EXTREMAL HIGHER CODIMENSION CYCLES ON MODULI SPACES OF CURVES

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ABSTRACT. We show that certain geometrically defined higher codimension cycles are extremal in the effective cone of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g curves with n ordered marked points. In particular, we prove that codimension two boundary strata are extremal and exhibit extremal boundary strata of higher codimension. We also show that the locus of hyperelliptic curves with a marked Weierstrass point in $\overline{\mathcal{M}}_{3,1}$ and the locus of hyperelliptic curves in $\overline{\mathcal{M}}_4$ are extremal cycles. In addition, we exhibit infinitely many extremal codimension two cycles in $\overline{\mathcal{M}}_{1,n}$ for $n \geq 5$ and in $\overline{\mathcal{M}}_{2,n}$ for $n \geq 2$.

CONTENTS

1. INTRODUCTION

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford-Knudsen moduli space of stable genus g curves with n ordered marked points. In this paper, we study the effective cones of higher codimension cycles on $\overline{\mathcal{M}}_{g,n}$. We work over the field of complex numbers $\mathbb{C}$.

Motivated by the problem of determining the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$, the cone of effective divisors of $\overline{\mathcal{M}}_{g,n}$ has been studied extensively, see e.g. [HMu, Harr, EH, Far, Lo, V, CT, CC2]. In contrast, little is known about higher codimension cycles on $\overline{\mathcal{M}}_{g,n}$, in part because their positivity properties are not as well-behaved. For instance, higher codimension nef cycles may fail to be pseudoeffective [DELV]. Furthermore, unlike the case of divisors, we lack simple numerical, cohomological and geometric conditions for determining whether

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higher codimension cycles are nef or pseudoeffective. Nevertheless, there has been growing interest in understanding the structure of effective cones of higher codimension cycles, see e.g. [FL1] for recent progress and the current state of the art in this field.

In this paper, we study the effective cone of higher codimension cycles in $\overline{\mathcal{M}}_{g,n}$. In particular, we show that certain geometrically defined higher codimension cycles span extremal rays of the effective cone of $\overline{\mathcal{M}}_{g,n}$.

Main Results.

(i) Every codimension two boundary stratum of $\overline{\mathcal{M}}_g$ and of $\overline{\mathcal{M}}_{0,n}$ is extremal (Theorems 4.3, 5.3 and 6.1).
(ii) The higher codimension boundary strata associated to certain dual graphs described in §5 are extremal in $\overline{\mathcal{M}}_g$ (Theorem 5.6).
(iii) Every codimension k boundary stratum of $\overline{\mathcal{M}}_{0,n}$ parameterizing curves with k marked tails attached to an unmarked $\mathbb{P}^1$ is extremal (Theorem 6.2).
(iv) There exist infinitely many extremal effective codimension two cycles in $\overline{\mathcal{M}}_{1,n}$ for every $n \geq 5$ (Theorem 7.2) and in $\overline{\mathcal{M}}_{2,n}$ for every $n \geq 2$ (Theorem 8.2).
(v) The locus of hyperelliptic curves with a marked Weierstrass point is a non-boundary extremal codimension two cycle in $\overline{\mathcal{M}}_{3,1}$ (Theorem 4.6).
(vi) The locus of hyperelliptic curves is a non-boundary extremal codimension two cycle in $\overline{\mathcal{M}}_4$ (Theorem 5.9).

These results illustrate that the effective cone of higher codimension cycles on $\overline{\mathcal{M}}_{g,n}$ can be very complicated even for small values of g and n.

In order to verify the extremality of a codimension k cycle, we use two criteria. First, we find a criterion that shows the extremality of loci that drop the largest possible dimension under a morphism (Proposition 2.2). We then apply the criterion to morphisms from $\overline{\mathcal{M}}_{g,n}$ to different modular compactifications of $\mathcal{M}_{g,n}$. Second, we use induction on dimension. To prove that a cycle Z is extremal, we first show that Z is extremal in a divisor $D \subset \overline{\mathcal{M}}_{g,n}$ containing Z, and then show that effective cycles representing Z must be contained in D (Proposition 2.5). In certain cases we can strengthen the result by showing the extremality of Z in the pseudoeffective cone (Remark 2.7). We point out at various places when it applies.

Most of the extremal cycles we consider are contained in the boundary of $\overline{\mathcal{M}}_{g,n}$. However, in Theorem 4.6 and Theorem 5.9, we show the extremality of some non-boundary cycles. The proofs of these theorems are more delicate, relying on a detailed analysis of canonical curves in genus three and four.

The paper is organized as follows. In Section 2 and Section 3 we review the basic properties of effective cycles and moduli spaces of curves, respectively. Then we carry out the study of the effective cones of $\overline{\mathcal{M}}_{g,n}$ according to the values of g and n: $g = 3$ and $n \leq 1$ (Section 4); $g \geq 4$ and $n = 0$ (Section 5); $g = 0$ and arbitrary n (Section 6); $g = 1$ and $n \geq 5$ (Section 7); $g = 2$ and $n \geq 2$ (Section 8). Sections 4–8 are basically independent, so the reader may read them in any order.

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2. Preliminaries on effective cycles

In this section, we review basic properties of the effective cone of cycles on an algebraic variety and develop some criteria for proving the extremality of an effective cycle. Throughout the paper, all varieties are defined over $\mathbb{C}$ and all linear combinations of cycles are with $\mathbb{R}$-coefficients.

Let X be a complete variety. A *cycle* on X is a formal sum of closed subvarieties of X. A cycle is *k-dimensional* if all subvarieties in the sum are k-dimensional. A cycle is *effective* if all coefficients in the sum are nonnegative. Two k-dimensional cycles A and B on X are *numerically equivalent*, if $A \cap P = B \cap P$ under the degree map for all weight k polynomials P in Chern classes of vector bundles on X, where $\cap$ is the cap product, see [Fu, Chapter 19]. When X is nonsingular, this is equivalent to requiring $A \cdot C = B \cdot C$ for all subvarieties C of codimension k, where $\cdot$ is the intersection product. The focus of the paper is the moduli space of curves, which is $\mathbb{Q}$-factorial but may have finite quotient singularities. Nevertheless, the intersection product is still compatible with the cup product, see [E, Section 1].

Denote by $[Z]$ the *numerical class* of a cycle Z. Let $N_k(X)$ (resp. $N^k(X)$) denote the $\mathbb{R}$-vector space of cycles of dimension k (resp. codimension k) modulo numerical equivalence. It is a finite dimensional vector space. Let $\mathrm{Eff}_k(X) \subset N_k(X)$ (resp. $\mathrm{Eff}^k(X) \subset N^k(X)$) denote the *effective cone* of dimension k (resp. codimension k) cycles generated by all effective cycle classes. Their closures $\overline{\mathrm{Eff}}_k(X)$ and $\overline{\mathrm{Eff}}^k(X)$ are called the *pseudoeffective cones*.

A closed convex cone is determined by its extremal rays; a general convex cone can be better understood by specifying its extremal rays. Recall that a ray R is called *extremal*, if for every $D \in R$ and $D = D_1 + D_2$ with D_1, D_2 in the cone, we have $D_1, D_2 \in R$. If an extremal ray is spanned by the class of an effective cycle D, we say that D is an *extremal effective cycle*.

There is a well-developed theory to study the cone $\mathrm{Eff}^1(X)$, including numerical, cohomological, analytic and geometric conditions for checking whether a divisor is in $\overline{\mathrm{Eff}}^1(X)$, see [La]. In contrast, $\mathrm{Eff}^k(X)$ for $k \geq 2$ is not well-understood. We first describe two simple criteria for checking extremality of cycles in $\mathrm{Eff}^k(X)$.

Let $f : X \to Y$ be a morphism between two complete varieties. To a subvariety $Z \subset X$ of dimension k we associate an index

$$e_f(Z) = \dim Z - \dim f(Z).$$

Note that $e_f(Z) > 0$ if and only if Z drops dimension under f.

Proposition 2.1. *Let $f : X \to Y$ be a morphism between two projective varieties and let $k > m \geq 0$ be two integers. Let Z be a k-dimensional subvariety of X such that $e_f(Z) \geq k-m$. If $[Z] = a_1[Z_1] + \cdots + a_r[Z_r] \in N_k(X)$, where Z_i is a k-dimensional subvariety of X and $a_i > 0$ for all i, then $e_f(Z_i) \geq k - m$ for every $1 \leq i \leq r$.*

Proof. Let A and B be two very ample divisor classes on X and on Y, respectively. Then $N = f^*B$ is base-point-free on X. In particular, if $U \subset X$ is an effective cycle of dimension k, then the intersection $N^j \cdot [U]$ is either zero or can be represented by an effective cycle of dimension $k - j$ for $1 \leq j \leq k$. In the latter case, by the projection formula and the very ampleness of A, we have $0 < A^{k-j} \cdot N^j \cdot [U] = B^j \cdot [f_*(W)]$, where W is an effective cycle representing $A^{k-j} \cdot [U]$. Since $e_f(Z) \geq k - m$, we conclude that $N^{m+1} \cdot [Z] = 0$. Therefore, $N^{m+1} \cdot [Z_i] = 0$ for $1 \leq i \leq r$. Otherwise, $N^{m+1} \cdot [Z_i]$ can be represented by an effective cycle.

Intersecting both sides of the equality $N^{m+1} \cdot [Z] = N^{m+1} \cdot (\sum_{i=1}^{r} a_i[Z_i])$ with A^{k-m-1} we would obtain a contradiction. By the projection formula, we conclude that $B^{m+1} \cdot (f_*[Z_i]) = 0$, hence $\dim f(Z_i) \le m$ as desired. □

As a corollary of Proposition 2.1, we obtain the following extremality criterion.

Proposition 2.2. *Let $f : X \to Y$ be a morphism between two projective varieties. Fix two integers $k > m \ge 0$. Among all k-dimensional subvarieties Z of X, assume that only finitely many of them, denoted by $Z_1, \ldots, Z_n$, satisfy $e_f(Z) \ge k - m$. If the classes of $Z_1, \ldots, Z_n$ are linearly independent, then each Z_i is an extremal effective cycle in $\mathrm{Eff}_k(X)$.*

Proof. Suppose that $[Z_i] = a_1[D_1] + \cdots + a_r[D_r]$ with D_j a k-dimensional subvariety of X and $a_j > 0$ for all j. Then, by Proposition 2.1, $e_f(D_j) \ge k - m$. Since $Z_1, \ldots, Z_n$ are the only k-dimensional subvarieties of X with index $e_f \ge k - m$, we conclude that D_j has to be one of $Z_1, \ldots, Z_n$. Since their classes are independent, we conclude that $[D_j]$ is proportional to $[Z_i]$ for all j. Therefore, Z_i is extremal in $\mathrm{Eff}_k(X)$. □

We will apply Propositions 2.1 and 2.2 to morphisms from $\overline{\mathcal{M}}_{g,n}$ to alternate modular compactifications of $\mathcal{M}_{g,n}$. We also remark that the classes of exceptional loci in higher codimension may fail to be linearly independent.

Remark 2.3. The following example, which is related to Hironaka's example of a complete but non-projective variety (see [Hart, Appendix B, Example 3.4.1]), was pointed out to the authors by John Lesieutre. Let X be a smooth threefold, and let $C, D \subset X$ be two smooth curves meeting transversally at two points p and q. Let Y be the blowup of X along C. The fibers F_r of the exceptional locus over C have the same numerical class for all $r \in C$. Next, let Z be the blowup of Y along the proper transform of D. The proper transforms E_p of F_p and E_q of F_q have classes different from (the transform of) F_r for $r \ne p, q$. However, E_p and E_q have the same class, and their normal bundles are isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Hence, they are flopping curves if we blow up D first and then blow up the proper transform of C. In particular, there exists a small contraction $Z \to W$ such that the exceptional locus consists of E_p and E_q only.

The next corollary will be useful when studying the effective cones of the boundary strata.

Corollary 2.4. *Let X and Y be projective varieties such that numerical equivalence and rational equivalence are the same for codimension k cycles in X, Y and $X \times Y$, respectively, with $\mathbb{R}$-coefficients. Suppose Z is an extremal effective cycle of codimension k in X. Then $Z \times Y$ is an extremal effective cycle of codimension k in $X \times Y$.*

Proof. Suppose $[Z \times Y] = \sum a_i[U_i]$, where $a_i > 0$ and U_i are irreducible subvarieties of $X \times Y$. Let π denote the projection of $X \times Y$ onto X. Then, by Proposition 2.1, $e_\pi(U_i) = \dim(Y)$. Hence, $U_i = V_i \times Y$ for a subvariety $V_i \subset X$ for each i. By the projection formula, $[Z] = \sum a_i[V_i]$. Since Z is extremal in X, the classes of Z and V_i are proportional. Hence, their pullbacks to $X \times Y$ are proportional under rational equivalence, and also proportional under numeral equivalence by our assumption. We thus conclude that $[U_i]$ is proportional to $[Z \times Y]$ for every i and $Z \times Y$ is extremal in $X \times Y$. □

Another useful criterion is the following. Let $A_k(X)$ (resp. $A^k(X)$) denote the *Chow group* of rationally equivalent cycle classes of dimension k (resp. codimension k) in X with $\mathbb{R}$-coefficients. If two cycles are rationally equivalent, they are also numerically equivalent, hence we have a map $A_k(X) \to N_k(X)$. For a morphism $Y \to X$, there is a pushforward map $A_k(Y) \to A_k(X)$.

