# Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$

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# Introduction

- Talk based on arXiv:1909.02985.
- Main result: an algorithm computing intersection cohomology Betti numbers of moduli spaces of semistable coherent sheaves on the complex projective plane $\mathbb{P}^2$.
- Form of the algorithm: a scattering diagram in the moduli space of Bridgeland stability conditions on the derived category of coherent sheaves on $\mathbb{P}^2$.

# Introduction

Related works I will not talk about today:

- The same algorithm computes Gromov-Witten invariants (genus 0: Tim Gräfnitz arXiv:2005.14018, higher genus: B. arXiv:1909.02992).
- In combination with the topic of the present talk, one gets a new sheaves/Gromov-Witten correspondence, which can be used to prove non-trivial results on the Gromov-Witten side (N. Takahashi's conjecture: B. arXiv:1909.02992) and on the sheaf side (construction of quasimodular forms from Betti numbers of moduli spaces of one-dimensional semistable coherent sheaves on $\mathbb{P}^2$: arXiv:2001.05347, with Honglu Fan, Shuai Guo, Longting Wu).

# Plan

- Coherent sheaves on $\mathbb{P}^2$ and moduli spaces.
- Bridgeland stability conditions.
- Scattering diagrams.
- Scattering diagram from stability conditions.

# Coherent sheaves on $\mathbb{P}^2$

- $\mathbb{P}^2$: complex projective plane.
- Line bundles on $\mathbb{P}^2$: $\mathcal{O}(n)$, $n \in \mathbb{Z}$.
- Higher rank vector bundles on $\mathbb{P}^2$?
- Direct sums of line bundles $\bigoplus_i \mathcal{O}(n_i)$.
- Tangent bundle $T_{\mathbb{P}^2}$: rank 2 vector bundle, not a sum of line bundles.
- Euler short exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \to 0 \,.$$

- Much more vector bundles, coming in families (more difficult to describe in a completely elementary way).

# Coherent sheaves on $\mathbb{P}^2$

- Consider vector bundles supported on subvarieties of $\mathbb{P}^2$.
- $p \in \mathbb{P}^2$ a point, $\mathcal{O}_p$ skyscraper sheaf.
- $C \subset \mathbb{P}^2$ curve $f = 0$, structure sheaf $\mathcal{O}_C$,

$$0 \to \mathcal{O}(-1) \xrightarrow{f} \mathcal{O} \to \mathcal{O}_C \to 0\,.$$

- Ideal sheaf of a point $I_p$, torsion free rank 1 not locally free coherent sheaf,

$$0 \to I_p \to \mathcal{O} \to \mathcal{O}_p \to 0\,.$$

- Coherent sheaves on $\mathbb{P}^2$ form an abelian category $\mathrm{Coh}(\mathbb{P}^2)$.

# Numerical invariants of coherent sheaves

- $E$ a coherent sheaf on $\mathbb{P}^2$.
- Rank $r(E) \in \mathbb{Z}_{\geq 0}$, degree $d(E) = c_1(E) \in H^2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$, holomorphic Euler characteristic $\chi(E) \in \mathbb{Z}$ $\chi(E) = \mathrm{ch}_2(E) + r(E) + \frac{3}{2} d(E)$.
- Define $\gamma(E) := (r(E), d(E), \chi(E)) \in \mathbb{Z}^3$.
- Additive invariant $\gamma(F) = \gamma(E) + \gamma(G)$ if $0 \to E \to F \to G \to 0$ exact.
- Universal additive invariant: $\gamma : E \mapsto \gamma(E)$ induces

$$\Gamma := K_0(\mathrm{Coh}(\mathbb{P}^2)) \simeq \mathbb{Z}^3 \,.$$

- $\gamma(\mathcal{O}(n)) = (1, n, \frac{(n+1)(n+2)}{2})$, $\gamma(T_{\mathbb{P}^2}) = (2, 3, 8)$, $\gamma(\mathcal{O}_L) = (0, 1, 1)$, $\gamma(I_p) = (1, 0, -1)$.

