EXTREMAL EFFECTIVE DIVISORS ON $\overline{\mathcal{M}}_{1,n}$

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Abstract. For every $n \geq 3$, we exhibit infinitely many extremal effective divisors on $\overline{\mathcal{M}}_{1,n}$, the Deligne-Mumford moduli space parameterizing stable genus one curves with n ordered marked points.

Contents

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli space of stable genus g curves with n ordered marked points. Understanding the cone of pseudo-effective divisors $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{g,n})$ is a central problem in the birational geometry of $\overline{\mathcal{M}}_{g,n}$. Since the 1980s, motivated by the problem of determining the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$, many authors have constructed families of effective divisors on $\overline{\mathcal{M}}_{g,n}$. For example, Harris, Mumford and Eisenbud [HMu, H, EH], using Brill-Noether and Gieseker-Petri divisors showed that $\overline{\mathcal{M}}_g$ is of general type for $g > 23$. Using Kozsul divisors, Farkas [F] extended this result to $g = 22$. Logan [Lo], using generalized Brill-Noether divisors, obtained similar results for the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ when $n > 0$.

Although we know many examples of effective divisors on $\overline{\mathcal{M}}_{g,n}$, the structure of the pseudo-effective cone $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{g,n})$ remains mysterious in general. Inspired by the work of Keel and Vermeire [V] on $\overline{\mathcal{M}}_{0,6}$, Castravet and Tevelev [CT1] constructed a sequence of non-boundary extremal effective divisors on $\overline{\mathcal{M}}_{0,n}$ for $n \geq 6$. For higher genera, Farkas and Verra [FV1, FV2] showed that certain variations of pointed Brill-Noether divisors are extremal on $\overline{\mathcal{M}}_{g,n}$ for $g-2 \leq n \leq g$. However, for fixed g and n, these constructions yield only finitely many extremal divisors. This raises the question whether there exist g and n such that $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{g,n})$ is not finitely generated.

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Motivated by this question, in this paper we study the moduli space of genus one curves with n ordered marked points and show that their effective cones are not finitely generated when $n \geq 3$.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a collection of n integers satisfying $\sum_{i=1}^n a_i = 0$, not all equal to zero. Define $D_{\mathbf{a}}$ in $\overline{\mathcal{M}}_{1,n}$ as the closure of the divisorial locus parameterizing smooth genus one curves with n ordered marked points $(E; p_1, \ldots, p_n)$ such that $\sum_{i=1}^n a_i p_i = 0$ in the Jacobian of E, which is E itself. Call $\mathbf{a} = (a_1, \ldots, a_n)$ the weights of $D_{\mathbf{a}}$. By definition, $D_{\mathbf{a}}$ is the same divisor as $D_{-\mathbf{a}}$, where $-\mathbf{a} = (-a_1, \ldots, -a_n)$. The number of irreducible components of $D_{\mathbf{a}}$ depends on $\mathbf{a}$. In Section 3.2, we will show that for $n \geq 3$, $D_{\mathbf{a}}$ is irreducible if and only if $\gcd(a_1, \ldots, a_n) = 1$.

A Mori dream space is a $\mathbb{Q}$-factorial projective variety X such that the Cox ring of X is finitely generated and the Néron-Severi space is equal to $\mathrm{Pic}(X) \otimes \mathbb{Q}$. The notion was introduced by Hu and Keel in [HK]. Mori dream spaces are the simplest varieties from the point of view of the minimal model program (MMP); one can run MMP for every effective divisor. Their nef cones coincide with their semi-ample cones and their effective cones are rational polyhedral [HK]. By exhibiting nef line bundles that are not semi-ample, Keel [K, Corollary 3.1] observed that $\overline{\mathcal{M}}_{g,n}$ cannot be a Mori dream space if $g \geq 3$ and $n \geq 1$. In a recent work [CT2], Castravet and Tevelev showed that $\overline{\mathcal{M}}_{0,n}$ is not a Mori dream space for $n \geq 134$. Our main theorem shows that the effective cone of $\overline{\mathcal{M}}_{1,n}$ is not finite polyhedral when $n \geq 3$ and, thus, $\overline{\mathcal{M}}_{1,n}$ is very far from being a Mori dream space. This provides further contrary evidence to expectations that the cones of divisors of $\overline{\mathcal{M}}_{g,n}$ are well-behaved.

We summarize our main result as follows.

Theorem 1.1. *Suppose that $n \geq 3$ and $\gcd(a_1, \ldots, a_n) = 1$. Then $D_{\mathbf{a}}$ is an extremal and rigid effective divisor on $\overline{\mathcal{M}}_{1,n}$. Moreover, these $D_{\mathbf{a}}$'s yield infinitely many extremal rays for $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$. Consequently, $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$ is not finite polyhedral and $\overline{\mathcal{M}}_{1,n}$ is not a Mori dream space.*

We prove that $D_{\mathbf{a}}$ is extremal by exhibiting irreducible curves C whose deformations are Zariski dense in $D_{\mathbf{a}}$ and satisfy $C \cdot D_{\mathbf{a}} < 0$.

The divisor class of $D_{\mathbf{a}}$ was first calculated by Hain [Ha, Theorem 12.1] using normal functions. The restriction of this class to the locus of curves with rational tails was worked out by Cavalieri, Marcus and Wise [CMW] using Gromov-Witten theory. Two other proofs were recently obtained by Grushevsky and Zakharov [GZ] and by Müller [M]. We remark that all of them considered more general cycle classes in $\mathcal{M}_{g,n}$ for $g \geq 1$, by pulling back the zero section of the universal Jacobian or the Theta divisor of the universal Picard variety of degree $g-1$. Pagani [P2, Lemma 3] calculated an extension of this divisor to the universal curve over $\mathcal{M}_{1,n-1}$.

The symmetric group $\mathfrak{S}_n$ acts on $\overline{\mathcal{M}}_{1,n}$ by permuting the labeling of the marked points. Denote the quotient, which parameterizes genus one curves with n unordered marked points, by $\widetilde{\mathcal{M}}_{1,n} = \overline{\mathcal{M}}_{1,n}/\mathfrak{S}_n$. In contrast to Theorem 1.1, in the last section, we show that $\overline{\mathrm{Eff}}(\widetilde{\mathcal{M}}_{1,n})$ is finitely generated. In fact, following an argument of Keel and M$^{\mathrm{c}}$Kernan [KM], we prove the following result.

Theorem 5.1. *The effective cone of $\widetilde{\mathcal{M}}_{1,n}$ is the closed, simplicial cone generated by the boundary divisors.*

Let G be a subgroup of $\mathfrak{S}_n$. Consider the action of G on $\overline{\mathcal{M}}_{1,n}$ by permuting the labelings of the marked points and denote the quotient space by $\overline{\mathcal{M}}_{1,n}/G$. If there are infinitely many extremal divisors $D_{\mathbf{a}}$, that are pairwise non-proportional and invariant under the action of G, then $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n}/G)$ is not finitely generated. For example, let $n = 6$ and let G be the subgroup of $\mathfrak{S}_6$ generated by three transpositions (12), (34) and (56). Then for any $a + b + c = 0$, the divisor $D_{(a,a,b,b,c,c)}$ descends to a well-defined divisor $D^G_{(a,a,b,b,c,c)}$ on $\overline{\mathcal{M}}_{1,6}/G$. Moreover, if $\gcd(a, b, c) = 1$ and a, b, c are nonzero, then $D^G_{(a,a,b,b,c,c)}$ is irreducible and extremal on $\overline{\mathcal{M}}_{1,6}/G$. More generally, as pointed out by the referee, our techniques show the following result.