Proposition 2.5. *Let $\gamma : Y \to X$ be a morphism between two projective varieties. Assume that $A_k(Y) \to N_k(Y)$ is an isomorphism and that the composite $\gamma_* : A_k(Y) \to A_k(X) \to N_k(X)$ is injective. Moreover, assume that $f : X \to W$ is a morphism to a projective variety W whose exceptional locus is contained in $\gamma(Y)$. If a k-dimensional subvariety $Z \subset Y$ is an extremal cycle in $\mathrm{Eff}_k(Y)$ and if $e_f(\gamma(Z)) > 0$, then $\gamma(Z)$ is also extremal in $\mathrm{Eff}_k(X)$.*

Proof. Suppose that $[\gamma(Z)] = a_1[Z_1] + \cdots + a_r[Z_r] \in N_k(X)$ for subvarieties $Z_i \subset X$ with $a_i > 0$ for all i. Since $e_f(\gamma(Z)) > 0$, by Proposition 2.1, Z_i drops dimension under f. Hence, Z_i is contained in $\gamma(Y)$ for all i. Let $Z'_i \subset Y$ such that $\gamma_*[Z'_i] = [Z_i]$. Then $\gamma_*([Z] - a_1[Z'_1] - \cdots - a_r[Z'_r]) = 0$. Since γ_* is injective, it follows that $[Z] = a_1[Z'_1] + \cdots + a_r[Z'_r]$ in $A_k(Y) \cong N_k(Y)$. Since Z is extremal in $\mathrm{Eff}_k(Y)$, we conclude that $[Z'_i]$ is proportional to $[Z]$ in $N_k(Y)$. Therefore, $[Z_i] = \gamma_*[Z'_i]$ is proportional to $\gamma_*[Z]$ in $N_k(X)$ as well. □

We will apply Proposition 2.5 to the case when X is the moduli space of curves and Y is an irreducible boundary divisor. Occasionally, we will be able to prove that effective cycles expressing a cycle class $[Z]$ are contained in a union of divisors. We will then use the following technical result to deduce the extremality of $[Z]$.

Proposition 2.6. *Let $Y = \bigcup_{i=0}^n D_i$ be a union of irreducible divisors $D_i \subset X$ such that $D_i \cap D_j$ consists of $m_{i,j}$ irreducible codimension two subvarieties $D_{i,j,k}$ for $0 \leq i < j \leq n$ and $k = 1, \ldots, m_{i,j}$. Assume for all i, j, k that $D_{i,j,k}$ is extremal in D_i and that a linear combination*

$$\sum_{\substack{j=0\\ j\neq i}}^{n} \sum_{k=1}^{m_{i,j}} a_{i,j,k}[D_{i,j,k}]$$

is effective in D_i if and only if $a_{i,j,k} \geq 0$. Further assume that $A^1(D_i) \to N^1(D_i)$ is an isomorphism and $A^1(Y) \to A^2(X) \to N^2(X)$ are injective. Let $Z \subset D_0$ be an effective divisor. Finally, assume that for every effective expression

$$[Z] = \sum_i a_i[Z_i] \in \mathrm{Eff}^2(X),$$

where $a_i > 0$ and $Z_i \subset X$ is a codimension two subvariety, Z_i is contained in Y for all i. If Z is extremal in $\mathrm{Eff}^1(D_0)$, then Z is also extremal in $\mathrm{Eff}^2(X)$.

Proof. Suppose that

$$[Z] = \sum_i a_i[Z_i] \in \mathrm{Eff}^2(X) \tag{1}$$

for $a_i > 0$ and $Z_i \subset X$ irreducible codimension two subvariety. We want to show that $[Z_i]$ is proportional to $[Z]$ in $N^2(X)$. By assumption, Z_i is contained in Y for all i. Reexpress the summation in (1) as

$$S_0 + \cdots + S_n, \quad \text{where} \quad S_j = \sum_{\substack{Z_i \subset D_j,\\ Z_i \not\subset D_k \text{ for } k<j}} a_i[Z_i].$$

By assumption, the second map in

$$\bigoplus_{i=1}^{n} N^1(D_i) \to N^1(Y) \to N^2(X)$$

is injective. The kernel of the first map is generated by elements of type

$$(0, \cdots, [D_{i,j,k}], \cdots, -[D_{i,j,k}], \cdots, 0)$$

for $0 \le i < j \le n$ and $1 \le k \le m_{i,j}$, where the nonzero entries occur in the ith and jth places. Lifting (1) to the direct sum, there exist $a_{i,j,k} \in \mathbb{R}$ such that

$$[Z] = S_0 + \sum_{j=1}^{n} \sum_{k=1}^{m_{0,j}} a_{0,j,k}[D_{0,j,k}] \in N^1(D_0) \tag{2}$$

and

$$0 = S_i + \sum_{j=0}^{i-1} \sum_{k=1}^{m_{j,i}} (-a_{j,i,k})[D_{j,i,k}] + \sum_{j=i+1}^{n} \sum_{k=1}^{m_{i,j}} a_{i,j,k}[D_{i,j,k}] \in N^1(D_i) \tag{3}$$

for $1 \le i \le n$.

In the expressions, each $a_{i,j,k}$ appears twice with opposite signs. Since S_i is effective, our assumption that the coefficients of $D_{i,j,k}$ in an effective sum have to be positive along with (3) implies that $a_{j,i,k} \ge 0$ for $0 \le j < i \le n$ and $a_{i,j,k} \le 0$ for $1 \le i < j \le n$. Therefore, we conclude that $a_{i,j,k} = 0$ for $1 \le i < j \le n$ and all k. Now (3) reduces to

$$S_i = \sum_{k=1}^{m_{0,j}} a_{0,j,k}[D_{0,i,k}] \in N^1(D_i) \tag{4}$$

for $1 \le i \le n$. First, assume that $[Z]$ is not proportional to any $[D_{0,j,k}]$ in $N^1(D_0)$. By assumption that Z is extremal in D_0 and $a_{0,i,k} \ge 0$ for $i > 0$, (2) and (4) imply that $a_{0,j,k} = 0$ for all j, k, $S_i = 0$ and the classes $[Z_i]$ in S_0 are proportional to $[Z]$. The classes $[D_{0,j,k}]$ are independent in D_0, otherwise it would contradict the assumption that $\sum_{j=1}^{n} \sum_{k=1}^{m_{0,j}} a_{0,j,k}[D_{0,j,k}]$ is effective in D_0 if and only if $a_{0,j,k} \ge 0$. Therefore, we conclude that the composite $N^1(D_0) \to N^1(Y) \to N^2(X)$ is injective, and hence Z is also extremal in $\mathrm{Eff}^2(X)$. If $[Z]$ is proportional to some $[D_{0,j,k}]$ in $N^1(D_0)$, then by (2) and (4), $a_{0,j',k'} = 0$ for all $(j', k') \ne (j, k)$ and classes in the summation S_i are proportional to $[Z]$ in $N^1(D_i)$ for all $i \ge 0$. We still conclude that Z is extremal in $\mathrm{Eff}^2(X)$. □

Remark 2.7. Under the assumptions of Proposition 2.2, one can further show that $[Z_i]$ is extremal in the *pseudoeffective* cone $\overline{\mathrm{Eff}}_k(X)$. The argument follows from very recent progress in the study of kernels of numerical pushforwards, see [FL2, Section 7]. We briefly explain the idea. For a morphism $f : X \to Y$ of projective varieties, recall that the index $e_f(Z)$ associated to a k-dimensional subvariety $Z \subset X$ measures the dimension decrease of Z under f. Equivalently, let H be an ample divisor class on Y. Then $e_f(Z) = 1 + c_f(Z)$, where $c_f(Z)$ is the largest integer $c \le k$ such that $[Z] \cdot f^*H^{k-c} = 0$. In [FL2], $c_f(Z)$ is called the *contractibility index* of Z, and can be defined in the same way for any *pseudoeffective* class in $\overline{\mathrm{Eff}}_k(X)$.

Returning to Proposition 2.2, there is no subvariety W of X such that $\dim W > k$ and $e_f(W) \ge k - m$. Otherwise there would exist infinitely many k-dimensional subvarieties Z_i satisfying $e_f(Z_i) \ge k - m$, contradicting the assumptions. Therefore by [FL2, Theorem 7.18], for any nonzero pseudoeffective class $\alpha \in \overline{\mathrm{Eff}}_d(X)$ such that $c_f(\alpha) \geq k-m-1$, we have $d \leq k$. Now suppose that $[Z_i] = \sum_j a_j \alpha_j$ such that $a_j > 0$, $\alpha_j \in \overline{\mathrm{Eff}}_k(X)$ and not proportional to $[Z_i]$. It is easy to see that $c_f(\alpha_j) \geq c_f(Z_i) = e_f(Z_i) - 1 = k-m-1$. By [FL2, Theorem 7.18] again, α_j is a nonnegative linear combination of $[Z_1], \ldots, [Z_n]$, hence $[Z_i]$ is a nonnegative linear combination of the other $[Z_l]$'s, contradicting the assumption of linear independence of their classes. So we have shown that $[Z_i]$ is extremal in $\overline{\mathrm{Eff}}_k(X)$.

In general, without the presence of a contraction morphism (e.g. Propositions 2.5 and 2.6), we do not know how to extend the extremality of an effective higher codimension cycle to the pseudoeffective cone. This is related to the following subtle and technical question. Let $f : X \to Y$ be a morphism and $\alpha \in \overline{\mathrm{Eff}}_k(X)$ such that $f_*\alpha = 0$. Is α in the closure of the cone generated by k-dimensional subvarieties that are contracted by f? The homological version of the question was first raised by [DJV], and was answered affirmatively for curves and divisors. The numerical analogue of the question was studied extensively in [FL2], and a number of new cases were established. We refer to these papers for further details. Another related question is the following. Does there exist a projective variety X and a k-dimensional subvariety Z such that $[Z]$ is extremal in $\mathrm{Eff}_k(X)$ but fails to be extremal in $\overline{\mathrm{Eff}}_k(X)$? We do not know any examples.

3. Preliminaries on moduli spaces of curves

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable genus g curves with n *ordered* marked points. The boundary Δ of $\overline{\mathcal{M}}_{g,n}$ consists of irreducible boundary divisors Δ_0 and $\Delta_{i;S}$ for $0 \leq i \leq [g/2]$, $S \subset \{1, \ldots, n\}$ such that $|S| \geq 2$ if $i = 0$. A general point of Δ_0 parameterizes an irreducible nodal curve of geometric genus $g-1$. A general point of $\Delta_{i;S}$ parameterizes a genus i curve containing the marked points labeled by S, attached at one point to a genus $g-i$ curve containing the marked points labeled by the complement S^c. We use λ to denote the first Chern class of the Hodge bundle. Let ψ_i be the first Chern class of the cotangent line bundle associated to the ith marked point and let $\psi = \sum_{i=1}^n \psi_i$.

We may also consider curves with unordered marked points. Let $\widetilde{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}/\mathfrak{S}_n$ be the moduli space of stable genus g curves with n *unordered* marked points. The boundary of $\widetilde{\mathcal{M}}_{g,n}$ consists of irreducible boundary divisors $\widetilde{\Delta}_0$ and $\widetilde{\Delta}_{i;k}$ for $0 \leq i \leq [g/2]$ and $0 \leq k \leq n$ such that $k \geq 2$ if $i = 0$.

Moduli spaces of curves of lower genera can be glued together to form the boundary of $\overline{\mathcal{M}}_{g,n}$. Set $\widehat{\Delta}_0 = \overline{\mathcal{M}}_{g-1,n+2}/\mathfrak{S}_2$, where $\mathfrak{S}_2$ interchanges the last two marked points. Identifying the last two marked points to form a node induces a gluing morphism

$$\alpha_0 : \widehat{\Delta}_0 \to \Delta_0.$$

For $0 < i < g/2$ or $i = g/2$ if g is even and $n > 0$, denote by $\widehat{\Delta}_{i;S} = \overline{\mathcal{M}}_{i,|S|+1} \times \overline{\mathcal{M}}_{g-i,n-|S|+1}$. Identifying the last marked points in the two factors induces

$$\alpha_{i;S} : \widehat{\Delta}_{i;S} \to \Delta_{i;S}.$$

For g even, $i = g/2$ and $n = 0$, set $\widehat{\Delta}_{g/2} = (\overline{\mathcal{M}}_{g/2,1} \times \overline{\mathcal{M}}_{g/2,1})/\mathfrak{S}_2$ and

$$\alpha_{g/2} : \widehat{\Delta}_{g/2} \to \Delta_{g/2}$$

is induced by identifying the two marked points to form a node. Restricted to the complement of the locus of curves with more than one node, the gluing morphisms are isomorphisms. Later on when studying codimension two boundary strata of $\overline{\mathcal{M}}_{g,n}$, this will help us identify them with boundary divisors on moduli spaces of curves of lower genera.

In order to study extremal higher codimension cycles on $\overline{\mathcal{M}}_{g,n}$ and $\widetilde{\mathcal{M}}_{g,n}$, we need to use the fact that the codimension one boundary strata are extremal. This is well-known (see e.g. [R1, 1.4]), and we explain it briefly as follows.

Proposition 3.1. *Every irreducible boundary divisor is extremal on $\overline{\mathcal{M}}_{g,n}$ and $\widetilde{\mathcal{M}}_{g,n}$.*

Proof. In general, one can exhibit a moving curve in a boundary divisor that has negative intersection with the divisor, which implies that it is extremal by [CC2, Lemma 4.1]. For Δ_0, fix a curve C of genus $g-1$ with $n+1$ distinct fixed points $p_1, \ldots, p_{n+1}$. One obtains the desired moving curve by gluing a varying point on C to p_{n+1}. For $\Delta_{i;S}$, fix a curve C of genus i with distinct marked points labeled by S and a curve C' of genus $g-i$ with distinct marked points labeled by S^c. One obtains the desired moving curve by gluing a fixed point on C' distinct from the previously chosen points to a varying point on C. The only exceptional case is Δ_0 in $g=1$. Its class equals 12λ, which is semi-ample. However, λ induces a fibration over $\overline{\mathcal{M}}_{1,1}$, hence it spans an extremal ray in both the nef cone and the effective cone. □

We also need to use several other compactifications of $\mathcal{M}_{g,n}$. Let $\tau : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{sat}}$ denote the *extended Torelli map* from $\overline{\mathcal{M}}_g$ to the Satake compactification of the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties (see e.g. [BHPV, III. 16]). It maps a stable curve to the product of the Jacobians of the irreducible components of its normalization. The exceptional locus of τ is the total boundary Δ.