# Classical stability

- Want to parametrize coherent sheaves of class $\gamma \in \mathbb{Z}^3$. Need to restrict the class of objects to get finite type moduli spaces (e.g. $\mathcal{O}(-n) \oplus \mathcal{O}(n)$).
- Let $E$ be a coherent sheaf on $\mathbb{P}^2$. The reduced Hilbert polynomial is the monic polynomial

$$p_E(n) := \frac{\chi(E(n))}{\alpha_E} \,,$$

  where $\alpha_E$ is the leading coefficient of the Hilbert polynomial $\chi(E(n))$.
- A coherent sheaf $E$ on $\mathbb{P}^2$ is *Gieseker semistable* (respectively *stable*) if $E$ is of pure dimension (that is, every nonzero subsheaf of $E$ has a support of dimension equal to the dimension of the support of $E$), and, for every nonzero strict subsheaf $F$ of $E$, we have $p_F(n) \leqslant p_E(n)$ (respectively $p_F(n) < p_E(n)$) for $n$ large enough.

# Moduli spaces

- $\gamma = (r, d, \chi) \in \mathbb{Z}^3$
- $M_\gamma$ "coarse" moduli space of Gieseker semistable coherent sheaves on $\mathbb{P}^2$ of class $\gamma$ (whose points parametrize $S$-equivalence classes of semistable sheaves).
- $M_\gamma$ projective scheme, constructed by geometric invariant theory.
- $M_\gamma$ is smooth if $\gamma$ is primitive.
- $M_\gamma$ is singular in general.
- Drézet-Le Potier (1985): $M_\gamma$ is irreducible. Precise determination of classes $\gamma$ such that $M_\gamma$ is non-empty.
- When there exists a stable object of class $\gamma$, $M_\gamma$ has dimension $r^2 + 3dr + d^2 - 2r\chi + 1$.

# Betti numbers

- Want to compute the Betti numbers $b_j(M_\gamma) := \dim H^j(M_\gamma, \mathbb{Q})$.
- We will compute the Betti numbers $b_j(M_\gamma)$ when $M_\gamma$ is smooth.
- We will compute the intersection Betti numbers $Ib_j(M_\gamma) := \dim IH^j(M_\gamma, \mathbb{Q})$ in general.
- The intersection Betti numbers $Ib_j(M_\gamma)$ are (refined) Donaldson-Thomas invariants of the non-compact Calabi-Yau 3-fold $K_{\mathbb{P}^2}$, local $\mathbb{P}^2$, total space of the canonical line bundle $\mathcal{O}(-3)$ of $\mathbb{P}^2$.

# Strategy

- View Gieseker stability as a particular point in a larger space of more general notions of stability conditions $\sigma$.
- Consider moduli spaces $M_\gamma^\sigma$ of $\sigma$-semistable objects.
- Study $M_\gamma^\sigma$ as a function of $\sigma$.
- Wall-crossing phenomenon: across codimension 1 loci in the space of stability conditions, the topology of $M_\gamma^\sigma$ jumps.

Two things to understand:

- How do the intersection Betti numbers of $M_\gamma^\sigma$ change across the walls? Wall-crossing formula? Answer (specific to $\mathbb{P}^2$ and using that $K_{\mathbb{P}^2}$ is a Calabi-Yau 3-fold): Kontsevich-Soibelman wall-crossing formula for Donaldson-Thomas invariants of Calabi-Yau categories of dimension 3.
- How to move nicely in the space of stability conditions? Answer: scattering diagram.

# Bridgeland stability conditions

To extend the notion of stability, need to replace the abelian category $\mathrm{Coh}(\mathbb{P}^2)$ of coherent sheaves on $\mathbb{P}^2$ by the bounded derived category $D^b\,\mathrm{Coh}(\mathbb{P}^2)$ of coherent sheaves on $\mathbb{P}^2$. Roughly, need to consider bounded complexes of coherent sheaves.

# Bridgeland stability conditions

## Definition (Bridgeland)

A *prestability condition* $\sigma$ on $D^b\operatorname{Coh}(\mathbb{P}^2)$ consists of a pair $\sigma = (Z, \mathcal{A})$, such that:

- $\mathcal{A}$ is the heart of a bounded t-structure on $D^b\operatorname{Coh}(\mathbb{P}^2)$ (in particular, an abelian category inside $D^b\operatorname{Coh}(\mathbb{P}^2)$).
- $Z$ is a linear map $Z\colon \Gamma \to \mathbb{C}$, called the *central charge*.
- For every nonzero object $E$ of $\mathcal{A}$, we have $Z(E) = \rho(E)e^{i\pi\phi(E)}$ with $\rho(E) \in \mathbb{R}_{>0}$, and $0 < \phi(E) \leqslant 1$, that is $Z(E)$ is contained in the upper half-plane minus the nonnegative real axis.
- A nonzero object $F$ of $\mathcal{A}$ is *$\sigma$-semistable* if for every nonzero subobject $F'$ of $F$ in $\mathcal{A}$, we have $\phi(F') \leqslant \phi(F)$. We require the Harder-Narasimhan property, that is, that every nonzero object $E$ of $\mathcal{A}$ admits a finite filtration $0 \subset E_0 \subset E_1 \cdots \subset E_n = E$ in $\mathcal{A}$, with each factor $F_i := E_i/E_{i-1}$ $\sigma$-semistable and $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$.

