MATH 330 FINAL

This is the take home final for Math 330. It is due **Wednesday December 4** at the beginning of class. You may use your textbook and class notes, but you may not use any other sources or discuss the problems with anyone but the instructor of the course. There are 10 problems and an extra credit problem. Be sure to show all your work. Good luck!

Problem 1. (5 points)

(1) Calculate $\gcd(1090, 220)$ and $\operatorname{lcm}(1090, 220)$.
(2) Express $\gcd(1090, 220)$ as a linear combination of 1090 and 220.

Problem 2. (15 points)

(1) Find all the subgroups of $\mathbb{Z}/120\mathbb{Z}$ and determine their orders.
(2) Find all the subgroups of order 9 in $\mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. (Caution: Do they have to be cyclic?)

Problem 3. (10 points) Classify all abelian groups of order $21600 = 2^5 \times 3^3 \times 5^2$ up to isomorphism.

Problem 4. (10 points)

(1) List all the group homomorphisms $f : \mathbb{Z}/35\mathbb{Z} \to \mathbb{Z}/25\mathbb{Z}$.
(2) For each of the homomorphism in part (1) determine the kernel and the image of the homomorphism. Express each kernel and image as $\mathbb{Z}/n\mathbb{Z}$ for an appropriate n.
(3) Determine which of these group homomorphisms are ring homomorphisms.

Problem 5. (10 points) Consider the group homomorphism $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ with the property that $f((1, 0)) = (6, 12)$ and $f(0, 1) = (2, 4)$.

(1) Determine the image of f.
(2) Determine the kernel of f.
(3) Consider the factor group $\mathbb{Z} \oplus \mathbb{Z}/f(\mathbb{Z} \oplus \mathbb{Z})$. Does this group contain any elements of order 5? Does it contain any elements of order 2? Does it contain any elements of infinite order? Is it cyclic?

Problem 6. (10 points) Let R be the ring $\mathbb{Z}/30\mathbb{Z}$.

(1) Find all the zero divisors of R.
(2) Find all the units of R.
(3) Find all the maximal ideals in R.
(4) What is the characteristic of R?
(5) Is R an integral domain?

Problem 7. (10 points) Prove that the ring $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}[x]/ < x^2 - 2 >$.

Problem 8. (10 points) Are the rings $\mathbb{R}[x, y]/ < x - y >$ and $\mathbb{R}[x, y]/ < x^2 - y^2 >$ isomorphic? Prove your answer.

Problem 9. (10 points) Is $\mathbb{R}[x]/ < x^3 + x + 1 >$ a field? Is it an integral domain? How about $\mathbb{R}[x]/ < x^2 + 3 >$?

Problem 10. (10 points) Prove that following polynomials are irreducible over the rational numbers.

(1) $2x^5 + 21x^3 + 49x^2 - 14x + 7$.
(2) $x^3 + 2x + 1$

Problem 11. (Extra Credit: 10 points) In this problem you will classify all groups of order 8. Let G be a group of order 8.

(1) Using Lagrange's Theorem, determine the possible orders of an element $g \in G$.
(2) Show that if every non-identity element has order 2, then G is abelian. Conclude that then G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(3) If G has an element of order 8, show that G is isomorphic to $\mathbb{Z}/8\mathbb{Z}$.
(4) For the rest of the problem, assume that G is a group of order 8 which is not isomorphic to one of the groups in parts (2) or (3). Conclude that G must have an element g of order 4. Let H be the subgroup generated by g. Show that H is a normal subgroup of G.
(5) If G is abelian, then conclude that G is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(6) For the rest of the problem, assume that G is not abelian. Show that you can write G as a disjoint union of two cosets of H, say H and xH. Prove that $xgx^{-1} = g^{-1}$.
(7) Conclude that if x has order 2, then G is the dihedral group of order 8 given by $\{x, g | x^2 = g^4 = e, xgx^{-1} = g^{-1}\}$
(8) On the other hand, if x has order 4, then G is the quaternion group given by $\{x, g | g^4 = x^4 = e, x^2 = g^2, xgx^{-1} = g^{-1}\}$.
(9) Conclude that up to isomorphism there are 5 groups of order 8.