Let $\mathrm{ps} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ be the *first divisorial contraction* of the log minimal model program for $\overline{\mathcal{M}}_{g,n}$ ([HH, AFSV]), where $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is the moduli space of genus g *pseudostable* curves with n ordered marked points. It contracts $\Delta_{1;\emptyset}$ only, replacing an unmarked elliptic tail by a cusp.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a collection of n rational numbers such that $0 < a_i \leq 1$ for all i and $2g-2+\sum_{i=1}^n a_i > 0$. The moduli space of *weighted* stable curves $\overline{\mathcal{M}}_{g,\mathcal{A}}$ parameterizes the data $(C, p_1, \ldots, p_n, \mathcal{A})$ such that

- C is a connected, reduced, at-worst-nodal arithmetic genus g curve.
- $p_1, \ldots, p_n$ are smooth points of C assigned the weights $a_1, \ldots, a_n$, respectively, where the total weight of any points that coincide is at most one.
- for every irreducible component X of C, the divisor class $K_C + \sum_{i=1}^n a_i p_i$ is ample restricted to X, i.e. numerically

$$2g_X - 2 + \#(X \cap \overline{C\backslash X}) + \sum_{p_i \in X} a_i > 0,$$

  where g_X is the arithmetic genus of X.

If $\mathcal{A} = \{1, \ldots, 1\}$, then $\overline{\mathcal{M}}_{g,\mathcal{A}} = \overline{\mathcal{M}}_{g,n}$. Hassett constructs a morphism $f_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ that modifies the locus of curves violating the stability inequality in the above [Hass]. Note that if $g_X \geq 1$ or $\#(X \cap \overline{C\backslash X}) \geq 2$ for C parameterized by $\overline{\mathcal{M}}_{g,n}$, the above inequality always holds for any permissible $\mathcal{A}$. Hence, the exceptional locus of $f_{\mathcal{A}}$ consists of curves that have a rational tail with at least three marked points of total weight at most one.

4. The effective cones of $\overline{\mathcal{M}}_3$ and $\overline{\mathcal{M}}_{3,1}$

We want to study the effective cone of higher codimension cycles on $\overline{\mathcal{M}}_{g,n}$. First, consider the case of codimension two and $n = 0$. Since $\dim \overline{\mathcal{M}}_2 = 3$, a cycle of codimension two on $\overline{\mathcal{M}}_2$ is a curve, and the cone of curves of $\overline{\mathcal{M}}_2$ is known to be spanned by the one-dimensional topological strata (F-curves). In this section, we introduce our methods by studying the first interesting case $g = 3$ in detail. In later sections, we will generalize some of the results to $g \geq 4$ or $n > 0$.

The Chow ring of $\overline{\mathcal{M}}_3$ was computed in [Fab1]. For ease of reference, we preserve Faber's notation introduced in [Fab1, p. 340–343]:

- Let $\Delta_{00} \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing irreducible curves with two nodes.
- Let $\Delta_{01a} \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing a rational nodal curve attached to a genus two curve at one point.
- Let $\Delta_{01b} \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing an elliptic nodal curve attached to an elliptic curve at one point.
- Let $\Delta_{11} \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing a chain of three elliptic curves.
- Let $\Xi_0 \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing irreducible nodal curves in which the normalization of the node consists of two conjugate points under the hyperelliptic involution.
- Let $\Xi_1 \subset \overline{\mathcal{M}}_3$ be the closure of the locus parameterizing two elliptic curves attached at two points.
- Let $H_1 \subset \overline{\mathcal{M}}_3$ be the closure of the locus of curves consisting of an elliptic tail attached to a genus two curve at a Weierstrass point.

The codimension two boundary strata of $\overline{\mathcal{M}}_3$ consist of Δ_{00}, Δ_{01a}, Δ_{01b}, Δ_{11} and Ξ_1. We denote the cycle class of a locus by the corresponding small letter, such as δ_{00} for the class of Δ_{00}. By [Fab1], the Chow group $A^2(\overline{\mathcal{M}}_3)$ is isomorphic to $N^2(\overline{\mathcal{M}}_3)$, with a basis given by δ_{00}, δ_{01a}, δ_{01b}, δ_{11}, ξ_0, ξ_1 and h_1 over $\mathbb{R}$.

The interior of Δ_0 parameterizing irreducible curves with exactly one node is given by

$$\operatorname{Int} \Delta_0 = \Delta_0 - \Delta_{00} - \Delta_{01a} - \Delta_{01b} - \Xi_1.$$

Faber shows that $A^1(\operatorname{Int} \Delta_0)$ is generated by ξ_0 ([Fab1, Lemma 1.12]). In particular, $A^1(\Delta_0)$ is generated by δ_{00}, δ_{01a}, δ_{01b}, ξ_0 and ξ_1, and $A^1(\Delta_0) \to N^2(\overline{\mathcal{M}}_3)$ is injective.

The interior of Δ_1 parameterizing the union of a smooth genus one curve and a smooth genus two curve attached at one point, is given by

$$\operatorname{Int} \Delta_1 = \Delta_1 - \Delta_{01a} - \Delta_{01b} - \Delta_{11}.$$

Faber shows that $A^1(\operatorname{Int} \Delta_1)$ is generated by h_1 ([Fab1, Lemma 1.11]). In particular, $A^1(\Delta_1)$ is generated by δ_{01a}, δ_{01b}, δ_{11} and h_1, and $A^1(\Delta_1) \to N^2(\overline{\mathcal{M}}_3)$ is injective.

Before studying effective cycles of codimension two in $\overline{\mathcal{M}}_3$, we need to study effective divisors in Δ_0 and Δ_1. Recall the gluing maps

$$\alpha_0 : \widehat{\Delta}_0 = \widetilde{\mathcal{M}}_{2,2} \to \Delta_0,$$

$$\alpha_1 : \widehat{\Delta}_1 = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{2,1} \to \Delta_1.$$

By the divisor theory of $\overline{\mathcal{M}}_{g,n}$ and $\widetilde{\mathcal{M}}_{g,n}$, we know that $A^1(\widehat{\Delta}_0) = N^1(\widehat{\Delta}_0)$ is generated by the classes of inverse images of Δ_{00}, Δ_{01a}, Δ_{01b}, Ξ_0 and Ξ_1 under α_0. Similarly, $A^1(\widehat{\Delta}_1) = N^1(\widehat{\Delta}_1)$ is generated by the classes of inverse images of Δ_{01a}, Δ_{01b}, Δ_{11} and H_1 under α_1. In particular, $(\iota \circ \alpha_i)_* : A^1(\widehat{\Delta}_i) \to A^2(\overline{\mathcal{M}}_3)$ is injective for $i = 0, 1$, where $\iota : \Delta \hookrightarrow \overline{\mathcal{M}}_3$ is the inclusion. For ease of notation, we denote the inverse image of a cycle class under α_i by the same symbol.

Lemma 4.1. *The classes δ_{01a}, δ_{01b}, δ_{11} and h_1 are extremal in* $\mathrm{Eff}^1(\widehat{\Delta}_1)$.

Proof. The effective cone of divisors on $\overline{\mathcal{M}}_{2,1}$ is computed in [R1, Section 3.3]. Note that δ_{01a} corresponds to the fiber class of $\widehat{\Delta}_1$ over $\overline{\mathcal{M}}_{1,1}$, while δ_{01b}, δ_{11} and h_1 are the pullbacks of the extremal divisor classes δ_0, δ_1 and w from $\overline{\mathcal{M}}_{2,1}$, respectively, where w is the divisor class of the locus of curves with a marked Weierstrass point in $\overline{\mathcal{M}}_{2,1}$. By Corollary 2.4, δ_{01a}, δ_{01b}, δ_{11} and h_1 are extremal. □

Let $BN_2^1 \subset \widetilde{\mathcal{M}}_{2,2}$ be the closure of the locus of curves such that the two unordered marked points are conjugate under the hyperelliptic involution. It has divisor class

$$[BN_2^1] = -\frac{1}{2}\lambda + \frac{1}{2}\psi - \frac{3}{2}\delta_{0;2} - \delta_{1;0},$$

see e.g. [Lo].

Lemma 4.2. *The classes δ_0, $\delta_{0;2}$, $\delta_{1;0}$, $\delta_{1;1}$ and $[BN_2^1]$ are extremal in* $\mathrm{Eff}^1(\widehat{\Delta}_0)$. *Consequently δ_{00}, δ_{01a}, δ_{01b}, ξ_1 and ξ_0 are extremal as classes in* $\mathrm{Eff}^1(\widehat{\Delta}_0)$.

Proof. By Proposition 3.1, the boundary divisors are known to be extremal. To obtain a moving curve B in BN_2^1 that has negative intersection with it, fix a general genus two curve C and vary a pair of conjugate points (p_1, p_2) in C. Since $[B]\cdot\lambda = 0$, $[B]\cdot\psi = 16$, $[B]\cdot\delta_{0;2} = 6$ and $[B]\cdot\delta_{1;0} = 0$, we have $[B]\cdot[BN_2^1] < 0$. Hence, BN_2^1 is extremal. Note that the classes δ_{00}, δ_{01a}, δ_{01b}, ξ_1 and ξ_0 in $\widehat{\Delta}_0$ correspond to δ_0, $\delta_{0;2}$, $\delta_{1;0}$, $\delta_{1;1}$ and $[BN_2^1]$ in $\widetilde{\mathcal{M}}_{2,2}$, respectively, thus proving the claim. □

Theorem 4.3. *The classes δ_{00}, δ_{01a}, δ_{01b}, ξ_1, ξ_0, δ_{11} and h_1 are extremal in* $\mathrm{Eff}^2(\overline{\mathcal{M}}_3)$.

Proof. Recall the Torelli map $\tau : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{sat}}$ and the first divisorial contraction $\mathrm{ps} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{\mathrm{ps}}$ discussed in Section 3. For $g = 3$, note that Δ_{01b}, Δ_{11} and H_1 are contained in Δ_1, and $e_{\mathrm{ps}}(\Delta_{01b}), e_{\mathrm{ps}}(\Delta_{11}), e_{\mathrm{ps}}(H_1) > 0$. Moreover, by Lemma 4.1 their classes are extremal in $\mathrm{Eff}^1(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{2,1})$. Proposition 2.5 implies that they are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_3)$.

The strata Δ_{00} and Ξ_1 are contained in Δ_0 and have index $e_\tau \geq 2$. If Z is a subvariety of codimension two in $\overline{\mathcal{M}}_3$ with $e_\tau(Z) \geq 2$, then Z is either contained in Δ_0 or in Δ_1. Since $e_\tau(\Delta_1) = 1$, if Z is contained in Δ_1 but not in Δ_0, then Z has to be Δ_{11}. By Lemma 4.2, δ_{00} and ξ_1 are extremal in $\mathrm{Eff}^1(\widetilde{\mathcal{M}}_{2,2})$. Moreover, δ_{00}, ξ_1 and δ_{11} are linearly independent in $N^2(\overline{\mathcal{M}}_3)$. We thus conclude that δ_{00} and ξ_1 are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_3)$ by Proposition 2.5.

The remaining cases are Δ_{01a} and Ξ_0. A rational nodal tail does not have moduli. Consequently, $e_\tau(\Delta_{01a}) = 1$ and $e_{\mathrm{ps}}(\Delta_{01a}) = 0$. Similarly we have $e_\tau(\Xi_0) = 1$ and $e_{\mathrm{ps}}(\Xi_0) = 0$. The dimension drops are too small to directly apply Proposition 2.2. Nevertheless, their e_τ indices are positive, hence we may apply Proposition 2.6 to $Y = \Delta_0 \cup \Delta_1$. Recall that $N^1(\widehat{\Delta}_i) \cong A^1(\widehat{\Delta}_i) \to N^2(\overline{\mathcal{M}}_3)$ is injective for $i = 0, 1$. Moreover, $\Delta_0 \cap \Delta_1 = \Delta_{01a} \cup \Delta_{01b}$ and both components are extremal in $\widehat{\Delta}_0$ and in $\widehat{\Delta}_1$ by Lemmas 4.1 and 4.2. Finally, ξ_0

is extremal in $\widehat{\Delta}_0$ by Lemma 4.2. By Proposition 2.6, we conclude that Δ_{01a} and Ξ_0 are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_3)$. □

Remark 4.4. It would be interesting to find an extremal cycle of codimension two that is not contained in the boundary of $\overline{\mathcal{M}}_3$. Some natural geometric non-boundary cycles fail to be extremal. For instance, the closure B_3 of the locus of bielliptic curves in $\overline{\mathcal{M}}_3$ is not extremal. The class of B_3 was calculated in terms of another basis of $A^2(\overline{\mathcal{M}}_3)$ ([FP]):

$$[B_3] = \frac{2673}{2}\lambda^2 - 267\lambda\delta_0 - 651\lambda\delta_1 + \frac{27}{2}\delta_0^2 + 69\delta_0\delta_1 + \frac{177}{2}\delta_1^2 - \frac{9}{2}\kappa_2.$$

By [Fab1, Theorem 2.10], we can rewrite it as follows:

$$[B_3] = \frac{1}{2}\delta_{00} + \frac{25}{8}\delta_{01a} + \frac{19}{4}\delta_{01b} + 18\delta_{11} + \frac{15}{8}\xi_0 + 15\xi_1 + 10h_1.$$

Therefore, B_3 is not extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_3)$.

However, if we consider $\overline{\mathcal{M}}_{3,1}$ instead, we are able to find an extremal non-boundary codimension two cycle. First, let us introduce some notation:

- Let $H \subset \overline{\mathcal{M}}_3$ be the closure of the locus of hyperelliptic curves.
- Let $HP \subset \overline{\mathcal{M}}_{3,1}$ be the closure of the locus of hyperelliptic curves with a marked point.
- Let $HW \subset \overline{\mathcal{M}}_{3,1}$ be the closure of the locus of hyperelliptic curves with a marked Weierstrass point.

Clearly HW is a subvariety of codimension two in $\overline{\mathcal{M}}_{3,1}$, which is not contained in the boundary. Moreover, $HW \subset HP = \pi^{-1}(H)$ and $\pi : HW \to H$ is generically finite of degree eight, where $\pi : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_3$ is the morphism forgetting the marked point.