# Bridgeland stability conditions

## Definition (Bridgeland)

A stability condition $\sigma = (Z, \mathcal{A})$ on $D^b \operatorname{Coh}(\mathbb{P}^2)$ is a prestability condition satisfying the *support property*, that is, such that there exists a quadratic form $Q$ on the $\mathbb{R}$-vector space $\Gamma \otimes \mathbb{R}$ such that:

- The kernel of $Z$ in $\Gamma \otimes \mathbb{R}$ is negative definite with respect to $Q$,
- For every $\sigma$-semistable object, we have $Q(\gamma(E)) \geqslant 0$.

We denote $\operatorname{Stab}(\mathbb{P}^2)$ the set of stability conditions on $D^b \operatorname{Coh}(\mathbb{P}^2)$. According to Bridgeland, $\operatorname{Stab}(\mathbb{P}^2)$ has a natural structure of complex manifold of dimension 3, such that the map

$$\operatorname{Stab}(\mathbb{P}^2) \to \operatorname{Hom}(\Gamma, \mathbb{C}) \simeq \mathbb{C}^3$$

$$\sigma = (Z, \mathcal{A}) \mapsto Z$$

is a local isomorphism of complex manifolds (locally on $\operatorname{Stab}(\mathbb{P}^2)$).

# Bridgeland stability conditions

- For every $(s,t) \in \mathbb{R}^2$ with $t > 0$, let $Z^{(s,t)}\colon \Gamma \to \mathbb{C}$ be the linear map defined

$$\gamma = (r,d,\chi) \mapsto Z_\gamma^{(s,t)},$$

$$Z_\gamma^{(s,t)} := -\frac{1}{2}(s^2 - t^2)r + ds + r + \frac{3}{2}d - \chi + i(d - sr)t .$$

- If $E$ is an object of $D^b \operatorname{Coh}(\mathbb{P}^2)$ of class $\gamma(E) \in \Gamma$, then we can write

$$Z_{\gamma(E)}^{(s,t)} = -\int_{\mathbb{P}^2} e^{-(s+it)H} \operatorname{ch}(E),$$

where $H := c_1(\mathcal{O}(1))$.

- For every $(s,t) \in \mathbb{R}^2$ with $t > 0$, the pair $(Z^{(s,t)}, \operatorname{Coh}^{\#s}(\mathbb{P}^2))$ is a stability condition on $D^b \operatorname{Coh}(\mathbb{P}^2)$ (Bridgeland, Bayer-Macri, Arcara-Bertram-Coskun-Huizenga). In particular, we get an embedding of the upper half-plane $\{(s,t) \in \mathbb{R}^2 | \, t > 0\}$ into $\operatorname{Stab}(\mathbb{P}^2)$.

# Bridgeland stability conditions

- For every $\sigma = (s, t)$ and $\gamma \in \mathbb{Z}^3$, get a moduli space $M_\gamma^\sigma$ of $\sigma$-semistable objects of class $\gamma$.
- For every given $\gamma \in \mathbb{Z}^3$, we have $M_\gamma^\sigma = M_\gamma$ for $t$ large enough.
- Gieseker stability is the limit of $\sigma = (s, t)$ Bridgeland stability conditions for $t$ large enough.
- This picture has been used to study the birational geometry of the moduli spaces $M_\gamma$ (Ohkawa, Aracara-Bertram-Coskun-Huizenga, Bertram-Martinez-Wang, Coskun-Huizenga, Coskun-Huizenga-Woolf, Li-Zhao).
- For the birational geometry: one crosses finitely many walls, corresponding to finitely birational modifications, and then the moduli space becomes empty.
- For the Betti numbers $Ib_j(M_\gamma)$: need to follow the other pieces of the moduli space. Problem: no $\sigma$ such that $\{M_\gamma^\sigma\}_\gamma$ is simple.