Theorem 5.2. *Let $G \subset \mathfrak{S}_n$ be a subgroup whose permutation action on the set $\{1, \ldots, n\}$ has at least three orbits. Then $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n}/G)$ is not finitely generated.*

It would be interesting to classify all subgroups $G \subset \mathfrak{S}_n$ for which $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n}/G)$ is not finitely generated. When the action of G on $\{1, \ldots, n\}$ has fewer than three orbits, we do not know such a classification.

This paper is organized as follows. In Section 2, we review the divisor theory of $\overline{\mathcal{M}}_{1,n}$. In Section 3, we discuss the geometry of $D_{\mathbf{a}}$, including its divisor class and irreducible components. In Section 4, we prove our main result Theorem 1.1. In Section 5, we study effective divisors on $\overline{\mathcal{M}}_{1,n}/G$ and prove Theorems 5.1 and 5.2. Finally, in the appendix, we analyze the singularities of $\overline{\mathcal{M}}_{1,n}$ and show that a canonical form defined on its smooth locus extends holomorphically to an arbitrary resolution.

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2. Preliminaries on $\overline{\mathcal{M}}_{1,n}$

In this section, we recall basic facts concerning the geometry of $\overline{\mathcal{M}}_{1,n}$. We refer the reader to [AC, BF, S] for the facts quoted below.

Let λ be the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{1,n}$. Let δ_{irr} be the divisor class of the locus in $\overline{\mathcal{M}}_{1,n}$ that parameterizes curves with a non-separating node. The general point of δ_{irr} parameterizes a rational nodal curve with one node and n ordered marked points. Let S be a subset of $\{1, \ldots, n\}$ with cardinality $|S| \geq 2$ and let S^c denote its complement. Let $\delta_{0;S}$ denote the divisor class of the locus in $\overline{\mathcal{M}}_{1,n}$ parameterizing curves with a node that separates the curve into a stable genus zero curve marked by S and a stable genus one curve marked by S^c. In addition, let ψ_i be the first Chern class of the cotangent bundle on $\overline{\mathcal{M}}_{1,n}$ associated to the ith marked point for $1 \leq i \leq n$. Here we consider the divisor classes on the moduli stack instead of the coarse moduli scheme, see e.g. [HMo, Section 3.D] for more details.

The rational Picard group of $\overline{\mathcal{M}}_{1,n}$ is generated by λ and $\delta_{0;S}$ for all $|S| \geq 2$. The divisor classes δ_{irr} and ψ_i can be expressed in terms of the generators as

$$\delta_{\mathrm{irr}} = 12\lambda,$$

$$\psi_i = \lambda + \sum_{i \in S} \delta_{0;S}.$$

Since $\delta_{\rm irr}$ and λ are proportional, we will use them interchangeably in calculations depending on convenience. The canonical class of $\overline{\mathcal{M}}_{1,n}$ is

$$K_{\overline{\mathcal{M}}_{1,n}} = (n-11)\lambda + \sum_{|S|\geq 2} (|S|-2)\delta_{0;S}.$$

For $n \leq 10$, $\overline{\mathcal{M}}_{1,n}$ is rational [Be, Theorem 1.0.1]. Moreover, the Kodaira dimension of $\overline{\mathcal{M}}_{1,11}$ is zero and the Kodaira dimension of $\overline{\mathcal{M}}_{1,n}$ for $n \geq 12$ is one [BF, Theorem 3].[1]

3. Geometry of $D_{\mathbf{a}}$

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a sequence of integers, not all equal to zero, such that $\sum_{i=1}^n a_i = 0$. The divisor $D_{\mathbf{a}}$ in $\overline{\mathcal{M}}_{1,n}$ is defined as the closure of the locus parameterizing smooth genus one curves E with n distinct marked points $p_1, \ldots, p_n$ satisfying $\sum_{i=1}^n a_i p_i = 0$ in $\mathrm{Jac}(E) = E$. Equivalently, let $\mathcal{J}$ denote the universal Jacobian and let η be the zero section of $\mathcal{J}$. We have a map $F_{\mathbf{a}} : \mathcal{M}_{1,n} \to \mathcal{J}$ given by

$$(E; p_1, \ldots, p_n) \mapsto \mathcal{O}_E\Big(\sum_{i=1}^n a_i p_i\Big).$$

Then $D_{\mathbf{a}}$ is the closure of the pullback $F_{\mathbf{a}}^*\eta$. Call $\mathbf{a} = (a_1, \ldots, a_n)$ the weights of $D_{\mathbf{a}}$. By definition, $D_{\mathbf{a}} = D_{-\mathbf{a}}$, where $-\mathbf{a} = (-a_1, \ldots, -a_n)$. Define the subset of indices corresponding to nonzero entries of $\mathbf{a}$ as

$$N = \{1 \leq i \leq n \mid a_i \neq 0\}.$$

The divisor class of $D_{\mathbf{a}}$ was first calculated by Hain [Ha, Theorem 12.1]. About the same time, several other calculations of the divisor class were carried out in [CMW, GZ, M]. We remark that our setting is slightly different from *loc. cit.* The morphism $F_{\mathbf{a}}$ extends naturally as a morphism $F'_{\mathbf{a}}$ to the locus $\mathcal{M}^{\rm ct}_{1,n}$ parameterizing curves of compact type in $\overline{\mathcal{M}}_{1,n}$ (see [Ha]). Let E_0 be the elliptic component of a curve of compact type $(E, p_1, \ldots, p_n)$ in $\overline{\mathcal{M}}_{1,n}$. Let $q_1, \ldots, q_r$ be the points on E_0 corresponding to the nodes of E and let Q_i be the set of indices of the marked points contained in the rational tail of E attached to E_0 at q_i. Then

$$F'_{\mathbf{a}}(E; p_1, \ldots, p_n) = \Big(\sum_{s \notin \cup_{i=1}^r Q_i} a_s p_s\Big) + \sum_{i=1}^r \Big(\sum_{s \in Q_i} a_s\Big) q_i \in \mathcal{J}(E_0) = \mathcal{J}(E).$$

Consequently, $F'^*_{\mathbf{a}}\eta$ contains boundary divisors $\delta_{0;S}$ for all $S \supset N$, because the condition $\sum_{i=1}^n a_i p_i = 0$ automatically holds by the assumption $\sum_{i\in N} a_i = 0$ when p_i's coincide in E_0 for $i \in N$. On the other hand, it does not contain any of the other boundary divisors $\delta_{0;S}$ for $S \not\supset N$. Hence, in $\mathcal{M}^{\rm ct}_{1,n} \setminus \cup_{S \supset N} \delta_{0;\{S\}}$, the two divisors $D_{\mathbf{a}}$ and $F'^*_{\mathbf{a}}\eta$ agree. In contrast, they disagree along $\delta_{0;S}$ for $S \supset N$, since $D_{\mathbf{a}}$ does not contain any boundary divisor by definition.

[1]In order to study the Kodaira dimension of a singular variety, one needs to ensure that a canonical form defined in its smooth locus extends holomorphically to a resolution. Farkas informed the authors that such a verification for $\overline{\mathcal{M}}_{1,n}$ seems not to be easily accessible in the literature. Although the Kodaira dimension of $\overline{\mathcal{M}}_{1,n}$ is irrelevant for our results, we will treat this issue in the appendix by a standard argument based on the Reid-Tai criterion.

