i=0,\, a_i\in\mathbb{Z}+\frac{m}{r}} \mathbb{L}^{\sum_{i<j}\binom{a_j-a_i}{2}+2r\sum_{i<j}a_ia_j}$$

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$$F_m(\mathbb{L}^{-2r}) = \prod_{k=1}^{\infty}\frac{1}{1-\mathbb{L}^{-k}}$$

The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$, $e \neq 1$.

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Every rational surface can be obtained by successively blowing up one of these surfaces.

We first understand the moduli spaces on $\mathbb{F}_1$.

Theorem

Assume $r \mid c \cdot F$. *Then the classes* $[\mathcal{M}_{\mathbb{F}_1, E+F}(r, c, \Delta)]$ *stabilize in* A^- *to*

$$\prod_{k=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^4}.$$

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Joyce's wall-crossing starting with Mozgovoy's calculation of $[\mathcal{M}_{\mathbb{F}_1, F}(r, c, \Delta)]$. In fact, Mozgovoy calculates $[\mathcal{M}_{X,F}(r, c, \Delta)]$, for any ruled surface X.

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1. If $r \nmid c \cdot F$, then the stack is empty.
2. If $r \mid c \cdot F$ and X is a rational ruled surface, then the classes $[\mathcal{M}_{X,F}(r,c,\Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-i})^4}.$$

1. Using the blowup formula, we conclude that the classes $[\mathcal{M}_{\mathbb{P}^2}(r,c,\Delta)]$ stabilize in A^- to

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2. Using the blowup formula and wall-crossing, we get that the classes $[\mathcal{M}_{\mathbb{F}_1,H}(r, c, \Delta)]$ stabilize in A^- to

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$$\prod_{i=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-i})^4}.$$
3. $\mathbb{F}_e$ is obtained from $\mathbb{F}_{e-1}$ by an elementary modification. Using the blowup formula twice and induction, the same holds for $\mathbb{F}_e$.

Finally, by induction on the number of blowups, the blowup formula and wall-crossing, we obtain:

Theorem (C-Woolf)

Let X be a rational surface and let H be a polarization such that $H \cdot K_X < 0$. Then the classes $[\mathcal{M}_{X,H}(r,c,\Delta)]$ stabilize in A^- to

$$\prod_{i=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-i})^{\chi_{\mathrm{top}}(X)}}.$$

Thank you!