# Main idea

- Do not try to follow the full set $\{M_\gamma^\sigma\}_\gamma$.
- Couple $\gamma$ and $\sigma$ via the central charge $Z_\gamma^\sigma$: for a given $\gamma$, focus on the codimension 1 set of the stability conditions $\sigma$ such that $Z_\gamma^\sigma$ has a given phase $\theta$.
- Show that the resulting picture form a consistent scattering diagram in the sense of Kontsevich-Soibelman and Gross-Siebert.
- Key point: for $\theta = \pi/2$, it is possible to identify the initial data of the scattering diagrams, i.e. there exists $\sigma$ such that $\{M_\gamma^\sigma | \mathrm{Arg} Z_\gamma^\sigma = \frac{\pi}{2}\}_\gamma$ is simple enough.
- Previous explicit connection between stability conditions and scattering diagrams: Bridgeland (2016). Main differences: Bridgeland considers a fixed abelian category (e.g. abelian category of quiver representations) and the scattering diagram lives in a "quotient" of the space of stability conditions. For us: abelian hearts of stability conditions on $D^b \mathrm{Coh}(\mathbb{P}^2)$ are moving and the scattering diagram lives in a "slice" of the space of stability conditions.

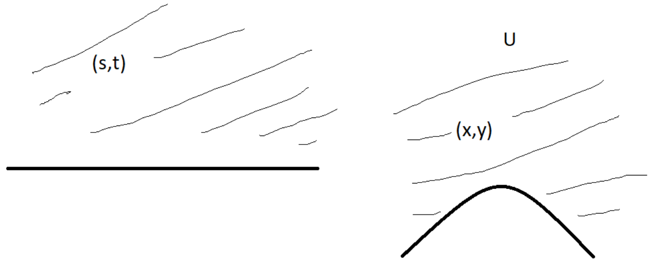
# Change of variables

- Draw the upper half-plane $\{(s,t) \in \mathbb{R}^2 | t > 0\}$ in different coordinates.
- Define $x = s, y = -\frac{1}{2}(s^2 - t^2)$.
- The map $(s,t) \mapsto (x,y)$ identifies the upper half-plane $\{(s,t) \in \mathbb{R}^2 | t > 0\}$ with the upper-parabola $U := \{(x,y) \in \mathbb{R}^2 | y > -\frac{x^2}{2}\}$.
- For $\sigma = (x,y) \in U$, the central charge becomes

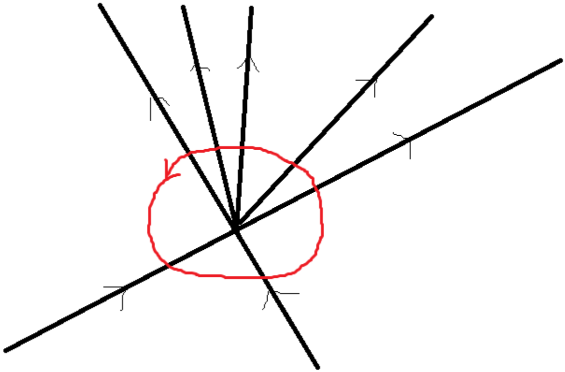
$$Z_\gamma^\sigma := ry + dx + r + \frac{3}{2}d - \chi + i(d - rx)\sqrt{x^2 + 2y}\,.$$

- Key reason for the $(x,y)$ coordinates: $\mathrm{Re}\, Z_\gamma^\sigma = 0$ is an affine equation in $x$ and $y$, defining a line in $U$.
- Main claim: the collections of half-lines $\{\sigma | \mathrm{Re}\, Z_\gamma^\sigma = 0, M_\gamma^\sigma \neq \varnothing\}$ locally decorated by the Betti numbers $Ib_j(M_\gamma^\sigma)$ defines a consistent scattering diagram.