In particular, if all a_i's are nonzero, $D_{\mathbf{a}}$ and $F'^{*}_{\mathbf{a}}\eta$ differ by $\delta_{0;\{1,\dots,n\}}$. This issue was already observed by Cautis [Ca, Proposition 3.4.7] for the case $n = 2$. In order to clarify this distinction and for the convenience of the reader, in what follows we will carry out a direct calculation for the class of $D_{\mathbf{a}}$, and then verify that it matches with *loc. cit.* after adding $\delta_{0;S}$ for each $S \supset N$.

3.1. **Divisor class of $D_{\mathbf{a}}$.** Take a general one-dimensional family $\pi : \mathcal{C} \to B$ of genus one curves with n sections $\sigma_1, \dots, \sigma_n$ such that every fiber contains at most one node and the total space of the family is smooth. Suppose there are d_S fibers in which the sections labeled by S intersect simultaneously and pairwise transversally. Let d_{irr} be the number of rational nodal fibers. Let ω be the first Chern class of the relative dualizing sheaf associated to π and η the locus of nodes in $\mathcal{C}$. Then the following formulae are standard [HMo]:

$$\pi_*\eta = d_{\mathrm{irr}} + \sum_S d_S,$$

$$\omega^2 = -\sum_S d_S,$$

$$\sigma_i \cdot \sigma_j = \sum_{\{i,j\}\subset S} d_S,$$

$$\omega \cdot \sigma_i = -\sigma_i^2 = B \cdot \psi_i - \sum_{i \in S} d_S = \frac{1}{12} d_{\mathrm{irr}}.$$

Suppose $D_{\mathbf{a}}$ has class

$$D_{\mathbf{a}} = c_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{|S|\geq 2} c_S \delta_{0;\{S\}}$$

with unknown coefficients c_{irr} and c_S, where we have used the basis including δ_{irr} instead of λ for convenience. By [GZ, page 11], the zero section of $\mathcal{J}$ vanishes along the boundary divisors $\delta_{0;S}$ for $S \supset N$ with multiplicity one. Applying the Grothendieck-Riemann-Roch formula to the push-forward of the section $\sum_{i=1}^n a_i\sigma_i$, we conclude that

$$\begin{aligned} B \cdot D_{\mathbf{a}} + \sum_{S \supset N} d_S &= c_1\Big(R^1\pi_* \sum_{i=1}^n a_i \sigma_i\Big) \\ &= -\pi_*\bigg(\Big(1 + \sum_{i=1}^n a_i\sigma_i + \frac{1}{2}\Big(\sum_{i=1}^n a_i\sigma_i\Big)^2\Big)\Big(1 - \frac{\omega}{2} + \frac{\omega^2+\eta}{12}\Big)\bigg) \\ &= -\frac{1}{12}d_{\mathrm{irr}} + \frac{1}{24}\Big(\sum_{i=1}^n a_i^2\Big) d_{\mathrm{irr}} - \sum_S \sum_{\{i,j\}\subset S} a_i a_j d_S. \end{aligned}$$

By comparing coefficients on both sides of the equation, we obtain that

$$12 c_{\mathrm{irr}} = -1 + \frac{1}{2}\sum_{i=1}^n a_i^2,$$

$$c_S = -\sum_{\{i,j\}\subset S} a_i a_j, \ S \not\supset N,$$

$$c_S = -\sum_{1\leq i<j\leq n} a_i a_j - 1 = -1 + \frac{1}{2}\sum_{i=1}^n a_i^2, \ S \supset N,$$

where the last equality uses the assumption $\sum_{i=1}^n a_i = 0$. Hence, we conclude the following formula.

Proposition 3.1. *The divisor class of $D_{\mathbf{a}}$ is given by*

$$D_{\mathbf{a}} = \Big(-1 + \frac{1}{2}\sum_{i=1}^n a_i^2 \Big)\Big(\lambda + \sum_{S\supset N} \delta_{0;S} \Big) - \sum_{S\not\supset N} \Big(\sum_{\{i,j\}\subset S} a_i a_j \Big) \delta_{0;S}.$$

Therefore, adding $\delta_{0;S}$ for each $S \supset N$ to $D_{\mathbf{a}}$, we recover the divisor class calculated in [Ha, Theorem 12.1]. As explained before, the coefficient of $\delta_{0;S}$ for $S \supset N$ in our calculation differs by 1 from that in [Ha, CMW, GZ, M], because the pullback of the zero section of $\mathcal{J}$ contains such $\delta_{0;S}$ in $\mathcal{M}_{1,n}^{\mathrm{ct}}$, but in our setting we do not treat it as a component of $D_{\mathbf{a}}$.

3.2. **Irreducible components of $D_{\mathbf{a}}$.** The divisor $D_{\mathbf{a}}$ is not always irreducible. For instance for $D_{(4,-4)}$ on $\overline{\mathcal{M}}_{1,2}$, the condition is $4p_1 - 4p_2 = 0$. There are two possibilities, $2p_1 - 2p_2 = 0$ and $2p_1 - 2p_2 \neq 0$, each yielding a component for $D_{(4,-4)}$. In general for $n \geq 3$, if $\gcd(a_1, \ldots, a_n) = 1$, then $D_{\mathbf{a}}$ is irreducible. If $\gcd(a_1, \ldots, a_n) > 1$, then $D_{\mathbf{a}}$ contains more than one component. Below we will prove this statement and calculate the divisor class of each irreducible component.

First, consider the special case $n = 2$. Let $\eta(d)$ denote the number of positive integers that divide d.

Proposition 3.2. *Suppose a is an integer bigger than one. Then the divisor $D_{(a,-a)}$ in $\overline{\mathcal{M}}_{1,2}$ consists of $\eta(a) - 1$ irreducible components.*

Proof. By definition, $D_{(a,-a)}$ is the closure of the locus parameterizing $(E; p_1, p_2)$ such that $p_2 - p_1$ is a nonzero a-torsion. Take the square $[0, a] \times [0, ai]$ and glue its parallel edges to form a torus E. Fix p_1 as the origin of E. The number of a-torsion points p_2 is equal to a^2 and the coordinates (x, y) of each a-torsion point satisfy $x, y \in \mathbb{Z}/a$.

When varying the lattice structure of E, the monodromy group acts on (x, y). Suppose we fix the horizontal edge and shift the vertical edge to the right until we obtain a parallelogram spanned by $[0, a] \times [0, a(1 + i)]$. The resulting torus is isomorphic to E. Consequently the monodromy action sends an a-torsion point (x, y) to $(x + y, y)$. Similarly, we may also obtain the action sending (x, y) to $(x, x + y)$. Then each orbit of the monodromy action is uniquely determined by $k = \gcd(x, y, a)$. In other words, the monodromy is transitive on the primitive a'-torsion points for each divisor a' of a, where $a' = a/k$. Hence, the number of its orbits is $\eta(a)$. Moreover, for $a' \neq a''$, the loci of primitive a'-torsion points and primitive a''-torsion points are disjoint. Therefore, each of the $\eta(a)$ orbits gives rise to an irreducible component parameterizing (E, p_1, p_2) such that $p_2 - p_1$ is a primitive a'-torsion, where a is divisible by a'. When $a' = 1$, i.e., $p_2 = p_1$, the corresponding component is $\delta_{0;\{1,2\}}$, hence we need to exclude it by our setting. □

Next, we consider the case $n \geq 3$. If m entries of $\mathbf{a}$ are zero, drop them and denote by $\mathbf{a'}$ the resulting $(n - m)$-tuple. Then we have $D_{\mathbf{a}} = \pi^* D_{\mathbf{a'}}$, where $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-m}$ is the map forgetting the corresponding marked points. Since the fiber of π over a general point in $D_{\mathbf{a'}}$ is irreducible, we conclude that $D_{\mathbf{a}}$ and $D_{\mathbf{a'}}$ possess the same number of irreducible components. It remains to consider the case when all entries of $\mathbf{a}$ are nonzero.