We will show that HW is an extremal cycle in $\overline{\mathcal{M}}_{3,1}$. In order to prove this, we need to understand the divisor theory of HP as well as the boundary components $\Delta_{1;\{1\}}$ and $\Delta_{1;\emptyset}$ of $\overline{\mathcal{M}}_{3,1}$. We further introduce the following cycles:

- Let $HP_0 \subset HP$ be the closure of the locus of irreducible nodal hyperelliptic curves with a marked point.
- Let $HP_1 = HP \cap \Delta_{1;\{1\}}$ be the closure of the locus of curves consisting of a marked genus one component attached to a genus two component at a Weierstrass point.
- Let $HP_2 = HP \cap \Delta_{1;\emptyset}$ be the closure of the locus of curves consisting of a genus one component attached at a Weierstrass point to a marked genus two component.
- Let $\Theta \subset HP$ be the closure of the locus of two genus one curves attached at two points, one of the curves marked.
- Let $D_{01a} \subset \Delta_{1;\{1\}}$ be the closure of the locus of a marked genus one curve attached to an irreducible nodal curve of geometric genus one.
- Let $D_{02b} \subset \Delta_{1;\{1\}}$ be the closure of the locus of a marked rational nodal curve attached to a genus two curve.
- Let $D_{12} \subset \Delta_{1;\{1\}} \cap \Delta_{1;\emptyset}$ be the closure of the locus of a chain of three curves of genera 2, 0 and 1, respectively, such that the rational component contains the marked point.
- Let $D_{11a} \subset \Delta_{1;\{1\}} \cap \Delta_{1;\emptyset}$ be the closure of the locus of a chain of three genus one curves such that one of the two tails contains the marked point.
- Let $W_1 \subset \Delta_{1;\emptyset}$ be the closure of the locus of a genus one curve attached to a marked genus two curve such that the marked point is a Weierstrass point.

- Let $D_{02a} \subset \Delta_{1;\emptyset}$ be the closure of the locus of a rational nodal curve attached to a marked genus two curve.
- Let $D_{01b} \subset \Delta_{1;\emptyset}$ be the closure of the locus of a genus one curve attached to a marked irreducible nodal curve of geometric genus one.
- Let $D_{11b} \subset \Delta_{1;\emptyset}$ be the closure of the locus of a chain of three genus one curves such that the middle component contains the marked point.

Recall the gluing morphisms

$$\alpha_{1;\{1\}} : \widehat{\Delta}_{1;\{1\}} = \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{2,1} \to \Delta_{1;\{1\}},$$
$$\alpha_{1;\emptyset} : \widehat{\Delta}_{1;\emptyset} = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{2,2} \to \Delta_{1;\emptyset}.$$

For ease of notation, we denote the inverse image of a class under the gluing maps by the same symbol.

Lemma 4.5. (i) *$N^1(HP) \cong A^1(HP)$ is generated by hw, hp_0, hp_1, hp_2 and θ.*

(ii) *The divisor classes hw, hp_0, hp_1, hp_2 and θ are extremal in HP. Moreover, if a linear combination $a_1 \cdot hp_1 + a_2 \cdot hp_2$ is effective, then $a_1, a_2 \geq 0$.*

(iii) *$N^1(\widehat{\Delta}_{1;\{1\}}) \cong A^1(\widehat{\Delta}_{1;\{1\}})$ is generated by hp_1, d_{01a}, d_{02b}, d_{11a} and d_{12}. Moreover, if a linear combination $b_1 \cdot hp_1 + b_2 \cdot d_{11a} + b_3 \cdot d_{12}$ is effective, then $b_i \geq 0$ for all i.*

(iv) *$N^1(\widehat{\Delta}_{1;\emptyset}) \cong A^1(\widehat{\Delta}_{1;\emptyset})$ is generated by hp_2, w_1, d_{01b}, d_{02a}, d_{11a}, d_{11b} and d_{12}. Moreover, if a linear combination $c_1 \cdot hp_2 + c_2 \cdot d_{11a} + c_3 \cdot d_{12}$ is effective, then $c_i \geq 0$ for all i.*

(v) *The kernel of $N^1(HP) \oplus N^1(\widehat{\Delta}_{1;\{1\}}) \oplus N^1(\widehat{\Delta}_{1;\emptyset}) \to N^2(\overline{\mathcal{M}}_{3,1})$ is generated by*

$$(hp_1, -hp_1, 0), (hp_2, 0, -hp_2), (0, d_{11a}, -d_{11a}) \text{ and } (0, d_{12}, -d_{12}).$$

Proof. The first two claims essentially follow from [R2, 9.2, 10.1]. A curve parameterized in HP is a genus three hyperelliptic curve C with a marked point p. Hence, it can be identified with an admissible double cover ([HMo, 3.G]) of a stable rational curve R branched at eight unordered points $q_1, \ldots, q_8$ with a distinguished marked point q as the image of p. Marking the conjugate point p' of p gives (C, p'), which is isomorphic to (C, p) under the hyperelliptic involution. Therefore, the data $(R, q_1, \ldots, q_8, q)$ uniquely determines (C, p). The curve (C, p) is not stable if and only if R has a rational tail marked by exactly two of the unordered points or by an unordered point and q or has a rational bridge marked only by q. In the first case, the double cover has an unmarked rational bridge. In the second case, the double cover has a rational tail marked by q. In the last case, the double cover has a rational bridge with no marked points. In all three cases stabilization contracts the nonstable rational component. Furthermore, when R has a rational tail marked by exactly two unordered points and q, the double cover has a rational bridge marked by q. While a rational bridge marked by one point has no moduli, a rational tail with three marked points has one-dimensional moduli.

Following the notation in [R2], let $X_{9,1} = \overline{\mathcal{M}}_{0,9}/\mathfrak{S}_8$ be the moduli space of stable genus zero curves with nine marked points $q_1, \ldots, q_8, q$ such that $q_1, \ldots, q_8$ are unordered but q is distinguished. Let $B_k \subset X_{9,1}$ be the boundary divisor parameterizing curves with a rational tail marked by q and $k-1$ of the q_i for $2 \leq k \leq 7$. There is a morphism $X_{9,1} \to HP$ defined by taking the admissible double cover and stabilizing. This morphism contracts the boundary divisor B_3 to the locus of hyperelliptic curves with a marked rational bridge. Numerical equivalence, rational equivalence and linear equivalence over $\mathbb{R}$ coincide for divisors on moduli spaces of stable pointed genus zero curves, and the Picard group is generated by boundary divisors. Considering the admissible double covers corresponding to the boundary divisors B_2, B_4, B_5, B_6 and B_7 of $X_{9,1}$, we see that their images in HP correspond to HW, HP_1, Θ,

HP_2 and HP_0, respectively, hence (i) follows. The same curves proving the extremality of the boundary divisors in this case (see [R2, 10.1]) also prove (ii).

For (iii), hp_1 and d_{ij} correspond to the pullbacks of generators of $N^1(\overline{\mathcal{M}}_{1,2}) \cong A^1(\overline{\mathcal{M}}_{1,2})$ and of $N^1(\overline{\mathcal{M}}_{2,1}) \cong A^1(\overline{\mathcal{M}}_{2,1})$, where $\widehat{\Delta}_{1;\{1\}} = \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{2,1}$. Moreover, the interior $\mathcal{M}_{1,2} \times \mathcal{M}_{2,1}$ is affine. Hence, these classes generate $N^1(\widehat{\Delta}_{1;\{1\}}) \cong A^1(\widehat{\Delta}_{1;\{1\}})$. Suppose that $b_1 \cdot hp_1 + b_2 \cdot d_{11a} + b_3 d_{12}$ is an effective divisor class. Let π_1 and π_2 be the projections of $\widehat{\Delta}_{1;\{1\}}$ to $\overline{\mathcal{M}}_{1,2}$ and $\overline{\mathcal{M}}_{2,1}$, respectively. Then, we have $hp_1 = \pi_2^* w$, $d_{11a} = \pi_2^*\delta_{1;\{1\}}$ and $d_{12} = \pi_1^*\delta_{0;\{1,2\}}$, where w is the divisor class of Weierstrass points. Hence, $b_3 \geq 0$ and $b_1 \cdot w + b_2 \cdot \delta_{1;\{1\}}$ is effective in $\overline{\mathcal{M}}_{2,1}$. Since w and $\delta_{1;\{1\}}$ span a face of $\mathrm{Eff}^1(\overline{\mathcal{M}}_{2,1})$ ([R1, Corollary 3.3.2]), we conclude that $b_1, b_2 \geq 0$. A similar argument yields (iv).

For (v), since $HP \cap \Delta_{1;\{1\}} = HP_1$, $HP \cap \Delta_{1;\emptyset} = HP_2$ and $\Delta_{1;\{1\}} \cap \Delta_{1;\emptyset} = D_{12} \cup D_{11a}$, it suffices to prove that hw, hp_0, hp_1, hp_2, θ, w_1, d_{01a}, d_{01b}, d_{02a}, d_{02b}, d_{11a}, d_{11b} and d_{12} are independent in $N^2(\overline{\mathcal{M}}_{3,1})$. Suppose that they satisfy a relation

$$a \cdot hw + t \cdot \theta + s \cdot w_1 + \sum_i b_i \cdot hp_i + \sum_{i,j,k} c_{ijk} \cdot d_{ijk} = 0 \in N^2(\overline{\mathcal{M}}_{3,1}) \tag{5}$$

for $a, b_i, c_{ijk}, s, t \in \mathbb{R}$. The morphism $\pi : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_3$ contracts these cycles except HW, W_1 and D_{12}, where $\pi(HW) = H$ and $\pi(W_1) = \pi(D_{12}) = \Delta_1$. Since π is flat and h, δ_1 are independent in $\overline{\mathcal{M}}_3$, applying π_* to (5) we conclude that $a = 0$.

For the remaining cycles, we have $\pi(HP_1) = \pi(HP_2) = H_1$, $\pi(HP_0) = \Xi_0$, $\pi(\Theta) = \Xi_1$, $\pi(D_{02a}) = \pi(D_{02b}) = \Delta_{01a}$, $\pi(D_{01a}) = \pi(D_{01b}) = \Delta_{01b}$ and $\pi(D_{11a}) = \pi(D_{11b}) = \Delta_{11}$. The images of W_1 and D_{12} are contained in Δ_1. By [Fab1], the subspace spanned by ξ_0 and ξ_1 has zero intersection with $A^1(\Delta_1)$ in $N^2(\overline{\mathcal{M}}_3)$. Intersect (5) with an ample divisor class and apply π_*. We thus conclude that $t = b_0 = 0$.

Relation (5) reduces to

$$s \cdot w_1 + b_1 \cdot hp_1 + b_2 \cdot hp_2 + \sum_{i,j,k} c_{ijk} \cdot d_{ijk} = 0. \tag{6}$$

Recall the pseudostable map $\mathrm{ps} : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}^{\mathrm{ps}}_{3,1}$. Applying ps_*, we obtain that

$$b_1 \cdot (\mathrm{ps}_*\, hp_1) + c_{01a} \cdot (\mathrm{ps}_*\, d_{01a}) + c_{02a} \cdot (\mathrm{ps}_*\, d_{02a}) + c_{02b} \cdot (\mathrm{ps}_*\, d_{02b}) = 0 \tag{7}$$

in $N^2(\overline{\mathcal{M}}^{\mathrm{ps}}_{3,1})$, since the other summands in (6) are contracted by ps. Consider a family S_1 of plane quartics passing through 12 general points. Marking one of the points, S_1 gives rise to a general two-dimensional family of plane quartics with a marked base point. Then S_1 intersects $\mathrm{ps}(D_{02a})$ at finitely many points parameterizing cuspidal quartics and S_1 does not intersect the other cycles in (7), which implies that $c_{02a} = 0$. Let $\pi_{\mathrm{ps}} : \overline{\mathcal{M}}^{\mathrm{ps}}_{3,1} \to \overline{\mathcal{M}}^{\mathrm{ps}}_3$ be the morphism forgetting the marked point and pseudostablizing. Note that $e_{\pi_{\mathrm{ps}}}(\mathrm{ps}(D_{02b})) = 1$ and $e_{\pi_{\mathrm{ps}}}(\mathrm{ps}(HP_1)) = e_{\pi_{\mathrm{ps}}}(\mathrm{ps}(D_{01a})) = 2$. Take an ample divisor class in $\overline{\mathcal{M}}^{\mathrm{ps}}_{3,1}$, intersect (7) and apply $\pi_{\mathrm{ps}*}$. We thus conclude that $c_{02b} = 0$. Finally, take a pencil of plane cubics, mark a base point, take another base point and use it to attach the pencil to a varying point in a fixed genus two curve. We obtain a two-dimensional family S_2 in $\overline{\mathcal{M}}^{\mathrm{ps}}_{3,1}$ such that $[S_2] \cdot (\mathrm{ps}_*\, hp_1) = -6$ and $[S_2] \cdot (\mathrm{ps}_*\, d_{01a}) = 0$, which implies that $b_1 = c_{01a} = 0$.

Relation (5) further reduces to

$$s \cdot w_1 + b_2 \cdot hp_2 + c_{12} \cdot d_{12} + c_{11a} \cdot d_{11a} + c_{01b} \cdot d_{01b} + c_{11b} \cdot d_{11b} = 0. \tag{8}$$

Inside the parameter space $\mathbb{P}^{14}$ of plane quartics, take a general two-dimensional subspace S_3. As S_3 is general, it contains finitely many general cuspidal curves. Take a line L passing through a Weierstrass point of the normalization of one of the cuspidal quartics C and make L general other than that. Mark the intersection points of L with curves in S_3, make a base change, perform stable reduction and still denote by S_3 the resulting family in $\overline{\mathcal{M}}_{3,1}$. Note that S_3 does not intersect the summands in (8) except W_1, and it intersects W_1 along a one-dimensional locus parameterizing the normalization of C with a pencil of elliptic tails attached at the inverse image of the cusp of C, which implies that $[S_3]\cdot w_1 < 0$ since the normal bundle of elliptic tails in Δ_1 has negative degree restricted to this family. We thus conclude that $s = 0$.