# Local scattering diagram



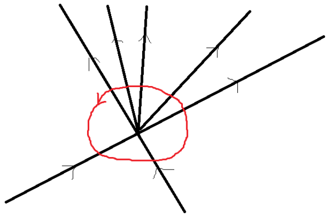
# Local scattering diagram



# Local scattering diagrams

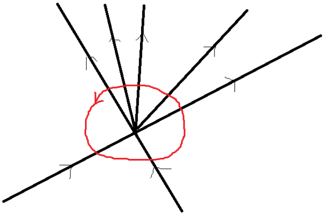
- $(M, \mathfrak{g})$, $M := \mathbb{Z}^2$, $\mathfrak{g} = \bigoplus_{m \in M} \mathfrak{g}_m$ a $M$-graded Lie algebra over $\mathbb{Q}$ (that is, with $[\mathfrak{g}_m, \mathfrak{g}_{m'}] \subset \mathfrak{g}_{m+m'}$) such that $[\mathfrak{g}_m, \mathfrak{g}_{m'}] = 0$ if $m$ and $m'$ are collinear.
- For every nonzero $m \in M$, a *local ray $\rho$ of class $m$* for $(M, \mathfrak{g})$ is a pair $(|\rho|, H_\rho)$, where:
  - $|\rho|$ is a subset of $M_{\mathbb{R}} := \mathbb{R}^2$ of the form either $\mathbb{R}_{\geqslant 0} m$ or $-\mathbb{R}_{\geqslant 0} m$.
  - $H_\rho \in \mathfrak{g}_m$.

  The local ray $\rho = (|\rho|, H)$ of class $m$ is *outgoing* if $|\rho| = -\mathbb{R}_{\geqslant 0} m$, and *ingoing* if $|\rho| = \mathbb{R}_{\geqslant 0} m$. We denote $m_\rho \in M$ the class of a local ray $\rho$.



# Local scattering diagram

A *local scattering diagram* for $(M, \mathfrak{g})$ is a collection $\mathfrak{D}$ of local rays $\rho = (|\rho|, H_\rho)$, such that for every nonzero $m \in M$, there is at most one ingoing local ray of class $m$ in $\mathfrak{D}$, and at most one outgoing ray of class $m$ in $\mathfrak{D}$.

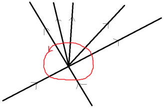


# Local scattering diagram: automorphisms

- Order $k$ automorphism attached to a ray $\rho = (|\rho|, H_\rho)$, the automorphism of $\mathfrak{g}$ given by: $\Phi_{\rho,k} := \exp([H_\rho, -])$.
- Let $\mathfrak{D}$ be a local scattering diagram for $(M, \mathfrak{g})$. We fix some smooth path $\alpha \colon [0,1] \to M_{\mathbb{R}} - \{0\}$, $t \mapsto \alpha(t)$, with transverse intersection with respect to all the rays $\rho = (|\rho|, H_\rho) \in \mathfrak{D}$. Let $\rho_1, \dots, \rho_N$ be the successive rays $\rho$ of $\mathfrak{D}$ intersected by the path $\alpha$ at times $t_1 \leqslant \cdots \leqslant t_N$. The *automorphism associated to* $\alpha$ is the automorphism of $\mathfrak{g}$ defined by

$$\Phi_\alpha^{\mathfrak{D}} := \Phi_{\rho_N}^{\epsilon_N} \circ \cdots \circ \Phi_{\rho_1}^{\epsilon_1} ,$$

  where, for every $j = 1, \dots, N$, $\epsilon_j := \mathrm{sign}(\det(\alpha'(t_j), m_{\rho_j})) \in \{\pm 1\}$.
- $\mathfrak{D}$ is *consistent* if $\Phi_\alpha^{\mathfrak{D}} = \mathrm{id}$ for every loop $\alpha$ (i.e. with $\alpha(0) = \alpha(1)$).