Proposition 3.3. *Suppose $n \geq 3$ and all entries of $\mathbf{a}$ are nonzero. Let $d = \gcd(a_1, \ldots, a_n)$. Then $D_{\mathbf{a}}$ consists of $\eta(d)$ irreducible components.*

Proof. If an entry of $\mathbf{a}$ equals 1 or -1, say $a_n = 1$, then we can freely choose $p_1, \ldots, p_{n-1}$ and a general choice uniquely determines p_n. In other words, $D_{\mathbf{a}}$ is birational to $\mathcal{M}_{1,n-1}$ which is irreducible.

Consider the remaining case when $|a_i| > 1$ for all i. Since $d|a_i$, we conclude that $|a_i| \geq d$ for all i. If $|a_i| > d$ for all i, without loss of generality, assume that $a_1 > 0 > a_n$ and $|a_1| > |a_n|$, otherwise we can use $-\mathbf{a}$ instead of $\mathbf{a}$. Fix $p_1, \ldots, p_{n-1}$ and replace p_n by $p'_n = 2p_1 - p_n$. Then $\mathbf{a} = (a_1, \ldots, a_n)$ becomes $\mathbf{a}' = (a_1 + 2a_n, a_2, \ldots, a_{n-1}, -a_n)$. Note that p_n and p'_n uniquely determine each other, and for general points in $D_{\mathbf{a}}$ we have $p'_n \neq p_i$ for $1 \leq i < n$, otherwise we would have $|a_i| = |a_n| = 1$. Moreover, $|a_1 + 2a_n| < |a_1|$ by assumption. Therefore, using such transformations, we can decrease $\sum_{i=1}^n |a_i|$ until one of the entries is equal to d (or $-d$). Note that none of the integers in the resulting sequence of numbers is zero.

Without loss of generality, assume that we have reduced to the case $a_n = d$. Fix $p_1, \ldots, p_{n-1}$ and set $\sum_{i=1}^{n-1} a_i p_i$ to be the origin of E. Then p_n is a d-torsion. Analyzing the monodromy associated to $D_{\mathbf{a}} \dashrightarrow \mathcal{M}_{1,n-1}$ as in the proof of Proposition 3.2, we see that $D_{\mathbf{a}}$ has at most $\eta(d)$ irreducible components. On the other hand for each positive factor s of d, the locus parameterizing $\sum_{i=1}^n b_i p_i = 0$ where $b_i = a_i/s$ gives rise to at least one component of $D_{\mathbf{a}}$. Hence $D_{\mathbf{a}}$ contains exactly $\eta(d)$ irreducible components. Since $n \geq 3$, none of the components is a boundary divisor of $\overline{\mathcal{M}}_{1,n}$. □

We remark that the results in Propositions 3.2 and 3.3 were also obtained by [Bo, Theorems 4.1] in the context of flat geometry and by [P2, Corollary 1] using abelian covers of elliptic curves.

Next, we calculate the classes of the components of $D_{\mathbf{a}}$ when $d > 1$. Let $D'_{\mathbf{a}}$ be the irreducible component of $D_{\mathbf{a}}$ such that at its general point $\sum_{i=1}^n a_i p_i = 0$ but $\sum_{i=1}^n (a_i/s) p_i \neq 0$ for any s dividing d and $s > 1$. Equivalently, $D'_{\mathbf{a}}$ parameterizes the closure of the locus of points where $\sum_{i=1}^n (a_i/d) p_i$ is d-torsion, but not torsion of smaller order. Recall that N is the set of indices i for $a_i \neq 0$.

Proposition 3.4. *Suppose $\gcd(a_1, \ldots, a_n) = d > 1$. Then the divisor $D'_{\mathbf{a}}$ has class*

$$D'_{\mathbf{a}} = \left(\prod_{p|d} \left(1 - \frac{1}{p^2}\right) \right) \left(D_{\mathbf{a}} + \lambda + \sum_{S \supset N} \delta_{0;S} \right),$$

where the product ranges over all primes p dividing d.

We remark that for $n = 2$ the above divisor class was calculated by Cautis [Ca, Proposition 3.4.7] and also communicated personally to the authors by Hain.

Proof. Let $b_i = a_i/d$ and $\mathbf{b} = (b_1, \ldots, b_n)$. For an integer m, use the notation $m\mathbf{b} = (mb_1, \ldots, mb_n)$. Note that

$$D_{\mathbf{a}} = D_{d\mathbf{b}} = \sum_{t|d} D'_{t\mathbf{b}},$$

where t ranges over all positive integers dividing d. By Proposition 3.1, we have

$$D_{\mathbf{a}} + \lambda + \sum_{S \supset N} \delta_{0;S} = d^2 \left(D_{\mathbf{b}} + \lambda + \sum_{S \supset N} \delta_{0;S} \right).$$

For an integer $t \geq 2$, define

$$\sigma(t) = t^2 \prod_{p|t} \Big(1 - \frac{1}{p^2}\Big),$$

where the product ranges over all primes p dividing t. We also set $\sigma(1) = 1$. Using the above observation, it suffices to prove that

$$\sum_{t|d} \sigma(t) = d^2$$

for all d. Note that

$$\sigma(t) = \sum_{m|t} \mu(m) \Big(\frac{t}{m}\Big)^2,$$

where μ is the Möbius function. The desired equality thus follows from the classic Möbius inversion formula, see [HW, §16.4]. □

Corollary 3.5. *If* $\gcd(a_1, \ldots, a_n) > 1$*, the divisor class* $D'_{\mathbf{a}}$ *is not extremal.*

Proof. By Proposition 3.4, $D'_{\mathbf{a}}$ is a positive linear combination of effective divisor classes, not all proportional. □

4. Extremality of $D_{\mathbf{a}}$

In this section, we will prove Theorem 1.1. Recall that an effective divisor D in a projective variety X is called extremal, if for any linear combination $D = a_1 D_1 + a_2 D_2$ with $a_i > 0$ and D_i pseudo-effective, D and D_i are proportional. In this case, we say that D spans an extremal ray of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$. Furthermore, we say that D is rigid, if for every positive integer m the linear system $|mD|$ consists of the single element mD. An irreducible effective curve contained in D is called a moving curve in D, if its deformations cover a dense subset of D.

The following well-known lemma gives a simple criterion for the extremality and rigidity of an effective divisor. We recall its proof for the convenience of the reader.

Lemma 4.1. *Suppose that* C *is a moving curve in an irreducible effective divisor* D *satisfying* $C \cdot D < 0$*. Then* D *is extremal and rigid.*

Proof. Let us first prove the extremality of D. Suppose that $D = a_1 D_1 + a_2 D_2$ with $a_i > 0$ and D_i pseudo-effective. If D_1 and D_2 are not proportional to D, we can assume that they lie on the boundary of $\overline{\mathrm{Eff}}(X)$ and moreover that $D_i - \epsilon D$ is not pseudo-effective for any $\epsilon > 0$. Otherwise, we can replace D_1 and D_2 by the intersections of their linear span with the boundary of $\overline{\mathrm{Eff}}(X)$.