Set $s = 0$, intersect (8) with the ψ-class of the marked point and apply π_*. Since the marked point lies in the rational bridge for curves in D_{12}, it implies that $\psi \cdot d_{12} = 0$. We thus obtain that

$$(3b_2)\cdot h_1 + (c_{11a} + 2c_{11b})\cdot \delta_{11} + c_{01b}\cdot \delta_{01b} = 0$$

in $N^2(\overline{\mathcal{M}}_3)$. Since h_1, δ_{11} and δ_{01b} are independent in $\overline{\mathcal{M}}_3$ ([Fab1]), we conclude that $b_2 = c_{01b} = 0$ and $c_{11a} + 2c_{11b} = 0$.

Now Relation (5) reduces to

$$c_{12}\cdot d_{12} - 2c_{11b}\cdot d_{11a} + c_{11b}\cdot d_{11b} = 0. \tag{9}$$

Attach two pencils of plane cubics to a smooth genus one curve E at two general points, and mark a third general point in E. We obtain a two-dimensional family S_4 in $\overline{\mathcal{M}}_{3,1}$ such that $[S_4]\cdot d_{12} = [S_4]\cdot d_{11a} = 0$ and $[S_4]\cdot d_{11b} = 1$. Plugging in (9), we obtain that $c_{11b} = 0$, hence $c_{12}\cdot d_{12} = 0$ and $c_{12} = 0$. □

Now we are ready to prove that HW is extremal in $\overline{\mathcal{M}}_{3,1}$. In fact, all extremal divisors in HP are extremal as codimension two cycles in $\overline{\mathcal{M}}_{3,1}$. First, observe that $\pi : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_3$ is flat of relative dimension one. As a consequence, if $Z \subset \overline{\mathcal{M}}_{3,1}$ is an irreducible subvariety of codimension two, then $\pi(Z) \subset \overline{\mathcal{M}}_3$ has codimension either one or two, i.e. $e_\pi(Z) = 0$ or 1.

Theorem 4.6. *The cycle classes of HW, HP_0, HP_1, HP_2 and Θ are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{3,1})$.*

Proof. We prove it for HW first. Suppose that

$$hw = \sum_{i=1}^{r} a_i[Y_i] + \sum_{j=1}^{s} b_j[Z_j] \in N^2(\overline{\mathcal{M}}_{3,1}) \tag{10}$$

where $a_i, b_j > 0$ and $Y_i, Z_j \subset \overline{\mathcal{M}}_{3,1}$ are irreducible subvarieties of codimension two such that $e_\pi(Y_i) = 0$ and $e_\pi(Z_j) = 1$. Since $\pi_*[Z_j] = 0$, we have

$$8\cdot h = \sum_{i=1}^{r} a_i \cdot \pi_*[Y_i] \in N^1(\overline{\mathcal{M}}_3).$$

Since H is an extremal and rigid divisor in $\overline{\mathcal{M}}_3$ (see e.g. [R1, 2.2]), it follows that $\pi(Y_i) = H$. Hence, $Y_i \subset \pi^{-1}(H) = HP$ for all i.

Next, we show that Z_j is contained in $HP \cup \Delta_{1;\{1\}} \cup \Delta_{1;\emptyset}$ for all j. Consider a family S of plane quartics passing through twelve general points. Marking one of the points, S can be regarded as a two-dimensional family of plane quartics with a marked base point. By taking the base points general enough, we can assume that S avoids any phenomenon of codimension three or higher. In particular, we may assume that S does not contain any reducible or nonreduced elements and S has finitely many cuspidal elements such that the base point is not any of the cusps. Furthermore, we may specify S to contain a specific smooth or non-hyperelliptic irreducible nodal quartic in Z_1 while preserving these properties. Then S induces a surface in $\overline{\mathcal{M}}_{3,1}$ such that $[S]\cdot hw = 0$, $[S]\cdot[Y_i]\geq 0$, $[S]\cdot[Z_j]\geq 0$ for $j\neq 1$ and $[S]\cdot[Z_1]>0$. Intersecting both sides of (10) with S leads to a contradiction. We thus conclude that $Y_i, Z_j \subset HP\cup\Delta_{1;\{1\}}\cup\Delta_{1;\emptyset}$ for all i,j. By Lemma 4.5, we can apply Proposition 2.6 to $Y = HP\cup\Delta_{1;\{1\}}\cup\Delta_{1;\emptyset}$, thus proving that hw is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{3,1})$.

For HP_0, HP_1, HP_2 and Θ, note that their e_π indices are all equal to one and their images under π are contained in H. If any of them satisfies a relation like (10), by Proposition 2.1 the Y_i terms cannot exist in the effective expression. Then the same argument in the previous paragraph implies that the remaining terms Z_j are contained in $HP\cup\Delta_{1;\{1\}}\cup\Delta_{1;\emptyset}$ for all j. Hence, we can apply Lemma 4.5 and Proposition 2.6 to conclude that they are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{3,1})$. □

5. The effective cones of $\overline{\mathcal{M}}_g$ for $g\geq 4$

In this section, we study effective cycles of codimension two on $\overline{\mathcal{M}}_g$ for $g\geq 4$. The method is similar to the case $g=3$. There is a topological stratification of $\overline{\mathcal{M}}_g$, where the strata are indexed by dual graphs of stable nodal curves. The codimension two boundary strata consist of curves with at least two nodes. We recall the notation introduced in [E] for these strata:

- Let Δ_{00} be the closure of the locus in $\overline{\mathcal{M}}_g$ parameterizing irreducible curves with two nodes.
- For $1\leq i\leq j\leq g-2$ and $i+j\leq g-1$, let Δ_{ij} be the closure of the locus in $\overline{\mathcal{M}}_g$ parameterizing a chain of three curves of genus i, $g-i-j$ and j, respectively.
- For $1\leq j\leq g-1$, let Δ_{0j} be the closure of the locus in $\overline{\mathcal{M}}_g$ parameterizing a union of a genus j curve and an irreducible nodal curve of geometric genus $g-1-j$, attached at one point.
- For $1\leq i\leq [(g-1)/2]$, let Θ_i be the closure of the locus in $\overline{\mathcal{M}}_g$ parameterizing a union of a curve of genus i and a curve of genus $g-i-1$, attached at two points.

In order to study higher codimension cycles on $\overline{\mathcal{M}}_g$, we need to understand the divisor theory of the boundary components. Recall the gluing morphisms $\alpha_i : \widehat{\Delta}_i \to \Delta_i$ for $0\leq i\leq [g/2]$. By [Fab3, p. 69] and [E, Section 4], $A^1(\Delta)\to N^2(\overline{\mathcal{M}}_g)$ is injective. Moreover, $(\iota\circ\alpha_i)_* : N^1(\widehat{\Delta}_i)\cong A^1(\widehat{\Delta}_i)\to N^2(\overline{\mathcal{M}}_g)$ is injective, where $\iota:\Delta\hookrightarrow\overline{\mathcal{M}}_g$ is the inclusion. As before, we denote the class of a locus in Δ_i and in $\widehat{\Delta}_i$ by the same symbol. Whenever we use $g/2$ as an index, the corresponding term exists if and only if g is even.

Lemma 5.1. (i) *The classes δ_{11} and δ_{1g-2} are extremal in* $\mathrm{Eff}^1(\widehat{\Delta}_1)$.
(ii) *The class δ_{0g-1} is extremal in* $\mathrm{Eff}^1(\widehat{\Delta}_0)$.

Proof. Note that $\widehat{\Delta}_1 = \overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{g-1,1}$. The classes δ_{11} and δ_{1g-2} are the pullbacks of boundary classes $\delta_{1;}$ and $\delta_{g-2;}$ from $\overline{\mathcal{M}}_{g-1,1}$, respectively. Now (i) follows from Corollary 2.4 and Proposition 3.1.

For (ii), δ_{0g-1} corresponds to the boundary class $\delta_{0;2}$ in $\widehat{\Delta}_0 = \widetilde{\mathcal{M}}_{g-1,2}$. Hence, the claim follows from Proposition 3.1. □

Lemma 5.2. (i) *The linear combination $a_0\delta_0 + \sum_{i=1}^{g-1} a_i\delta_{i,1}$ is effective on $\overline{\mathcal{M}}_{g,1}$ if and only if all the coefficients are nonnegative.*

(ii) *For $0 < i < g/2$, the linear combination*

$$\sum_{k=0}^{i-1}(c_{ki}\delta_{ki} + d_{kg-i}\delta_{kg-i}) + \sum_{k=i+1}^{[g/2]}(c_{ik}\delta_{ik} + d_{ig-k}\delta_{ig-k})$$

is effective on $\widehat{\Delta}_i$ if and only if all the coefficients are nonnegative.

(iii) *For g even and $i = g/2$, the linear combination*

$$\sum_{k=0}^{g/2-1} b_{kg/2}\delta_{kg/2}$$

is effective on $\widehat{\Delta}_{g/2}$ if and only if all the coefficients are nonnegative.

Proof. For (i), take a nodal curve of geometric genus $g-1$ and vary a marked point on it. We obtain a curve C_0 moving in Δ_0 such that $[C_0]\cdot\delta_0 < 0$ and $[C_0]\cdot\delta_{i,1} = 0$ for all i. For $1 \le i \le g-2$, sliding a genus i curve with a marked point along a genus $g-i$ curve, we obtain a curve C_i moving in $\Delta_{i,1}$ such that $[C_i]\cdot\delta_{i,1} < 0$, $[C_i]\cdot\delta_0 = 0$ and $[C_i]\cdot\delta_{j,1} = 0$ for $j \neq i$. Finally, attach a genus $g-1$ curve with a marked point to a genus one curve at a general point and vary the marked point in the component of genus $g-1$. We obtain a curve C_{g-1} moving in $\Delta_{g-1,1}$ such that $[C_{g-1}]\cdot\delta_0 = 0$, $[C_{g-1}]\cdot\delta_{1,1} = 1$, $[C_{g-1}]\cdot\delta_{g-1,1} = 4-2g < 0$ and $[C_{g-1}]\cdot\delta_{i,1} = 0$ for $2 \le i \le g-2$.

Suppose that D is an effective divisor on $\overline{\mathcal{M}}_{g,1}$ with class $[D] = a_0\delta_0 + \sum_{i=1}^{g-1} a_i\delta_{i,1}$. If the support of D is contained in the union of Δ_0 and the $\Delta_{i,1}$, we are done, because the boundary divisor classes are linearly independent. Suppose that D does not contain any boundary components in its support. Since the curves constructed are moving in the respective boundary components, we have $[D]\cdot[C_i] \ge 0$ for $0 \le i \le g-1$. We thus conclude that $a_0 \le 0$, $a_i \le 0$ for $1 \le i \le g-2$ and $(4-2g)a_{g-1} + a_1 \ge 0$, hence $a_{g-1} \le 0$. It follows that $[D] + (-a_0)\delta_0 + \sum_{i=1}^{g-1}(-a_i)\delta_{i,1} = 0$, leading to a contradiction, because the class of a positive sum of effective divisors cannot be zero. Indeed we have proved that an effective divisor with class $a_0\delta_0 + \sum_{i=1}^{g-1} a_i\delta_{i,1}$ has to be supported in the union of Δ_0 and the $\Delta_{i,1}$.

For (ii) and (iii), the boundary classes in the linear combination are the pullbacks of the boundary classes from $\overline{\mathcal{M}}_{i,1}$ and $\overline{\mathcal{M}}_{g-i,1}$, respectively, hence the claims follow from (i). $\square$

Theorem 5.3. *For $g \ge 2$, every codimension two boundary stratum of $\overline{\mathcal{M}}_g$ is extremal in* $\mathrm{Eff}^2(\overline{\mathcal{M}}_g)$.

Proof. When $g = 2$, the two F-curves Δ_{00} and Δ_{01} are dual to the two nef divisors λ and $12\lambda - \delta$, respectively, and form the extremal rays of the Mori cone of curves. The case $g = 3$ is covered by Theorem 4.3. Hence, we may assume that $g \ge 4$.

Recall that the extended Torelli map $\tau : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{sat}}$ sends a stable curve to the product of the Jacobians of the irreducible components of its normalization. Suppose that Z is a codimension two boundary stratum of $\overline{\mathcal{M}}_g$ such that a general curve C parameterized by Z does not have an irreducible component of geometric genus less than two. This assumption ensures that selecting two points (as the inverse image of a node) in the normalization of C has two-dimensional moduli. We thus conclude that

$$e_\tau(Z) = \dim Z - \dim \tau(Z) = 2 + 2 = 4,$$

because $\tau(C_1) = \tau(C_2)$ if C_1 and C_2 have the same normalization. If a general curve C parameterized by Z contains exactly one irreducible component E of geometric genus one and no other components of geometric genus less than two, then $e_\tau(Z) = 3$ since choosing n points in the normalization of E has $(n-1)$-dimensional moduli.

Conversely, if $Z' \subset \overline{\mathcal{M}}_g$ is a subvariety of codimension two such that $e_\tau(Z') \geq 3$, then every curve parameterized by Z' has at least two nodes. Therefore, Z' has to be one of the codimension two boundary strata. Since the classes of the codimension two boundary strata of $\overline{\mathcal{M}}_g$ are independent ([E, Theorem 4.1]), we conclude that $[Z]$ is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_g)$ by Proposition 2.2.

The remaining boundary strata are: Δ_{11} whose general point parameterizes a chain of three curves of genera 1, $g-2$ and 1; Δ_{1g-2} whose general point parameterizes a chain of three curves of genera 1, 1 and $g-2$; Δ_{0g-1} whose general point parameterizes a curve with a rational nodal tail. They are all contained in the boundary divisor Δ_1.

Recall that $\mathrm{ps} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{\mathrm{ps}}$ contracts the boundary divisor Δ_1 by replacing elliptic tails by cusps. A subvariety of $\overline{\mathcal{M}}_g$ contracted by ps has to be contained in Δ_1. Note that $e_{\mathrm{ps}}(\Delta_{11}), e_{\mathrm{ps}}(\Delta_{1g-2}) > 0$, and by Lemma 5.1 (i), δ_{11} and δ_{1g-2} are extremal in $\mathrm{Eff}^1(\widehat{\Delta}_1)$. Moreover, $N^1(\widehat{\Delta}_1) \cong A^1(\widehat{\Delta}_1) \to N^2(\overline{\mathcal{M}}_g)$ is injective. Therefore, we conclude that δ_{11} and δ_{1g-2} are extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_g)$ by Proposition 2.5.