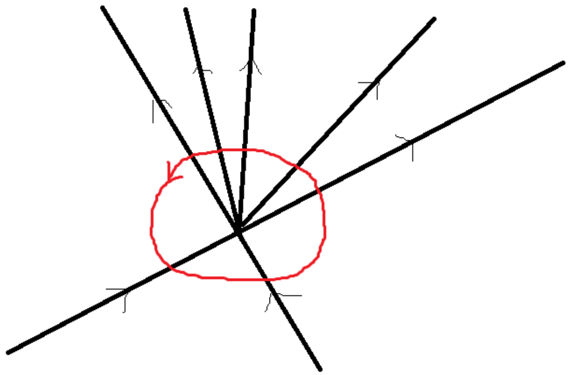


# Local scattering diagram: consistent completion

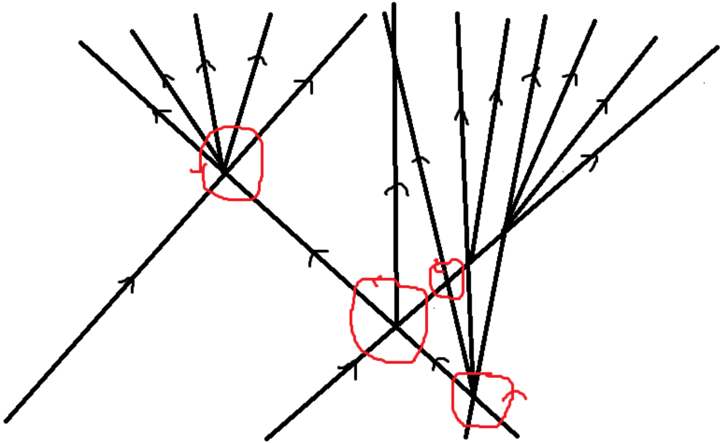
## Proposition

Let $\mathfrak{D}$ be a local scattering diagram for $(M, \mathfrak{g})$. Then, there exists a unique consistent local scattering diagram $S(\mathfrak{D})$ such that the set of incoming rays of $S(\mathfrak{D})$ and $\mathfrak{D}$ are identical.

# Local scattering diagram



# Scattering diagram



# Scattering diagram

- Still fix $(M, \mathfrak{g})$.
- $U$ open subset in $\mathbb{R}^2$, $\bar{U}$ its closure.
- For every $\sigma \in U$, think of $M_{\mathbb{R}}$ as the tangent space to $U$ at $\sigma$.

For every $m \in M$, a *ray* $\rho$ *of class* $m$ in $U$ for $(M, \mathfrak{g})$ is a pair $(|\rho|, H_\rho)$, where

- $|\rho|$ is a subset $|\rho|$ of $\bar{U}$ of the form $|\rho| = \mathrm{Init}(\rho) - \mathbb{R}_{\geqslant 0} m$ for some $\mathrm{Init}(\rho) \in \mathbb{R}^2$, or of the form $|\rho| = \mathrm{Init}(\rho) - [0, T_\rho] m$ for some $\mathrm{Init}(\rho) \in \mathbb{R}^2$ and some $T_\rho \in \mathbb{R}_{>0}$.,
- $H_\rho$ is a nonzero element of $\mathfrak{g}_m$.

We denote $m_\rho \in M$ the class of a ray $\rho$.

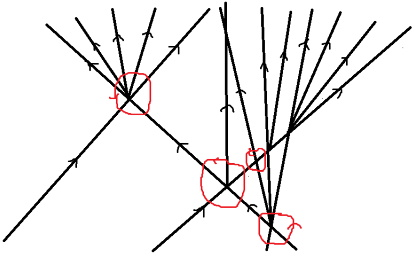
# Scattering diagram

A *scattering diagram* on $U$ for $(M, \mathfrak{g})$ is a collection $\mathfrak{D}$ of rays $\rho = (|\rho|, H_\rho)$ in $U$ for $(M, \mathfrak{g})$, such that:

- For every $\sigma \in U$ and for every nonzero $m \in M$, there is at most one ray $\rho$ in $\mathfrak{D}$ of class $m$ such that $\sigma$ belongs to the interior of $|\rho|$.
- There do no exist rays $\rho_1 = (|\rho_1|, H_{\rho_1})$ and $\rho_2 = (|\rho_2|, H_{\rho_2})$ in $\mathfrak{D}$ such that the endpoint of $|\rho_1|$ coincides with the initial point of $|\rho_2|$, and such that $H_{\rho_1} = H_{\rho_2}$.

# Scattering diagram

- Let $\mathfrak{D}$ be a scattering diagram, and let $\sigma \in U$. The local scattering diagram $\mathfrak{D}_\sigma$ for $(M, \mathfrak{g})$ is a local picture of $\mathfrak{D}$ around the point $\sigma$, $M_\mathbb{R} = M \otimes \mathbb{R}$ being identified with the tangent space to $U$ at $\sigma$.
- A scattering diagram $\mathfrak{D}$ on $U$ for $(M, \mathfrak{g})$ is *consistent* if, for every $\sigma \in U$, the local scattering diagram $\mathfrak{D}_\sigma$ for $(M, \mathfrak{g})$ is consistent.