Since $C \cdot D < 0$, at least for one of the D_i's, say D_1, we have $C \cdot D_1 < 0$. Without loss of generality, rescale the class of D_1 such that $C \cdot D_1 = -1$. Take a very ample divisor class A and consider the class $F_n = nD_1 + A$ for n sufficiently large. Then F_n can be represented by an effective divisor. Suppose $C \cdot A = a$ and $C \cdot D = -b$ for some $a, b > 0$. Note that if C has negative intersection with an effective divisor, then it is contained in that divisor. Since C is moving in D, it further implies that D is contained in that divisor. It is easy to check that $C \cdot (F_n - kD) < 0$ for any $k < (n-a)/b$. Moreover, the multiplicity of D in the base locus of F_n is at least equal to $(n-a)/b$. Consequently $E_n = F_n - (n-a)D/b$ is a pseudo-effective divisor class. As n goes to infinity, the limit of E_n/n has class $D_1 - D/b$. Since E_n is pseudo-effective, we conclude that $D_1 - D/b$ is also pseudo-effective, contradicting the assumption that $D_1 - \epsilon D$ is not pseudo-effective for any $\epsilon > 0$.

Next, we prove the rigidity. Suppose for some integer m there exists another effective divisor D' such that $D' \sim mD$. Without loss of generality, assume that D' does not contain D, for otherwise we just subtract D from both sides. Since $C \cdot D < 0$, we have $C \cdot D' < 0$, and hence D' contains C. But C is moving in D, hence D' has to contain D, contradicting the assumption. □

Although we can give a uniform proof of Theorem 1.1 as in Section 4.2, for the reader to get a feel, let us first discuss the case $n = 3$ in detail.

4.1. **Geometry of $\overline{\mathcal{M}}_{1,3}$.** Let $\mathbf{a} = (a_1, a_2, a_3)$. If $a_3 = 0$, then $a_2 = -a_1$ are not relatively prime unless they are 1 and -1. In that case, we require $p_1 = p_2$, which cannot hold in $\mathcal{M}_{1,n}$. Hence, $D_{(1,-1,0)}$ by definition is empty. Therefore, below we assume that $\gcd(a_1, a_2, a_3) = 1$ and none of the a_i's is zero.

Fix a smooth genus one curve E with a marked point p_1. Vary two points p_2, p_3 on E such that $\sum_{i=1}^{3} a_i p_i = 0$ in the Jacobian of E. Let X be the curve induced in $\overline{\mathcal{M}}_{1,n}$ by this one parameter family of three pointed genus one curves. We obtain deformations of X by varying the complex structure on E. Since these deformations cover a Zariski dense subset of $D_{\mathbf{a}}$, we obtain a moving curve in the divisor $D_{\mathbf{a}}$. We have the following intersection numbers:

$$X \cdot \delta_{\mathrm{irr}} = 0,$$
$$X \cdot \delta_{0;\{i,j\}} = a_k^2 - 1 \text{ for } k \neq i, j,$$
$$X \cdot \delta_{0;\{1,2,3\}} = 1.$$

The intersection numbers $X \cdot \delta_{\mathrm{irr}}$ and $X \cdot \delta_{0;\{i,j\}}$ are straightforward. At the intersection with $\delta_{0;\{1,2,3\}}$, p_1, p_2, p_3 coincide at the same point t in E. Blow up t and we obtain a rational tail $R \cong \mathbb{P}^1$ that contains the three marked points. Without loss of generality, suppose $a_1 > 0$ and $a_2, a_3 < 0$. The pencil induced by $a_1 p_1 \sim (-a_2)p_2 + (-a_3)p_3$ degenerates to an admissible cover π of degree a_1. By the Riemann-Hurwitz formula, π is totally ramified at p_1, has ramification order $(-a_i)$ at p_i for $i = 2, 3$, and is simply ramified at t. Suppose $\pi(p_1) = 0$, $\pi(p_2) = \pi(p_3) = \infty$ and $\pi(t) = 1$ in the target $\mathbb{P}^1$. Then in affine coordinates π is given by

$$\pi(x) = \prod_{i=1}^{3} (x - p_i)^{a_i}.$$

The condition imposed on t is that

$$(x - p_1)^{a_1} - (x - p_2)^{-a_2}(x - p_3)^{-a_3}$$

has a critical point at t and $\pi(t) = 1$. Solving for t, we easily see that t exists and is uniquely determined by p_1, p_2, p_3, namely, the four points t, p_1, p_2, p_3 have unique moduli in R.

We remark that similar test curve calculations were extensively used in [GZ, M]. The former takes advantage of the theta function, while the latter relies on the theory of limit linear series.

Now we can prove Theorem 1.1 for the case $n = 3$.

Proof. Using the divisor class $D_{\mathbf{a}}$ in Proposition 3.1 and the above intersection numbers, we see that

$$X \cdot D_{\mathbf{a}} = -1.$$

By assumption both X and $D_{\mathbf{a}}$ are irreducible. Moreover, X is a moving curve inside $D_{\mathbf{a}}$. Therefore, by Lemma 4.1 $D_{\mathbf{a}}$ is an extremal and rigid divisor.

To see that we obtain infinitely many extremal rays of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,3})$ this way, let us take $\mathbf{a} = (n+1, -n, -1)$. Then $D_{(n+1,-n,-1)}$ is irreducible and its divisor class lies on the ray

$$c\left(\lambda + \delta_{0,\{1,2,3\}} + \delta_{0,\{1,2\}} + \frac{1}{n}\delta_{0,\{1,3\}} - \frac{1}{n+1}\delta_{0,\{2,3\}}\right), \quad c > 0.$$

As n varies, we obtain infinitely many extremal rays. □

4.2. **Geometry of $\overline{\mathcal{M}}_{1,n}$ for $n \geq 4$.** In this section suppose $n \geq 4$. First, let us consider pulling back divisors from $\overline{\mathcal{M}}_{1,3}$.

Let $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,3}$ be the forgetful map forgetting $p_4, \ldots, p_n$. Assume that $\gcd(a_1, a_2) = 1$. In Section 4.1 we have shown that $D_{(a_1,a_2,-a_1-a_2)}$ is extremal. Now fix a smooth genus one curve E with fixed $p_3, p_4, \ldots, p_n$ in general position. Varying p_1, p_2 in E such that $\sum_{i=1}^{3} a_i p_i = 0$, we obtain a curve X moving inside $\pi^* D_{(a_1,a_2,-a_1-a_2)}$. We have also seen that $(\pi_* X) \cdot D_{(a_1,a_2,-a_1-a_2)} < 0$ on $\overline{\mathcal{M}}_{1,3}$, hence by the projection formula, we have $X \cdot (\pi^* D_{(a_1,a_2,-a_1-a_2)}) < 0$. Since $\pi^* D_{(a_1,a_2,-a_1-a_2)}$ is irreducible, we conclude the following.

Proposition 4.2. *Let* $\mathbf{a} = (a_1, a_2, -a_1 - a_2, 0, \ldots, 0)$ *for* $\gcd(a_1, a_2) = 1$. *Then the divisor class* $D_{\mathbf{a}}$ *is extremal in* $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$.