Finally, for Δ_{0g-1}, the argument is similar to that of Δ_{01a} in the proof of Theorem 4.3. Since $e_\tau(\Delta_{0g-1}) > 0$, we can apply Proposition 2.6 to the setting $Y = \Delta_0 \cup \cdots \cup \Delta_{[g/2]}$ and $D_0 = \Delta_0$. Note that $\Delta_i \cap \Delta_j = \Delta_{ij} \cup \Delta_{ig-j}$ and $\Delta_i \cap \Delta_{g/2} = \Delta_{ig/2}$ for $0 \leq i < g/2$. Lemmas 5.1 (ii) and 5.2 ensure that Proposition 2.6 applies in this case, thus proving that Δ_{0g-1} is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_g)$. □

Remark 5.4. If Z is a codimension two subvariety of $\overline{\mathcal{M}}_g$ such that $e_\tau(Z) \geq 3$, the above proof shows that Z has to be a codimension two boundary stratum. As a consequence of Remark 2.7, any codimension two boundary stratum with $e_\tau \geq 3$ is extremal in the pseudoeffective cone $\overline{\mathrm{Eff}}^2(\overline{\mathcal{M}}_g)$.

The techniques of Theorem 5.3 allow us to show that certain boundary strata of arbitrarily high codimension are extremal. In the next theorem, we will give the simplest examples. Given a stable dual graph Γ of arithmetic genus g, let Δ_Γ denote the closure of the stratum indexed by Γ in the topological stratification of $\overline{\mathcal{M}}_g$. Denote the class of Δ_Γ by δ_Γ. The codimension of Δ_Γ is the number of edges in Γ.

Let $\kappa(g,r)$ denote the set of stable dual graphs of arithmetic genus g with r edges such that the geometric genera of the curves associated to each node is at least two. Let $\kappa'(g,r)$ be the set of stable dual graphs of arithmetic genus g with r edges such that the geometric genera of the curves associated to all the nodes but one is at least two and the remaining genus is at least one. Clearly $\kappa(g,r) \subset \kappa'(g,r)$. With this notation, we have the following application of Propositions 2.1 and 2.2.

Lemma 5.5. *Let $\Gamma \in \kappa(g,r)$ (resp. $\Gamma \in \kappa'(g,r)$). If the class δ_Γ is not in the span of the classes δ_Ξ for $\Gamma \neq \Xi \in \kappa(g,r)$ (resp. $\Gamma \neq \Xi \in \kappa'(g,r)$), then δ_Γ is an extremal cycle of codimension r in $\overline{\mathcal{M}}_g$.*

Proof. If $\Gamma \in \kappa(g,r)$, then $e_\tau(\Delta_\Gamma) = 2r$. If $\Gamma \in \kappa'(g,r) - \kappa(g,r)$, then $e_\tau(\Delta_\Gamma) = 2r-1$. Conversely, if $Z \subset \overline{\mathcal{M}}_g$ is an irreducible variety of codimension r such that $e_\tau(Z) \geq 2r$, then $Z = \Delta_\Gamma$ for some $\Gamma \in \kappa(g,r)$. Similarly, if $e_\tau(Z) \geq 2r-1$, then $Z = \Delta_\Gamma$ for some $\Gamma \in \kappa'(g,r)$.

By Proposition 2.1, any effective linear combination expressing δ_Γ for $\Gamma \in \kappa(g,r)$ (resp. $\kappa'(g,r)$) must be of the form $\sum a_\Xi \delta_\Xi$ with $\Xi \in \kappa(g,r)$ (resp. $\kappa'(g,r)$). Since δ_Γ is not in the span of δ_Ξ for $\Gamma \neq \Xi \in \kappa(g,r)$ (resp. $\kappa'(g,r)$), all the coefficients except for a_Γ have to be zero. Therefore, δ_Γ is an extremal cycle of codimension r. □

In view of Lemma 5.5, it is interesting to determine when the classes δ_Γ are independent. Using test families, we give examples of independent classes, which by Lemma 5.5 are extremal.

Let $T_r(g) \subset \kappa(g,r)$ be the set of dual graphs whose underlying abstract graph is a tree with r leaves. If $\Gamma \in T_r(g)$, then the general point of Δ_Γ parameterizes curves with r components $C_1, \ldots, C_r$ each attached at one point to distinct points on a curve C_0. In addition, each of the curves C_i for $0 \leq i \leq r$ have genus at least 2. Let $T'_r(g) \subset \kappa'(g,r)$ be the set of dual graphs whose underlying abstract graph is a tree with r leaves and the central component C_0 has genus one. Let $T''_r(g) \subset \kappa'(g,r)$ be the set of dual graphs whose underlying abstract graph is a tree with r leaves, one of the components C_i, $i \neq 0$, has genus one and the genus g_0 of the central component satisfies $4g_0 + 9 \geq r$.

Let $L_{r-1,g_0}(g) \subset \kappa(g,r)$ denote the set of dual graphs whose underlying abstract graph consists of a vertex v_0 with a self-loop and $r-1$ leaves, where v_0 is assigned a curve of geometric genus g_0 satisfying $4g_0 + 9 \geq r$. If $\Gamma \in L_{r-1,g_0}(g)$, then the general point of Δ_Γ parameterizes curves with $r-1$ components $C_1, \ldots, C_{r-1}$ attached at one point to distinct smooth points on a one-nodal central curve C_0 (corresponding to v_0) of geometric genus g_0.

Theorem 5.6. *If Γ is a dual graph in $T_r(g) \cup T'_r(g) \cup T''_r(g) \cup L_{r-1,g_0}(g)$, then δ_Γ is independent from all the classes δ_Ξ for $\Gamma \neq \Xi \in \kappa'(g,r)$ and δ_Γ is an extremal codimension r class in $\overline{\mathcal{M}}_g$.*

Proof. Suppose there exists a linear relation

$$\sum_{\Xi \in \kappa'(g,r)} c_\Xi \delta_\Xi = 0 \tag{11}$$

among the classes with $\Xi \in \kappa'(g,r)$. Using test families, we will show that if $\Gamma \in T_r(g) \cup T'_r(g) \cup T''_r(g) \cup L_{r-1,g_0}(g)$, then the coefficient c_Γ in the linear relation has to be zero proving that the class δ_Γ is independent from the classes indexed by $\kappa'(g,r)$.

First, assume that $\Gamma \in T_r(g) \cup T'_r(g)$. Fix a general curve in Δ_Γ. Recall that the curve has r components $C_1, \ldots, C_r$ attached to a central component C_0. Let X be the r-dimensional test family obtained by varying the attachment point on the curves C_i for $1 \leq i \leq r$. Every curve in the family X has the same dual graph Γ. Hence, $[X] \cdot \delta_\Xi = 0$ for every $\Gamma \neq \Xi \in \kappa'(g,r)$. By [E, Lemma 3.5], the intersection number $[X] \cdot \delta_\Gamma = \prod_{i=1}^{r} (2 - 2g(C_i))$. Since $g(C_i) > 1$, $[X] \cdot \delta_\Gamma \neq 0$. Intersecting Relation (11), we conclude that $c_\Gamma = 0$. Hence, δ_Γ is independent of the classes δ_Ξ with $\Gamma \neq \Xi \in \kappa'(g,r)$ and by Lemma 5.5 is an extremal codimension r class.

Next, suppose $\Gamma \in L_{r-1,g_0}(g)$. A general point of Δ_Γ has $r-1$ curves $C_1, \ldots, C_{r-1}$ attached to a one-nodal curve C_0 of geometric genus g_0. Let P be a general pencil of curves of type $(2, g_0+2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Such a pencil has $4g_0+8$ base points. Let Y be the r-dimensional family obtained by varying C_0 in the pencil P and attaching $C_1, \ldots, C_{r-1}$ at distinct base-points of P along varying points on $C_1, \ldots, C_{r-1}$. We need the inequality $4g_0+9 \geq r$ to ensure that we can form this family. The family Y intersects the codimension r boundary components exactly when a member of the pencil becomes nodal. Hence, Y intersects the codimension r boundary stratum Δ_Γ and is disjoint from all other codimension r bounday strata. We conclude that $[Y] \cdot \delta_\Xi = 0$ for $\Gamma \neq \Xi \in \kappa'(g,r)$. By [E, Lemma 3.4], $[Y] \cdot \delta_\Gamma = (8g_0 + 12) \prod_{i=1}^{r-1} (1 - 2g(C_i))$, which is not zero since $g(C_i) \geq 2$ for $0 \leq i \leq r-1$. Intersecting Relation (11) with Y, we conclude that $c_\Gamma = 0$. Hence, δ_Γ is independent of the classes δ_Ξ with $\Gamma \neq \Xi \in \kappa'(g,r)$ and by Lemma 5.5 is an extremal codimension r class.

Finally, suppose that $\Gamma \in T''_r(g)$. A general point of Δ_Γ parameterizes a curve with r components $C_1, \dots, C_r$ attached to a central component C_0, where $4g(C_0) + 9 \geq r$. In addition, one of the components, say C_1, has genus one. Let Q be a general pencil of plane cubics. Let Z be the r-dimensional family obtained by attaching C_0 to the pencil Q at a base point and varying the points of attachments on $C_2, \dots, C_r$. The pencil Q has 12 nodal members. Since the geometric genus of the nodal curves in Q are zero, the only boundary strata δ_Ξ, with $\Xi \in \kappa'(g,r)$, that Z intersects are Δ_Γ and Δ_Ψ with $\Psi \in L_{r-1,g(C_0)}(g)$. By [E, Lemma 3.5], $[Z]\cdot\delta_\Gamma = (-1)\cdot\prod_{i=2}^{r}(2-2g(C_i)) \neq 0$. By the previous paragraph, the coefficients c_Ψ in the Relation (11) are zero. Hence, intersecting the relation with Z, we conclude that $c_\Gamma = 0$. Hence, δ_Γ is independent of the classes δ_Ξ with $\Gamma \neq \Xi \in \kappa'(g,r)$ and by Lemma 5.5 is an extremal codimension r class. This concludes the proof of the theorem. □

Remark 5.7. If $2r+2 \leq g$, then the sets $T_r(g), T'_r(g)$ and $T''_r(g)$ are non-empty. Hence, Theorem 5.6 gives examples of extremal cycles of arbitrarily high codimension. Note that the codimenions of these examples are $\leq \frac{g-2}{2}$, which is roughly one-sixth of the dimension of $\overline{\mathcal{M}}_g$. It would be interesting to construct extremal cycles of arbitrarily high codimension relative to the dimension of $\overline{\mathcal{M}}_g$.

Remark 5.8. By Remark 2.7, we further conclude that δ_Γ is extremal in the pseudoeffective cone.

As mentioned before, it would be interesting to find an extremal higher codimension cycle not contained in the boundary of $\overline{\mathcal{M}}_g$. Since for $g \geq 1$ any birational morphism from $\overline{\mathcal{M}}_{g,n}$ has its exceptional locus contained in the boundary ([GKM, Corollary 0.11]), one cannot directly apply Proposition 2.1. Nevertheless, we are able to find a non-boundary extremal codimension two cycle in $\overline{\mathcal{M}}_4$.

Let $H \subset \overline{\mathcal{M}}_4$ be the closure of the locus of hyperelliptic curves and let $GP \subset \overline{\mathcal{M}}_4$ be the closure of the Gieseker-Petri special curves, i.e. curves whose canonical embeddings are contained in a quadric cone in $\mathbb{P}^3$.

Theorem 5.9. *The cycle class of H is extremal in* $\mathrm{Eff}^2(\overline{\mathcal{M}}_4)$.

Proof. The proof relies on analyzing the canonical model of a genus four curve C. If C is 3-connected and non-hyperelliptic, then the canonical embedding of C is a complete intersection of a quadric surface Q and a cubic surface T in $\mathbb{P}^3$. If Q is smooth, consider the linear system $V = |\mathcal{O}(3,3)|$ on Q. Let $U \subset V$ be the open locus of smooth curves. Let $NI \subset V$ be the codimension one locus of nodal irreducible curves. Let $CP \subset V$ be the codimension two locus of irreducible curves with a cusp. Note that $V \backslash (U \cup NI \cup CP)$ has codimension three in V. Curves in $U \cup NI$ are stable and curves in CP are pseudostable. In addition, NI dominates Δ_0 under the moduli map.

If Q is a quadric cone with vertex v, let F_2 be the Hirzebruch surface obtained by blowing up v. Let E be the exceptional (-2)-curve with class e and let f be the ruling class of F_2. If $[C] \in GP$ is general, then C has class $3e+6f$ as a curve in F_2. Consider the linear system $V' = |3e+6f|$ on F_2. Let $U' \subset V'$ be the open locus of smooth curves. Let $NI'_1 \subset V'$ be the codimension one locus of irreducible nodal curves. Let $NI'_2 \subset V'$ be the codimension one locus of irreducible at-worst-nodal curves B of class $2e+6f$ union E, where B and E

intersect transversally at two distinct points. Let $NI'_3 \subset V'$ be the codimension two locus of irreducible at-worst-nodal curves B of class $2e+6f$ union E, where B and E are tangent at one point. Let $CP' \subset V'$ be the codimension two locus of irreducible curves with a cusp. Note that $V' \backslash (U' \cup NI'_1 \cup NI'_2 \cup NI'_3 \cup CP')$ has codimension three in V'. Curves in $U' \cup NI'_1$ are stable and curves in CP' are pseudostable. For a curve in NI'_2, its stabilization as a curve in Q has a node at v. For a curve in NI'_3, its pseudostabilization as a curve in Q has a cusp at v. In other words, blowing down E, curves in NI'_2 and in NI'_3 become stable and pseudostable, respectively.