# Scattering diagram from stability conditions

- Construct a scattering diagram on $U = \{(x,y) \in \mathbb{R}^2 | y > -\frac{x^2}{2}\} \subset \mathrm{Stab}(\mathbb{P}^2)$.
- $M = \mathbb{Z}^2 = \{(r, -d)\}$.
- $\langle -,- \rangle \colon \wedge^2 M \to \mathbb{Z}$, $\langle (a,b),(a',b') \rangle = 3(a'b - ab')$ (skew-symmetrized Euler form of $\mathbb{P}^2$, or Euler form of $K_{\mathbb{P}^2}$).
- $\mathfrak{g}$: the $\mathbb{Q}(q^{\pm\frac{1}{2}})$-Lie algebra

$$\mathfrak{g} := \bigoplus_{m \in M} \mathbb{Q}(q^{\pm\frac{1}{2}}) z^m$$

with Lie bracket given by

$$[z^m, z^{m'}] := (-1)^{\langle m,m' \rangle} (q^{\frac{\langle m,m' \rangle}{2}} - q^{-\frac{\langle m,m' \rangle}{2}}) z^{m+m'} .$$

# Scattering diagram from stability conditions

- Poincaré polynomial:

$$Ib_\gamma^\sigma(q^{\frac{1}{2}}) := (-q^{\frac{1}{2}})^{-\dim M_\gamma^\sigma} \sum_{j=0}^{2\dim M_\gamma^\sigma} (-1)^j Ib_j(M_\gamma^\sigma) q^{\frac{j}{2}} .$$

- Consider the set $\mathfrak{D}$ of rays

$$R_\gamma := \{\sigma \in U \,|\, Z_\gamma^\sigma \in i\mathbb{R}_{>0} \,, Ib_\gamma^\sigma(q^{\frac{1}{2}}) \neq 0\} \,,$$

of direction $m_\gamma = (r, -d) \in M$, with elements

$$H_{\rho_{\gamma,\sigma}} := \left( - \sum_{\substack{\gamma' \in \Gamma_\gamma \\ \gamma = \ell\gamma'}} \frac{1}{\ell} \frac{Ib_{\gamma'}^\sigma(q^{\frac{\ell}{2}})}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right) z^{m_\gamma} \in \mathfrak{g}_{m_\gamma} \,,$$

for every $\sigma \in R_\gamma$.

# Scattering diagram from stability conditions

**Lemma (B)**

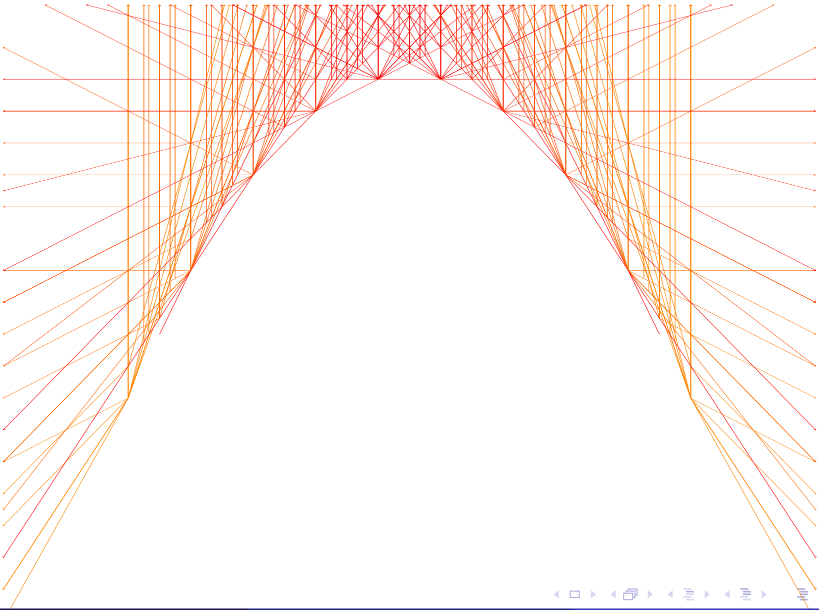
$\mathfrak{D}$ is a scattering diagram for $(M, \mathfrak{g})$.

**Theorem (B)**

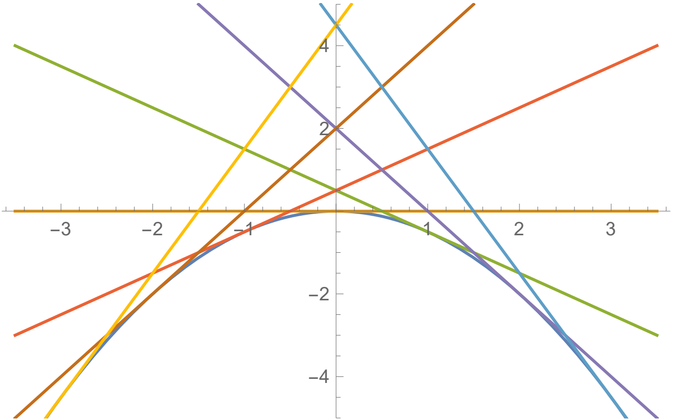
The scattering diagram $\mathfrak{D}$ is consistent.

