MATH 553: FINAL PROBLEM SET

This problem set is due May 3, 2019. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. Throughout assume that all the curves and surfaces are defined over an algebraically closed field of characteristic 0 or sufficiently large characteristic.

Problem 1. Compute the genus of the unique nonsingular projective model of the following curves:

(1) $y^2 = x^7 - 1$
(2) $y^3 = x^5 - 1$

Problem 2. Compute the genus of a smooth, complete intersection curve of degrees $d_1, \dots, d_{n-1}$ in $\mathbb{P}^n$. Deduce that a smooth curve of genus 8 cannot be realized as a complete intersection in projective space.

Problem 3. Prove that there does not exist a nonsingular curve of degree 9 and genus 11 in $\mathbb{P}^3$.

Problem 4.

(1) The bounded negativity conjecture asserts that the self-intersection of reduced, irreducible curves on a nonsingular, complex, projective surface is bounded below. Prove that if $-K_X$ is ample, then the self-intersection of any reduced, irreducible curve C satisfies $C^2 \geq -1$. Furthermore, $C^2 = -1$ if and only if C is an exceptional curve.
(2) Classify all the exceptional curves on the blowup of $\mathbb{P}^2$ at $r \leq 6$ general points. If you are feeling energetic, do also the cases $r = 7$ and 8.
(3) Let C be a nonsingular curve over an algebraically closed field k of characteristic p. Let F_r denote the r-th power of the Frobenius morphism. Compute the self-intersection of the graph of F_r in $C \times C$. What can you conclude about the bounded negativity conjecture in positive characteristic?

Problem 5. Compute the zeta function of the Grassmannian $G(2,4)$ and verify the Weil conjectures directly. Generalize to arbitrary Grassmannians $G(k,n)$.

Problem 6. Let a, b be integers. Compute the cohomology of $\mathcal{O}_{\mathbb{F}_e}(aE+bF)$. Show that $\mathcal{O}_{\mathbb{F}_e}(aE+bF)$ is ample if and only if $b > ae$, $a, b > 0$.

Problem 7. Compute the genus of a smooth curve in the class $aE + bF$, $b \geq ae$, on $\mathbb{F}_e$. Deduce the possible genera of smooth curves on a quadric cone in $\mathbb{P}^3$.

Problem 8.

(1) Show that the anticanonical image of the blowup of $\mathbb{P}^2$ at 6 points, three of which are collinear, has a singular point. How many lines does this surface contain?
(2) Show that a cubic surface in $\mathbb{P}^3$ with isolated singularities can have at most 4 singular points. (Hint: Show that the points have to be in linearly general position. If there are 5 singular points, consider the twisted cubic curves passing through them.) Find a configuration of 6 points on $\mathbb{P}^2$ such that the anticanonical image of the blowup of $\mathbb{P}^2$ at these 6 points has 4 singular points. How many lines does your surface have?