Corollary 4.3. *For* $n \geq 4$, *the cone* $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$ *is not finite polyhedral.*

Proof. For $\mathbf{a} = (a_1, a_2, -a_1 - a_2, 0, \ldots, 0)$ with $\gcd(a_1, a_2) = 1$, by Proposition 3.1 we have

$$D_{\mathbf{a}} = (-1 + a_1^2 + a_2^2 + a_1 a_2)\Big(\lambda + \sum_{\{1,2,3\}\subset S} \delta_{0;S}\Big) - a_1 a_2 \Big(\sum_{\substack{\{1,2\}\subset S\\ 3\notin S}} \delta_{0;S}\Big)$$

$$+ a_1(a_1 + a_2)\Big(\sum_{\substack{\{1,3\}\subset S\\ 2\notin S}} \delta_{0;S}\Big) + a_2(a_1 + a_2)\Big(\sum_{\substack{\{2,3\}\subset S\\ 1\notin S}} \delta_{0;S}\Big).$$

By varying a_1, a_2, we obtain infinitely many extremal rays. □

Next we consider $D_{\mathbf{a}}$ in general for $n \geq 4$ and $\gcd(a_1, \ldots, a_n) = 1$. Let $D_{\mathbf{a}}(E, \eta)$ be the locus in $\overline{\mathcal{M}}_{1,n}$ parameterizing $(E; p_1, \ldots, p_n)$ such that $\sum_{i=1}^{n} a_i p_i = \eta$ for fixed $\eta \in \mathrm{Jac}(E)$ on a fixed genus one curve E.

For $S = \{i_1, \ldots, i_k\}$, consider the locus $\delta_{0;S}(E)$ of curves parameterized in $\delta_{0;S}$ whose genus one component is E. Blow down the rational tails and $p_{i_1}, \ldots, p_{i_k}$ reduce to the same point q in E. For fixed $\eta \neq 0$, the condition

$$\Big(\sum_{j=1}^{k} a_{i_j}\Big) q + \sum_{j \notin S} a_j p_j = \eta$$

does not hold for q and p_j in general position in E. Hence $\delta_{0;S}(E)$ is not contained in $D_{\mathbf{a}}(E, \eta)$ for $\eta \neq 0$, and $D_{\mathbf{a}}(E, \eta)$ is irreducible of codimension-two in $\overline{\mathcal{M}}_{1,n}$.

If $\eta = 0$, the above argument still goes through with the exception when $S \supset N$, where N is the set of indices i for $a_i \neq 0$. This is because the condition $\sum_{i=1}^{n} a_i p_i =$

0 automatically holds if all the marked points p_i with $i \in N$ coincide, due to the assumption $\sum_{i\in N} a_i = 0$ and $a_j = 0$ for $j \notin N$. In other words, $D_{\mathbf{a}}(E, 0)$ is reducible. One of its components is $D_{\mathbf{a}}(E)$ whose general points parameterize n distinct points $p_1, \ldots, p_n$ in E such that $\sum_{i=1}^n a_i p_i = 0$ and the others are $\delta_{0;S}(E)$ for $S \supset N$ whose general points parameterize E attached to a rational tail that contains marked points labeled by S. Denote by $\delta_N(E)$ the union of $\delta_{0;S}(E)$ for all $S \supset N$. It is nonempty, because $\{1, \ldots, n\} \supset N$ for any N.

Now let us prove Theorem 1.1 for the case $n \geq 4$.

Proof. Note that for $\eta \neq 0$, $D_{\mathbf{a}}(E, \eta)$ is disjoint from $D_{\mathbf{a}}$. This is clear in the interior of $\overline{\mathcal{M}}_{1,n}$. At the boundary, if k marked points coincide, say $p_1 = \cdots = p_k = q$ in E, then

$$\Big(\sum_{i=1}^{k} a_i\Big) q + \sum_{j=k+1}^{n} a_j p_j$$

has to be η for $D_{\mathbf{a}}(E, \eta)$ and 0 for $D_{\mathbf{a}}$, which cannot hold simultaneously for $\eta \neq 0$.

Since $n \geq 4$, take $n - 3$ very ample divisors on $\overline{\mathcal{M}}_{1,n}$ and consider their intersection restricted to $D_{\mathbf{a}}(E, \eta)$, which gives rise to an irreducible curve $C_{\mathbf{a}}(E, \eta)$ moving in $D_{\mathbf{a}}(E, \eta)$. Restricting to $D_{\mathbf{a}}(E, 0)$, we see that $C_{\mathbf{a}}(E, \eta)$ specializes to $C_{\mathbf{a}}(E, 0)$ which consists of one component $C_{\mathbf{a}}(E)$ contained in $D_{\mathbf{a}}(E)$ and the other components $C_N(E)$ contained in $\delta_N(E)$. Moreover, $C_{\mathbf{a}}(E, 0)$ is connected, hence $C_{\mathbf{a}}(E)$ and $C_N(E)$ intersect each other. Therefore, we conclude that

$$\begin{aligned} (C_{\mathbf{a}}(E) + C_N(E)) \cdot D_{\mathbf{a}} = C_{\mathbf{a}}(E, \eta) \cdot D_{\mathbf{a}} = 0, \\ C_N(E) \cdot D_{\mathbf{a}} > 0, \\ C_{\mathbf{a}}(E) \cdot D_{\mathbf{a}} < 0. \end{aligned}$$

The curve $C_{\mathbf{a}}(E)$ is moving in $D_{\mathbf{a}}(E)$ and also varies with the complex structure of E, hence it is moving in $D_{\mathbf{a}}$. Since it has negative intersection with $D_{\mathbf{a}}$ and $D_{\mathbf{a}}$ is irreducible, by Lemma 4.1 we thus conclude that $D_{\mathbf{a}}$ is extremal and rigid. □

Corollary 4.4. *For $n \geq 3$ the moduli space $\overline{\mathcal{M}}_{1,n}$ is not a Mori dream space.*

Proof. By [HK, 1.11 (2)], if $\overline{\mathcal{M}}_{1,n}$ is a Mori dream space, its effective cone would be the affine hull spanned by finitely many effective divisors, which contradicts the fact that $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$ has infinitely many extremal rays. □

5. Effective divisors on $\overline{\mathcal{M}}_{1,n}/G$

Let G be a subgroup of $\mathfrak{S}_n$ permuting the labelings of the n ordered marked points. Let $\overline{\mathcal{M}}_{1,n}/G$ be the quotient of $\overline{\mathcal{M}}_{1,n}$ under the action of G. In this section, we study $\mathrm{Eff}(\overline{\mathcal{M}}_{1,n}/G)$.

First, consider the special case $G = \mathfrak{S}_n$. Denote the quotient space $\overline{\mathcal{M}}_{1,n}/\mathfrak{S}_n$ by $\widetilde{\mathcal{M}}_{1,n}$, the moduli space of stable genus one curves with n unordered marked points. The rational Picard group of $\widetilde{\mathcal{M}}_{1,n}$ is generated by $\widetilde{\delta}_{\mathrm{irr}}$ and $\widetilde{\delta}_{0;k}$ for $2 \leq k \leq n$, where $\widetilde{\delta}_{\mathrm{irr}}$ is the image of δ_{irr} and $\widetilde{\delta}_{0;k}$ is the image of the union of $\delta_{0;S}$ for all $|S| = k$.

In the case of genus zero, Keel and McKernan [KM] showed that the effective cone of $\widetilde{\mathcal{M}}_{0,n}$ is spanned by the boundary divisors. Here we establish a similar result for $\widetilde{\mathcal{M}}_{1,n}$.

Theorem 5.1. *The effective cone of $\widetilde{\mathcal{M}}_{1,n}$ is the closed simplicial cone spanned by the boundary divisors $\widetilde{\delta}_{\rm irr}$ and $\widetilde{\delta}_{0;k}$ for $2 \leq k \leq n$.*

Proof. It suffices to show that any irreducible effective divisor is a nonnegative linear combination of boundary divisors. Suppose D is an effective divisor different from any boundary divisor and has class

$$D = a\widetilde{\delta}_{\rm irr} + \sum_{k=2}^{n} b_k \widetilde{\delta}_{0;k}.$$

If C is a curve class whose irreducible representatives form a Zariski dense subset of a boundary divisor $\widetilde{\delta}_{0;k}$, then $C \cdot D \geq 0$. Otherwise, the curves in the class C and, consequently, the divisor $\widetilde{\delta}_{0;k}$ would be contained in D, contradicting the irreducibility of D. We first show that $b_k \geq 0$ by induction on k. Here the argument is exactly as in Keel and M$^{\rm c}$Kernan.