Let h be the class of H. Suppose that

$$h = \sum_i a_i [Z_i] \in N^2(\overline{\mathcal{M}}_4) \tag{12}$$

for $a_i > 0$ and $Z_i \subset \overline{\mathcal{M}}_4$ irreducible subvariety of codimension two, not equal to H. Recall the first divisorial contraction $\mathrm{ps} : \overline{\mathcal{M}}_4 \to \overline{\mathcal{M}}_4^{\mathrm{ps}}$ induced by replacing an elliptic tail by a cusp for curves in Δ_1. Applying ps_*, we obtain that

$$\mathrm{ps}_* h = \sum_i a_i (\mathrm{ps}_*[Z_i]) \in N^2(\overline{\mathcal{M}}_4^{\mathrm{ps}}). \tag{13}$$

We first show that Z_i is contained in $GP \cup \Delta$ for all i. By rearranging the indices if necessary, we concentrate on Z_1. If $\mathrm{ps}_*[Z_1] = 0$, then $Z_1 \subset \Delta_1$. Hence, we may assume that $\mathrm{ps}_*[Z_1] \neq 0$ in (13). Suppose that Z_1 is not contained in $GP \cup \Delta$. Then a general curve C parameterized by Z_1 is smooth and its canonical embedding is contained in a smooth quadric surface. Take a two-dimensional subspace S in V such that S is spanned by C and two other general $(3,3)$-curves. Then S is disjoint from $V \backslash (U \cup NI \cup CP)$. Hence, any curve parameterized in S is pseudostable. Furthermore, since all these curves lie on a smooth quadric surface, the curves parameterized by S are not contained in $\mathrm{ps}(GP)$ and none of the curves are hyperelliptic. For a curve D parameterized in $\mathrm{ps}(Z_i)$, if D does not admit a canonical embedding contained in a smooth quadric, then $[D] \notin S$. On the other hand if a general curve parameterized by $\mathrm{ps}(Z_i)$ admits a canonical embedding contained in a smooth quadric, then $\mathrm{ps}(Z_i)$ corresponds to a locus of codimension ≥ 2 in V. Therefore, $[S] \cdot (\mathrm{ps}_* h) = 0$, $[S] \cdot (\mathrm{ps}_*[Z_1]) > 0$ and $[S] \cdot (\mathrm{ps}_*[Z_i]) \geq 0$ for $i \neq 1$. Plugging in (13) leads to a contradiction. We thus conclude that $Z_i \subset GP \cup \Delta$ for all i.

Next, we show that if $Z_i \neq H$, then it is contained in Δ. Again we concentrate on Z_1. Suppose that a general curve $[C] \in Z_1 \neq H$ is contained in GP but not in Δ. In particular, C is smooth and non-hyperelliptic. Consequently, its canonical model is a complete intersection of a singular quadric Q and a cubic T. Let B_1 and B_2 be a pencil of quadrics and a pencil of cubics containing Q and T, respectively, and otherwise general. Let $S = B_1 \times B_2$ be the two-dimensional family of canonical curves of genus four arising from the intersection of each pair of these quadrics and cubics. Based on the previous analysis of V and V', we can assume that all curves parameterized by S are pseudostable and none of them are hyperelliptic. Note that B_1 contains finitely many singular quadrics (including Q), hence S contains finitely many one-dimensional families of curves that are parameterized in GP. Moreover, these one-dimensional families are moving curves in GP by the construction of S. Similarly, the intersection of S with a boundary divisor consists of moving curves in the boundary. Hence, we can assume that S intersects each of the $\mathrm{ps}(Z_i)$ in at most finitely many points. It follows that $[S] \cdot (\mathrm{ps}_* h) = 0$, $[S] \cdot (\mathrm{ps}_*[Z_1]) > 0$ and $[S] \cdot (\mathrm{ps}_*[Z_i]) \geq 0$ for $i \neq 1$. Plugging in (13) leads to a contradiction. We thus conclude that if $Z_i \neq H$, then Z_i is contained in Δ.

Now it follows from (12) that $h = 0 \in A^2(\mathcal{M}_4)$, which contradicts the fact that h is a nonzero multiple of λ^2 in $\mathcal{M}_4$ ([Fab2]). □

6. The effective cones of $\overline{\mathcal{M}}_{0,n}$

In this section, we study effective cycles on $\overline{\mathcal{M}}_{0,n}$. Since $\dim \overline{\mathcal{M}}_{0,n} = n-3$, there are no interesting higher codimension cycles for $n \leq 6$. Hence, from now on we assume that $n \geq 7$.

Let S be a subset of $\{1,\ldots,n\}$ such that $|S|, |S^c| \geq 2$. To make the notation symmetric, denote by Δ_{S,S^c} the boundary divisor of $\overline{\mathcal{M}}_{0,n}$ parameterizing genus zero curves that have a node that separates the curve into two curves, one marked by S and the other by S^c. By definition, $\Delta_{S,S^c} = \Delta_{S^c,S}$. The codimension one boundary strata of $\overline{\mathcal{M}}_{0,n}$ consist of the divisors Δ_{S,S^c}, where the pair $\{S, S^c\}$ varies over subsets of $\{1,\ldots,n\}$ with $|S|, |S^c| \geq 2$.

Consider the codimension two boundary strata of $\overline{\mathcal{M}}_{0,n}$. Let S_1, S_2, S_3 be an *ordered* decomposition of $\{1,\ldots,n\}$ with $s_i = |S_i|$ such that $s_1 + s_2 + s_3 = n$, $s_1, s_3 \geq 2$ and $s_2 \geq 1$. Let D_{S_1,S_2,S_3} be the codimension two boundary stratum of $\overline{\mathcal{M}}_{0,n}$ whose general point parameterizes a chain of three smooth rational curves C_1, C_2, C_3 such that C_i is marked by S_i. By definition, $D_{S_1,S_2,S_3} = D_{S_3,S_2,S_1}$.

Theorem 6.1. *The cycle class of D_{S_1,S_2,S_3} is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{0,n})$.*

Proof. By Proposition 2.1, it suffices to exhibit morphisms f such that $e_f(D_{S_1,S_2,S_3}) > e_f(Z)$ for any subvariety Z of codimension two that is not D_{S_1,S_2,S_3}. We will use the morphisms $f_{\mathcal{A}} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,\mathcal{A}}$ introduced in Section 3.

First, suppose $s_1, s_3 > 2$. Let $\mathcal{A}$ be the weight parameter assigning $\frac{1}{s_1}$ to the marked points in S_1 and 1 to the other marked points. Then $e_{f_{\mathcal{A}}}(D_{S_1,S_2,S_3}) = s_1 - 2 > 0$. By Proposition 2.1, if $e_{f_{\mathcal{A}}}(Z) \geq s_1 - 2$, then Z has to be contained in Δ_{S_1,S_1^c}. Similarly, let $\mathcal{B}$ assign $\frac{1}{s_3}$ to the marked points in S_3 and 1 to the other marked points. Then $e_{f_{\mathcal{B}}}(D_{S_1,S_2,S_3}) = s_3 - 2 > 0$. If $e_{f_{\mathcal{B}}}(Z) \geq s_3 - 2$, then Z is contained in Δ_{S_3,S_3^c}. Since the intersection of Δ_{S_1,S_1^c} and Δ_{S_3,S_3^c} is exactly D_{S_1,S_2,S_3}, we conclude that D_{S_1,S_2,S_3} is extremal when $s_1, s_3 > 2$.

Next, suppose that $s_1 = s_3 = 2$. Since $n \geq 7$, we have $s_2 = n - 4 \geq 3$. Without loss of generality, let $S_1 = \{1,2\}$, $S_3 = \{3,4\}$ and $S_2 = \{5,\ldots,n\}$. Let $\mathcal{C}$ assign $\frac{1}{n-2}$ to the marked points in $S_1 \cup S_2$ and 1 to the marked points in S_3. We have $e_{f_{\mathcal{C}}}(D_{S_1,S_2,S_3}) = s_1 + s_2 - 3 = n-5$. If $e_{f_{\mathcal{C}}}(Z) \geq n-5$, then Z is contained in Δ_{S,S^c}, where $S \subset \{1,2,5,\ldots,n\}$ and $|S| = n-2$ or $n-3$. Similarly let $\mathcal{D}$ assign $\frac{1}{n-2}$ to the marked points in $S_3 \cup S_2$ and 1 to the marked points in S_1. We have $e_{f_{\mathcal{D}}}(D_{S_1,S_2,S_3}) = s_2 + s_3 - 3 = n-5$. If $e_{f_{\mathcal{D}}}(Z) \geq n-5$, then Z is contained in Δ_{T,T^c} where $T \subset \{3,4,5,\ldots,n\}$ and $|T| = n-2$ or $n-3$. When $|S| = |T| = n-2$, the intersection of Δ_{S,S^c} and Δ_{T,T^c} is exactly D_{S_1,S_2,S_3}. If $|S| = n-2$ and $|T| = n-3$, then the intersection of Δ_{S,S^c} and Δ_{T,T^c} is of the type $D_{S_1',S_2',S_3'}$ where $S_1' = \{3,4\}$ and $S_3' = \{1,2,i\}$ for $i \in S_2$. We have $e_{f_{\mathcal{D}}}(D_{S_1',S_2',S_3'}) = n-6 < e_{f_{\mathcal{D}}}(D_{S_1,S_2,S_3})$. Finally, if $|S| = |T| = n-3$, the intersection of Δ_{S,S^c} and Δ_{T,T^c} is of the type $D_{S_1',S_2',S_3'}$ where $S_1' = \{3,4,i\}$ and $S_3' = \{1,2,j\}$ for $i \neq j \in S_2$. We have $e_{f_{\mathcal{D}}}(D_{S_1',S_2',S_3'}) = n-6 < e_{f_{\mathcal{D}}}(D_{S_1,S_2,S_3})$. This proves that D_{S_1,S_2,S_3} is extremal when $s_1 = s_3 = 2$.

The case $s_1 = 2$ and $s_3 > 2$ can be verified by a similar (and simpler) argument as in the above paragraph, so we omit the details. □

Next, we give explicit examples of extremal higher codimension cycles in $\overline{\mathcal{M}}_{0,n}$. Let $S_1, \ldots, S_k$ be an *unordered* decomposition of $\{1,\ldots,n\}$ such that $k \geq 3$ and $|S_i| \geq 2$ for each i. Denote by $B_{S_1,\ldots,S_k}$ the subvariety of $\overline{\mathcal{M}}_{0,n}$ whose general point parameterizes a rational curve R attached to k rational tails $C_1,\ldots,C_k$ such that C_i is marked by S_i. The codimension of $B_{S_1,\ldots,S_k}$ in $\overline{\mathcal{M}}_{0,n}$ equals k.

Theorem 6.2. *The cycle class of $B_{S_1,\ldots,S_k}$ is extremal in $\mathrm{Eff}^k(\overline{\mathcal{M}}_{0,n})$.*

Proof. Suppose that $[B_{S_1,\ldots,S_k}] = \sum_{j=1}^r a_j[Z_j] \in N^k(\overline{\mathcal{M}}_{0,n})$, where $a_j > 0$ and $Z_j \subset \overline{\mathcal{M}}_{0,n}$ is an irreducible subvariety of codimension k. First, consider the case $|S_i| = s_i > 2$ for all i. Let $\mathcal{A}_i$ assign $\frac{1}{s_i}$ to the marked points in S_i and 1 to the other marked points. Then $e_{f_{\mathcal{A}_i}}(B_{S_1,\ldots,S_k}) = s_i - 2 > 0$. If $e_{f_{\mathcal{A}_i}}(Z) \geq s_i - 2$, then Z has to be contained in Δ_{S_i,S_i^c}. Hence by Proposition 2.1, Z_j is contained in $\Delta_{S_1,S_1^c} \cap \cdots \cap \Delta_{S_k,S_k^c}$ for all j. The intersection locus is exactly $B_{S_1,\ldots,S_k}$, which is irreducible. We thus conclude that $Z_j = B_{S_1,\ldots,S_k}$.

Next, consider the case $k \geq 4$ and some $s_i = 2$. Without loss of generality, assume that $S_i = \{2i-1, 2i\}$ for $1 \leq i \leq l$ and $s_{l+1},\ldots,s_k > 2$. Let $\overline{\mathcal{B}}_{0,n}$ be the moduli space of n-pointed genus zero curves with rational k-fold singularities (without unmarked components). A rational k-fold singularity is locally isomorphic to the intersection of the k concurrent coordinate axes in $\mathbb{A}^k$. There is a birational morphism $f : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{B}}_{0,n}$ contracting unmarked components X to a rational k-fold singularity, where $k = \#(X \cap \overline{C\backslash X}) \geq 3$, see e.g. [CC1, 3.7–3.11]. Note that $e_f(B_{S_1,\ldots,S_k}) = k-3 > 0$. By Proposition 2.1, we have $e_f(Z_j) \geq k-3$. Since an unmarked component with m nodes loses $(m-3)$-dimensional moduli under f, we conclude that Z_j has to be one of the $B_{T_1,\ldots,T_k}$. Using the morphism $f_{\mathcal{A}_i}$ in the previous paragraph for $i > l$, we further conclude that Z_j is of the type $B_{T_1,\ldots,T_l,S_{l+1},\ldots,S_k}$, where $T_1,\ldots,T_l$ yield a decomposition of $\{1,\ldots,2l\}$. Since $|T_i| \geq 2$, it implies that $|T_i| = 2$ for each i. Therefore, Z_j is the image of $B_{S_1,\ldots,S_k}$ under the automorphism of $\overline{\mathcal{M}}_{0,n}$ induced by relabeling $\{2i-1, 2i\}$ as T_i for each i. Note that the numerical classes of cycles in the orbit of $B_{T_1,\ldots,T_l,S_{l+1},\ldots,S_k}$ by relabeling the marked points are not all proportional, which can be easily checked using test families. Therefore, by symmetry of the marked points, any one of them cannot be a nonnegative linear combination of the others.