- Expression of the Kontsevich-Soibelman formula for Donaldson-Thomas invariants of local $\mathbb{P}^2$.
- A key technical point (Li-Zhao): if $E$ is $\sigma$-semistable and $\gamma(E) \neq (0,0,*)$, then $\mathrm{Ext}^2(E,E) = 0$.
- Then, use results of Meinhardt relating intersection cohomology and Donaldson-Thomas theory, and make the wall-crossing argument in the motivic Hall algebra.

# The scattering diagram $\mathfrak{D}$ (Figure due to Tim Gräfnitz)



$\mathfrak{D}_{in}$: the scattering diagram consisting uniquely of rays defined by the line bundles $\mathcal{O}(n)$ and their shifts $\mathcal{O}(n)[1]$.



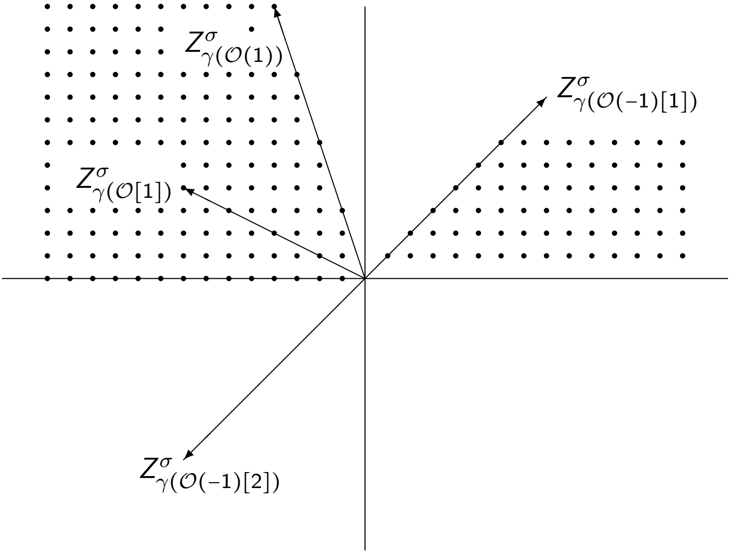
# Main result

## Theorem (B)

We have $\mathfrak{D} = S(\mathfrak{D}_{in})$: $\mathfrak{D}$ is the consistent completion of $\mathfrak{D}_{in}$. In particular, $\mathfrak{D}$ can be algorithmically reconstructed from $\mathfrak{D}_{in}$.

- Explicit version, at the level of cohomology of moduli spaces of semistable objects, of the classical fact that the derived category $D^b \mathrm{Coh}(\mathbb{P}^2)$ is generated by the line bundles $\mathcal{O}(n)$ (Beilinson).
- Key point: need to show that $\mathfrak{D}$ coincides with $\mathfrak{D}_{in}$ near the parabola.
- Use a quiver description of the stability conditions near the parabola.

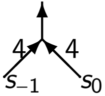
# Using the quiver description



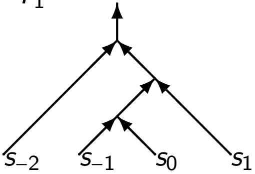
# An example: $\gamma = (0, 4, 4)$

$M_{(0,4,4)}$: a 17-dimensional projective variety, singular compactification of the family of 3-dimensional Jacobians over the part of the 14-dimensional linear system of quartic curves in $\mathbb{P}^2$ parametrizing smooth curves.

$T_0 =$




$T_1 =$




$$P(M_{(0,4,4)}) = [12]_q(1 + q + 4q^2 + 4q^3 + 4q^4 + q^5 + q^6)\,.$$

# Application

Working with Hodge numbers rather than with Betti numbers:

**Theorem (B)**

For every $\gamma$, the natural pure Hodge structure on $IH^{\bullet}(M_\gamma)$ is Hodge-Tate, i.e. with $h^{p,q} = 0$ for $p \neq q$.

For $M_\gamma$ smooth, this was known (Ellingsrud-Strømme, Beauville) using that Künneth components of the Chern classes of the universal sheaf generate the cohomology. It is new for $M_\gamma$ singular.

Thank you for your attention !