Let C be the curve class in $\widetilde{\mathcal{M}}_{1,n}$ induced by fixing a genus one curve E with $n-1$ fixed marked points and letting an n-th point vary along E. Since the general n-pointed genus one curve occurs on a representative of C, C is a moving curve class. We conclude that $C \cdot D \geq 0$ for any effective divisor. On the other hand, since $C \cdot \widetilde{\delta}_{0;2} = n-1$ and $C \cdot \widetilde{\delta}_{\rm irr} = C \cdot \widetilde{\delta}_{0;k} = 0$, for $2 < k \leq n$, we conclude that $b_2 \geq 0$.

By induction assume that $b_k \geq 0$ for $k \leq j$. We would like to show that $b_{j+1} \geq 0$. Let E be a genus one curve with $n-j$ fixed points. Let R be a rational curve with $j+1$ fixed points $p_1, \ldots, p_{j+1}$. Let C_j be the curve class in $\widetilde{\mathcal{M}}_{1,n}$ induced by attaching R at p_{j+1} to a varying point on E. Since the general point on $\widetilde{\delta}_{0;j}$ is contained on a representative of the class C_j, we conclude that C_j is a moving curve in $\widetilde{\delta}_{0;j}$. Hence, $C_j \cdot D \geq 0$. On the other hand, C_j has the following intersection numbers with the boundary divisors:

$$C_j \cdot \widetilde{\delta}_{\rm irr} = 0,$$

$$C_j \cdot \widetilde{\delta}_{0;i} = 0 \text{ for } i \neq j, j+1,$$

$$C_j \cdot \widetilde{\delta}_{0;j+1} = n-j,$$

$$C_j \cdot \widetilde{\delta}_{0;j} = -(n-j).$$

Hence, we conclude that $b_{j+1} \geq b_j \geq 0$ by induction.

There remains to show that the coefficient a is nonnegative. Fix a general pencil of plane cubics and a rational curve R with $n+1$ fixed marked points $p_1, \ldots, p_{n+1}$. Let C_n be the curve class in $\widetilde{\mathcal{M}}_{1,n}$ induced by attaching R at p_{n+1} to a base-point of the pencil of cubics. The class C_n is a moving curve class in the divisor $\widetilde{\delta}_{0;n}$. Consequently, $C_n \cdot D \geq 0$. Since $C_n \cdot \widetilde{\delta}_{\rm irr} = 12$, $C_n \cdot \widetilde{\delta}_{0;k} = 0$ for $k < n$ and $C_n \cdot \widetilde{\delta}_{0;n} = -1$, we conclude that $12a \geq b_n \geq 0$. This concludes the proof that the effective cone of $\widetilde{\mathcal{M}}_{1,n}$ is generated by boundary divisors. $\square$

Next, as suggested by the referee, we consider the case when the action of G on $\{1, \ldots, n\}$ has at least three orbits.

Theorem 5.2. *Let $G \subset \mathfrak{S}_n$ be a subgroup whose permutation action on the set $\{1, \ldots, n\}$ has at least three orbits. Then $\overline{\rm Eff}(\overline{\mathcal{M}}_{1,n}/G)$ is not finitely generated.*

Proof. Without loss of generality, assume that three orbits of G are given by $\{1,\ldots,k\}$, $\{k+1,\ldots,k+l\}$ and $\{k+l+1,\ldots,k+l+m\}$, respectively. Use a^k to denote a k-tuple whose entries are all equal to a. Take three nonzero integers a, b and c such that $ka+lb+mc=0$ and $\gcd(a,b,c)=1$. Denote by $D^G_{(a^k,b^l,c^m)}$ the closure of the locus parameterizing smooth pointed genus one curves $(E;p_1,\ldots,p_n)$ in $\overline{\mathcal{M}}_{1,n}/G$ such that

$$a\Big(\sum_{i=1}^{k} p_i\Big) + b\Big(\sum_{j=k+1}^{k+l} p_j\Big) + c\Big(\sum_{h=k+l+1}^{k+l+m} p_h\Big) = 0$$

in E. By assumption, $D^G_{(a^k,b^l,c^m)}$ is a well-defined effective divisor on $\overline{\mathcal{M}}_{1,n}/G$.

Note that

$$f^* D^G_{(a^k,b^l,c^m)} = D_{(a^k,b^l,c^m)}$$

under the quotient map $f:\overline{\mathcal{M}}_{1,n}\to\overline{\mathcal{M}}_{1,n}/G$. In Theorem 1.1 and its proof, we have shown a curve C moving in $D_{(a^k,b^l,c^m)}$ satisfying $C\cdot D_{(a^k,b^l,c^m)}<0$. The image of C is a moving curve in $D^G_{(a^k,b^l,c^m)}$ and by the projection formula we have $(f_*C)\cdot D^G_{(a^k,b^l,c^m)}<0$. As a consequence, we conclude that $D^G_{(a^k,b^l,c^m)}$ spans an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n}/G)$.

Finally, we claim that varying the values of a, b and c, the divisors $D^G_{(a^k,b^l,c^m)}$'s give rise to infinitely many distinct extremal rays. It suffices to prove the same claim for their pullbacks $D_{(a^k,b^l,c^m)}$ on $\overline{\mathcal{M}}_{1,n}$. When the $\gcd(a,b,c)=1$, these divisors are extremal and rigid. Hence, it suffices to exhibit infinitely many relatively prime solutions to the equation $ka+lb+mc=0$. Let

$$k'=\frac{k}{\gcd(k,l)},\qquad l'=\frac{l}{\gcd(k,l)}.$$

Taking $a=l'$, $b=-k'$ and $c=0$ evidently gives a relatively prime triple satisfying the equation. Now we can obtain infinitely many relatively prime solutions by setting $a=l', b=-k'-tml', c=tll'$ and letting t vary over integers. □

Appendix A. Singularities of $\overline{\mathcal{M}}_{1,n}$

Let $\overline{M}_{1,n}$ be the underlying coarse moduli scheme of $\overline{\mathcal{M}}_{1,n}$. Denote by $\overline{M}^{\mathrm{reg}}_{1,n}$ its smooth locus. Below we will show that a canonical form defined on $\overline{M}^{\mathrm{reg}}_{1,n}$ extends holomorphically to any resolution of $\overline{M}_{1,n}$.

Since $\overline{M}_{1,n}$ is rational when $n\le 10$ [Be], in this case there are no nonzero holomorphic forms on any resolution. We may, therefore, assume that $n\ge 11$ as needed. The standard reference on the singularities of $\overline{M}_{g,n}$ dates back to [HMu] and some recent generalizations include [Lo, Lu, FV1, CF, BFV].