The remaining case is that $k = 3$ and some $s_i = 2$. Since $n \geq 7$, without loss of generality, assume that $S_1 = \{1,2\}$, $S_2 = \{3,\ldots,m\}$ and $S_3 = \{m+1,\ldots,n\}$ with $4 \leq m \leq n-3$ so that $s_3 = n-m > 2$. As above, we know that Z_j is contained in Δ_{S_3,S_3^c}. By [K, p. 549], we know that $N^2(\Delta_{S_3,S_3^c}) \to N^3(\overline{\mathcal{M}}_{0,n})$ is injective. Moreover, for any curve parameterized in Δ_{S_3,S_3^c}, there exists a unique node that separates the curve into two connected components, containing the marked points in S_3 and in S_3^c, respectively. It follows that $\Delta_{S_3,S_3^c} \cong \overline{\mathcal{M}}_{0,m+1} \times \overline{\mathcal{M}}_{0,n-m+1}$ (see also [K, p. 548]). Similarly B_{S_1,S_2,S_3} can be identified with $D_{S_1,\{p\},S_2} \times \overline{\mathcal{M}}_{0,n-m+1} \subset \Delta_{S_3,S_3^c} \cong \overline{\mathcal{M}}_{0,m+1} \times \overline{\mathcal{M}}_{0,n-m+1}$, where p denotes the unique node that separates the curve into two components, containing the marked points in S_3 and in S_3^c, respectively. By Theorem 6.1, $D_{S_1,\{p\},S_2}$ is an extremal codimension two cycle in $\overline{\mathcal{M}}_{0,m+1}$, hence B_{S_1,S_2,S_3} is an extremal codimension two cycle in Δ_{S_3,S_3^c} by Corollary 2.4. It follows that B_{S_1,S_2,S_3} is an extremal codimension three cycle in $\overline{\mathcal{M}}_{0,n}$ by Proposition 2.5. □

Finally, we point out that not all boundary strata are extremal cycles on moduli spaces of pointed stable curves.

Remark 6.3. Recall that $\widetilde{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n}/\mathfrak{S}_n$ is the moduli space of stable genus zero curves with n unordered marked points. The one-dimensional strata of $\widetilde{\mathcal{M}}_{0,n}$ (F-curves) correspond to partitions of n into four parts. For $n = 7$, there are three F-curves on $\widetilde{\mathcal{M}}_{0,7}$: $F_{1,1,1,4}$, $F_{1,1,2,3}$

and $F_{1,2,2,2}$. However, their numerical classes satisfy that $2[F_{1,1,2,3}] = [F_{1,1,1,4}] + [F_{1,2,2,2}]$ (see e.g. [M, Table 1]). In particular, $F_{1,1,2,3}$ is not extremal in the Mori cone of curves on $\widetilde{\mathcal{M}}_{0,7}$.

7. The effective cones of $\overline{\mathcal{M}}_{1,n}$

In this section, using the infinitely many extremal effective divisors in $\overline{\mathcal{M}}_{1,n-2}$ constructed in [CC2], we show that $\mathrm{Eff}^2(\overline{\mathcal{M}}_{1,n})$ is not finite polyhedral for $n \geq 5$.

Let $p_1, \ldots, p_n$ be the n marked points and let $T = \{p_{n-2}, p_{n-1}, p_n\}$. The gluing morphism

$$\widehat{\Delta}_{0;T} = \overline{\mathcal{M}}_{1,n-2} \times \overline{\mathcal{M}}_{0,4} \to \Delta_{0;T} \subset \overline{\mathcal{M}}_{1,n}$$

is induced by gluing a pointed genus one curve $(E, p_1, \ldots, p_{n-3}, q)$ to a pointed rational curve $(C, p_{n-2}, p_{n-1}, p_n, q)$ by identifying q in E and C to form a node. Denote by Γ_S (resp. Γ_0) the image of $\Delta_{0;S} \times \overline{\mathcal{M}}_{0,4}$ (resp. $\Delta_0 \times \overline{\mathcal{M}}_{0,4}$) in $\overline{\mathcal{M}}_{1,n}$ for $S \subset \{p_1, \ldots, p_{n-3}, q\}$ and $|S| \geq 2$. Let Γ be the image of $\overline{\mathcal{M}}_{1,n-2} \times \Delta_{0;\{p_{n-1},p_n\}}$. Note that Γ_0, Γ_S and Γ are the codimension two boundary strata of $\overline{\mathcal{M}}_{1,n}$ whose general point parameterizes a two-nodal curve that has a rational tail marked by T. The cases $q \in S$ and $q \notin S$ correspond to the middle component having genus zero and one, respectively.

Since $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, it follows that $A^1(\widehat{\Delta}_{0;T}) \cong N^1(\widehat{\Delta}_{0;T})$, generated by the classes of Γ_S, Γ_0 and Γ.

Lemma 7.1. *The cycle classes of Γ_S, Γ_0 and Γ are independent in $N^2(\overline{\mathcal{M}}_{1,n})$.*

Proof. Denote by γ_S, γ_0 and γ the classes of Γ_S, Γ_0 and Γ, respectively. Suppose that they satisfy a relation

$$a \cdot \gamma_0 + \sum_S b_S \cdot \gamma_S + c \cdot \gamma = 0 \tag{14}$$

in $N^2(\overline{\mathcal{M}}_{1,n})$. Below we will show that all the coefficients in (14) are zero.

Let $\mathcal{A}$ be the weight parameter assigning $\frac{1}{3}$ to p_{n-2}, p_{n-1}, p_n and 1 to p_k for $k \leq n-3$. Then $e_{f_\mathcal{A}}(\Gamma) = 0$ and $e_{f_\mathcal{A}}(\Gamma_0) = e_{f_\mathcal{A}}(\Gamma_S) = 1$. Applying $f_{\mathcal{A}*}$ to (14), we conclude that $c = 0$. Relation (14) reduces to

$$a \cdot \gamma_0 + \sum_S b_S \cdot \gamma_S = 0. \tag{15}$$

Let $\psi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-2}$ be the morphism forgetting p_1 and p_2. We have $e_\psi(\Gamma_{\{p_1,p_2\}}) = 0$ and $e_\psi(\Gamma_0), e_\psi(\Gamma_S) > 0$ for $S \neq \{p_1, p_2\}$. Applying ψ_* to (15), we conclude that $b_{\{p_1,p_2\}} = 0$, hence by symmetry $b_S = 0$ for $S = \{p_i, p_j\}$ for $i, j \leq n-3$. Next, let $\phi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-1}$ be the morphism forgetting p_1. Among the remaining cycles, $e_\phi(\Gamma_{\{p_1,q\}}) = 0$ and $e_\phi(\Gamma_0) = e_\phi(\Gamma_S) > 0$ for $S \neq \{p_1, q\}$. Applying ϕ_*, we conclude that $b_{\{p_1,q\}} = 0$, hence by symmetry $b_{\{p_i,q\}} = 0$ for all $i \leq n-3$. In sum, we obtain that $b_S = 0$ for all $|S| = 2$.

Apply induction on $|S|$. Suppose that $b_S = 0$ for all $|S| \leq k-1$. Relation (15) reduces to

$$a \cdot \gamma_0 + \sum_{|S| \geq k} b_S \cdot \gamma_S = 0. \tag{16}$$

Let $\varphi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-k}$ be the morphism forgetting $p_1, \ldots, p_k$. Then we have $e_\varphi(\Gamma_{\{p_1,\ldots,p_k\}}) = k-2$ and $e_\varphi(\Gamma_0), e_\varphi(\Gamma_S) > k-2$ for $|S| \geq k$ and $S \neq \{p_1, \ldots, p_k\}$. Take an ample divisor class A in $\overline{\mathcal{M}}_{1,n}$, intersect (16) with A^{k-2} and apply φ_*. We obtain that $b_{\{p_1,\ldots,p_k\}} = 0$, hence by symmetry $b_S = 0$ for $S \subset \{p_1, \ldots, p_{n-3}\}$ and $|S| = k$. Next, let $\eta : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-k+1}$ be the morphism forgetting $p_1, \ldots, p_{k-1}$. Among the remaining cycles, $e_\eta(\Gamma_{\{p_1,\ldots,p_{k-1},q\}}) = k-2$ and $e_\eta(\Gamma_0), e_\eta(\Gamma_S) \geq k-1$ for $S \neq \{p_1, \ldots, p_{k-1}, q\}$. Intersecting with A^{k-2} and applying η_*, we obtain that $b_{\{p_1,\ldots,p_{k-1},q\}} = 0$, hence $b_S = 0$ for all $|S| = k$. By induction we thus conclude that $b_S = 0$ for all S.

Finally, Relation (16) reduces to $a \cdot \gamma_0 = 0$, hence $a = 0$. □

Theorem 7.2. *For $n \geq 5$, $\mathrm{Eff}^2(\overline{\mathcal{M}}_{1,n})$ is not finite polyhedral.*

Proof. Let $\mathcal{A}$ be the weight parameter assigning $\frac{1}{3}$ to marked points in $T = \{p_{n-2}, p_{n-1}, p_n\}$ and 1 to the other marked points. The exceptional locus of $f_{\mathcal{A}} : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,\mathcal{A}}$ is $\Delta_{0;T}$. By Corollary 2.4, any extremal effective divisor D' on $\overline{\mathcal{M}}_{1,n-2}$ pulls back to an extremal divisor D on $\widehat{\Delta}_{0;T} = \overline{\mathcal{M}}_{1,n-2} \times \overline{\mathcal{M}}_{0,4}$. Moreover, $e_{f_{\mathcal{A}}}(D) > 0$ because the moduli of the rational tail marked by T is forgotten under $f_{\mathcal{A}}$. By Lemma 7.1, we can apply Proposition 2.5 to conclude that the class of D is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{1,n})$. Now the claim follows from the fact that there are infinitely many extremal effective divisors on $\overline{\mathcal{M}}_{1,n-2}$ for every $n \geq 5$ ([CC2]). □

8. The effective cones of $\overline{\mathcal{M}}_{2,n}$

Applying the same idea as in Section 7, in this section we show that $\mathrm{Eff}^2(\overline{\mathcal{M}}_{2,n})$ is not finite polyhedral for $n \geq 2$.

Let $p_1, \ldots, p_n$ be the n marked points. The gluing morphism

$$\widehat{\Delta}_{1;\emptyset} = \overline{\mathcal{M}}_{1,n+1} \times \overline{\mathcal{M}}_{1,1} \to \Delta_{1;\emptyset} \subset \overline{\mathcal{M}}_{2,n}$$

is induced by gluing two pointed genus one curves $(E_1, p_1, \ldots, p_n, q)$ and (E_2, q) by identifying q in both curves to form a node. Denote by Γ_S (resp. Γ_0) the image of $\Delta_{0;S} \times \overline{\mathcal{M}}_{1,1}$ (resp. $\Delta_0 \times \overline{\mathcal{M}}_{1,1}$) in $\overline{\mathcal{M}}_{2,n}$ for $S \subset \{1, \ldots, n+1\}$ and $|S| \geq 2$. Let Γ be the image of $\overline{\mathcal{M}}_{1,n+1} \times \Delta_0$. Note that Γ_0, Γ_S and Γ are the codimension two boundary strata of $\overline{\mathcal{M}}_{2,n}$ whose general point parameterizes a curve with two nodes and an unmarked tail of arithmetic genus one. The cases $q \in S$ and $q \notin S$ correspond to the middle component having genus zero and one, respectively.

Since $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$, it follows that $A^1(\widehat{\Delta}_{1;\emptyset}) \cong N^1(\widehat{\Delta}_{1;\emptyset})$, generated by the classes of Γ_S, Γ_0 and Γ.

Lemma 8.1. *The cycle classes of Γ_S, Γ_0 and Γ are independent in $N^2(\overline{\mathcal{M}}_{2,n})$.*

Proof. Denote by γ_S, γ_0 and γ the classes of Γ_S, Γ_0 and Γ in $\overline{\mathcal{M}}_{2,n}$, respectively. Suppose that they satisfy

$$a \cdot \gamma_0 + \sum_S b_S \cdot \gamma_S + c \cdot \gamma = 0. \tag{17}$$

Recall that $\mathrm{ps} : \overline{\mathcal{M}}_{2,n} \to \overline{\mathcal{M}}_{2,n}^{\mathrm{ps}}$ contracts $\Delta_{1;\emptyset}$, replacing an elliptic tail by a cusp. Hence we have $e_{\mathrm{ps}}(\Gamma) = 0$ and $e_{\mathrm{ps}}(\Gamma_0) = e_{\mathrm{ps}}(\Gamma_S) = 1$ for all S. Applying ps_* to (17), we conclude that $c = 0$. Now the same induction procedure in the proof of Lemma 7.1 implies that $a = b_S = 0$ for all S. □

Theorem 8.2. *For $n \geq 2$, $\mathrm{Eff}^2(\overline{\mathcal{M}}_{2,n})$ is not finite polyhedral.*

Proof. The exceptional locus of $\mathrm{ps} : \overline{\mathcal{M}}_{2,n} \to \overline{\mathcal{M}}^{\mathrm{ps}}_{2,n}$ is $\Delta_{1;\emptyset}$. By Corollary 2.4, any extremal effective divisor D' on $\overline{\mathcal{M}}_{1,n+1}$ pulls back to an extremal divisor D on $\widehat{\Delta}_{1;\emptyset} = \overline{\mathcal{M}}_{1,n+1} \times \overline{\mathcal{M}}_{1,1}$. Moreover, $e_{\mathrm{ps}}(D) > 0$ because the unmarked genus one tail is forgotten under ps. By Lemma 8.1, we can apply Proposition 2.5 to conclude that the class of D is extremal in $\mathrm{Eff}^2(\overline{\mathcal{M}}_{2,n})$. Now the claim follows from the fact that there are infinitely many extremal effective divisors on $\overline{\mathcal{M}}_{1,n+1}$ for every $n \geq 2$ ([CC2]). □

Remark 8.3. Since $N^1(\widehat{\Delta}_1) \cong A^1(\widehat{\Delta}_1) \to N^2(\overline{\mathcal{M}}_g)$ is injective and $\widehat{\Delta}_1 \cong \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,1}$, the same proof as that of Theorem 8.2 implies that the pullback of any extremal effective divisor from $\overline{\mathcal{M}}_{g-1,1}$ to $\widehat{\Delta}_1$ gives rise to an extremal codimension two cycle in $\overline{\mathcal{M}}_g$. For instance, the divisor W of Weierstrass points is known to be extremal in $\overline{\mathcal{M}}_{g-1,1}$ for $3 \leq g \leq 6$ (see e.g. [C, Theorem 4.3 and Remark 4.4]). Hence W yields an extremal codimension two cycle in $\mathrm{Eff}^2(\overline{\mathcal{M}}_g)$ for $3 \leq g \leq 6$.

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