Let $(C;\overline{x})=(C;x_1,\ldots,x_n)$ be a stable curve with n ordered marked points. Let ϕ be a non-trivial automorphism of C such that $\phi(x_i)=x_i$ for all i, and suppose that the order of ϕ is k. If the eigenvalues of the induced action of ϕ on $H^0(C,\omega_C\otimes\Omega^1_C(x_1+\cdots x_n))^\vee$ are $e^{2\pi i k_j/k}$ with $0\le k_j<k$, then the age of ϕ is defined as

$$\mathrm{age}(\phi)=\sum_j\frac{k_j}{k}.$$

If ϕ acts trivially on a codimension-one subspace of the deformation space of $(C;\overline{x})$, we say that ϕ is a quasi-reflection. For a quasi-reflection, all but one of the eigenvalues of ϕ are equal to one and $\mathrm{age}(\phi) = 1/k$. By the Reid-Tai Criterion, see e.g. [HMu, p. 27], if $\mathrm{age}(\phi) \geq 1$ for any $\phi \in \mathrm{Aut}(C;\overline{x})$, then a canonical form defined on the smooth locus of the moduli space extends holomorphically to any resolution. Moreover, suppose that $\mathrm{Aut}(C;\overline{x})$ does not contain any quasi-reflections, then the resulting singularity is canonical if and only if $\mathrm{age}(\phi) \geq 1$ for any $\phi \in \mathrm{Aut}(C,\overline{x})$, see e.g. [Lu, Theorem 3.4]. The quasi-reflections form a normal subgroup of $\mathrm{Aut}(C,\overline{x})$. One can consider the action modulo this subgroup and use the Reid-Tai Criterion, see [Lu, Proposition 3.5]. In particular, no singularities arise if and only if $\mathrm{Aut}(C,\overline{x})$ is generated by quasi-reflections.

According to the above, the upshot is to carry out the age calculation. For $\overline{\mathcal{M}}_{1,n}$, its age has been worked out explicitly in [P1, Corollary 4.8] in the context of inertia stack and twisted sector. Alternatively, one can perform an elementary calculation as follows to figure out the locus of points whose age is possibly smaller than one.

The automorphism ϕ induces an action on $H^0(C,\omega_C\otimes\Omega_C^1(x_1+\cdots+x_n))^\vee$. We have an exact sequence:

$$0 \to \bigoplus_{p\in C_{\mathrm{sing}}} \mathrm{tor}_p \to H^0\Big(C, \omega_C\otimes\Omega_C^1\Big(\sum_{i=1}^n x_i\Big)\Big) \to \bigoplus_\alpha H^0\Big(C_\alpha, \omega_{C_\alpha}^{\otimes 2}\Big(\sum_\beta p_{\alpha\beta}\Big)\Big) \to 0,$$

where C_α's are the components of the normalization of C and $p_{\alpha\beta}$'s are the inverse images of nodes in C_α.

First, we show that for an irreducible elliptic curve E with n distinct marked points, we have $\mathrm{age}(\phi) \geq 1$. The automorphism group of E has order 2 if $j(E) \neq 0, 1728$, has order 4 if $j(E) = 1728$, and has order 6 if $j(E) = 0$. Since ϕ fixes all $x_1, \ldots, x_n$, if $n \geq 3$, then ϕ has order $k = 2$ or 3. If $k = 2$, then $n = 3$ or 4, and hence by [HMu, p. 37, Case c2)] we have $\mathrm{age}(\phi) = \frac{n-1}{2} \geq 1$. If $k = 3$, then $n = 3$, and hence [HMu, p. 38, Case c3)] implies that $\mathrm{age}(\phi) \geq 1$.

Next, consider a stable nodal genus one curve $(C;\overline{x})$ with n ordered marked points. Let C_0 be its core curve of genus one. Then C_0 is either irreducible elliptic, or consists of a circle of copies of $\mathbb{P}^1$. It is easy to see that ϕ acts trivially on every component of $C\backslash C_0$. Let C_0 be a circle of l copies of $\mathbb{P}^1$, i.e. $B_1, \ldots, B_l$ are glued successively at the nodes $p_1, \ldots, p_l$, where $B_i \cong \mathbb{P}^1$, $B_i \cap B_{i+1} = p_{i+1}$ and $p_{l+1} = p_1$. By the stability of $(C;\overline{x})$, each B_i contains at least one more node or marked point, which has to be fixed by ϕ. Therefore, ϕ acts non-trivially on B_i only if it acts as an involution, switching p_i and p_{i+1} and fixing the other nodes and marked points on B_i. This implies that $l = 2$ and $k = 2$. By [HMu, p. 34], either $\mathrm{age}(\phi) \geq 1$ or $\mathrm{Aut}(C,\overline{x})$ is generated by this elliptic involution, which is a quasi-inflection and does not induce a singularity. Thus, we are left with the case when C_0 is an irreducible elliptic curve E and ϕ is induced by a non-trivial automorphism of E fixing all marked points and acting trivially on the other components of C.

If E contains at least one marked point x, [FV1, proof of Theorem 1.1 (ii)] says that $\mathrm{age}(\phi) \geq 1$. We can also see this directly using [HMu, p. 37-39, Case c)] as follows. If the order n of ϕ is 2, then the action restricted to $H^0(K_E^{\otimes 2}(x))$ contributes 1/2 to $\mathrm{age}(\phi)$. At a node p of E, suppose that the two branches have coordinates y and z. Then tor_p is generated by $ydz^{\otimes 2}/z = zdy^{\otimes 2}/y$, see [HMu, p. 33]. The action of ϕ locally is given by $y \to -y$ and $z \to z$, hence tor_p also contributes 1/2. Consequently we get $\mathrm{age}(\phi) \geq 1$. If $k = 3$, at p the action is locally given by $y \to \zeta y$ and $z \to z$, where ζ is a cube root of unity, hence tor_p contributes 1/3. At x, take a translation invariant differential dz. Then locally $dz^{\otimes 2}$ is an eigenvector of $H^0(K_E^{\otimes 2}(x+p))$. The action ϕ is locally given by $x \to \zeta x$, hence it contributes $2/3$. We still get $\mathrm{age}(\phi) \geq 1/3 + 2/3 = 1$. If $k = 4$, similarly tor_p contributes $1/4$. Locally take $dz^{\otimes 2}$ and $dz^{\otimes 2}/z$ as eigenvectors of $H^0(K_E^{\otimes 2}(x+p))$. We get an additional contribution equal to $2/4 + 1/4$. In total we still have $\mathrm{age}(\phi) \geq 1$. Finally, since ϕ cannot fix both x and p, the case $k = 6$ does not occur. Similarly, if E contains more than one node, ϕ fixes all the nodes, and hence the same analysis implies that $\mathrm{age}(\phi) \geq 1$.

Based on the above analysis, we conclude that the locus of non-canonical singularities of $\overline{M}_{1,n}$ is contained in the locus of curves $(C, \overline{x})$ where the core curve of C is an unmarked irreducible elliptic tail E attached to the rest of C at a node p. Moreover, $G = \mathrm{Aut}(C, \overline{x}) = \mathrm{Aut}(E, p)$ fixes all marked points and acts trivially on the other components of C. Harris and Mumford [HMu, p. 40-42] proved that any canonical form defined in $\overline{M}_{g,n}^{\mathrm{reg}}$ extends holomorphically to any resolution over the locus of curves of this type. Strictly speaking, Harris and Mumford discussed the case $\overline{M}_g$. They constructed a suitable neighborhood of a point in $\overline{M}_g$ parameterizing an elliptic curve attached to a curve C_1 of genus $g-1$ without any automorphisms. In their construction, the only property of C_1 they need is that C_1 does not have any non-trivial automorphisms. Hence, their construction is applicable to the case when C_1 is replaced by an arithmetic genus zero curve with n marked points for $n \geq 2$. Therefore, there is a neighborhood of $(C, \overline{x})$ in $\overline{M}_{1,n}$ such that any canonical form defined in the smooth locus of this neighborhood extends holomorphically to a desingularization of the neighborhood. This thus completes the proof.

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