THE BIRATIONAL GEOMETRY OF THE MODULI SPACES OF SHEAVES ON $\mathbb{P}^2$

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Abstract. In this paper, we survey recent developments in the birational geometry of the moduli spaces $M(\xi)$ of semistable sheaves on $\mathbb{P}^2$ following [ABCH], [CH], [CH2], [CHW] and [H]. We discuss the cones of effective, ample and movable divisors on $M(\xi)$. We introduce Bridgeland stability conditions, the main technical tools that precipitated the recent advances, and explain their applications to interpolation problems.

1. Introduction

In this paper, we survey recent developments in the birational geometry of moduli spaces of semistable sheaves on $\mathbb{P}^2$ following [ABCH], [CH], [CH2], [CHW], [H]. There have been parallel developments in the study of moduli spaces of sheaves on other surfaces such as K3 surfaces [BM], [BM2], [MYY1], [MYY2] abelian surfaces [MM], [YY], [Y2], Enriques surfaces [Ne], Hirzebruch surfaces and del Pezzo surfaces [BC]. For the purposes of exposition, we concentrate on $\mathbb{P}^2$. This survey is an expanded version of the first author's lecture in the Twenty First Gökova Geometry/Topology Conference. It is intended as a guide to already existing literature.

Motivational Example. Different problems may require different birational models of a moduli space. For example, conics on $\mathbb{P}^2$ can be parameterized by coefficients of degree 2 polynomials up to scaling:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0 \longrightarrow [a : b : c : d : e : f].$$

The resulting parameter space is the *Hilbert scheme of conics* $H_{2t+1}(\mathbb{P}^2)$ and is isomorphic to $\mathbb{P}^5$. In the Hilbert scheme, the divisor of conics tangent to a given conic contains the locus V of double lines. Consequently, $H_{2t+1}(\mathbb{P}^2)$ is not a convenient parameter space for enumerative geometry involving tangency conditions. The *moduli space of complete conics* $C_{2t+1}(\mathbb{P}^2)$, which is the blowup of $\mathbb{P}^5$ along V, is much better suited for the purpose. While $C_{2t+1}(\mathbb{P}^2)$ is a more complicated space than $H_{2t+1}(\mathbb{P}^2)$, the precise relation between them allows one to understand the cohomology of $C_{2t+1}(\mathbb{P}^2)$. Furthermore, the modular interpretation of $C_{2t+1}(\mathbb{P}^2)$ in terms of complete conics facilitates doing enumerative geometry on $C_{2t+1}(\mathbb{P}^2)$.

Birational geometry of moduli spaces. The aim of the program is to understand a moduli or parameter space $\mathcal{M}$ by relating it to simpler birational models. The program consists of the following steps.

(1) Determine the cones of ample, movable and effective divisors on $\mathcal{M}$ and describe the stable base locus decomposition of the effective cone.

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(2) Assuming that the section ring is finitely generated, for every effective integral divisor D on $\mathcal{M}$, describe the model

$$\mathcal{M}(D) = \operatorname{Proj}\left(\bigoplus_{m\geq 0} H^0(\mathcal{M}, mD)\right)$$

and determine an explicit sequence of flips and divisorial contractions that relate $\mathcal{M}$ to $\mathcal{M}(D)$. Using these relations compute invariants of $\mathcal{M}$.

(3) If possible, find a modular interpretation of $\mathcal{M}(D)$.

There has been significant progress in the program for many important moduli spaces, including the moduli space of curves (see [AFSV], [HH1], [HH2]), the Kontsevich moduli spaces of genus-zero stable maps (see [Ch], [CC1], [CC2]) and the moduli space of genus-one stable quotients [Coo]. In this paper, we survey the progress for moduli spaces of semistable sheaves on $\mathbb{P}^2$.

Contents. After reviewing the basic definitions and results of birational geometry in §2, we will discuss the geometry of the Hilbert scheme $\mathbb{P}^{2[n]}$ of points on $\mathbb{P}^2$. The Hilbert scheme $\mathbb{P}^{2[n]}$ parameterizes zero-dimensional subschemes of $\mathbb{P}^2$ of length n. In §3, we will recall its basic geometric properties, describe the ample cone $\operatorname{Amp}(\mathbb{P}^{2[n]})$ and state the second author's theorem describing the effective cone $\operatorname{Eff}(\mathbb{P}^{2[n]})$. All ideal sheaves of points are stable. Consequently, $\mathbb{P}^{2[n]}$ is more concrete than arbitrary moduli spaces of sheaves and provides a good test example for general constructions.

The need for higher rank vector bundles. Even when focusing on $\mathbb{P}^{2[n]}$, one naturally encounters moduli spaces of higher rank vector bundles $M(\xi)$. The description of $\operatorname{Eff}(\mathbb{P}^{2[n]})$ for general n is intimately tied to the geometry of $M(\xi)$. Brill-Noether divisors (which will be discussed in detail in §7) give natural sections of tautological line bundles on $M(\xi)$ and, in particular, on $\mathbb{P}^{2[n]}$. Determining the existence of effective Brill-Noether divisors makes crucial use of the classification of Gieseker semistable sheaves on $\mathbb{P}^2$ due to Drézet and Le Potier [DLP]. In §4, we will recall this classification. We will also discuss basic geometric properties of $M(\xi)$.

Bridgeland stability. Much of the recent progress in the birational geometry of moduli spaces of sheaves is inspired by Bridgeland stability. In §5, we will recall the definition. Bridgeland [Br1] proves that the space of stability conditions $\operatorname{Stab}(X)$ on the derived category of a smooth projective variety X is a complex manifold. Given a Chern character ξ, there is a wall and chamber decomposition of $\operatorname{Stab}(X)$, where the space of semistable objects with Chern character ξ in a chamber remains constant.

The main examples of Bridgeland stability conditions for $\mathbb{P}^2$ are parameterized by an upper-half-plane $(s,t) \in \mathbb{R}^2, t > 0$. In §5, we will discuss the wall and chamber decomposition in the (s,t)-plane. For each Chern character ξ, there are finitely many Bridgeland walls [ABCH]. Let $\mu(\xi)$ denote the slope of ξ. Except for a vertical wall at $s = \mu(\xi)$, the Bridgeland walls are nested semicircles centered along the real axis. We call the outermost semicircular wall to the left of $s = \mu(\xi)$ the *Gieseker wall* and the inner most semicircular wall the *collapsing wall*. They correspond to the extremal rays of the ample and effective cone of $M(\xi)$, respectively. Computing these walls is the main topic of this survey.

For each stability condition σ in the (s,t) plane, there is a moduli space $M_\sigma(\xi)$ of Bridgeland semistable objects. These moduli spaces are projective varieties that can be constructed via GIT [ABCH]. They provide modular interpretations of the birational models of $\mathbb{P}^{2[n]}$ [ABCH], [LZ]. For small n, the sequence of flips relating the different models can be fully computed [ABCH].

The ample cone. In §6, we will give the description of the ample cone of the moduli spaces of sheaves $M(\xi)$ when the rank and the first Chern class are coprime and the second Chern class is sufficiently negative following [CH2]. We will also describe the ample cone when the rank is at most 6, but we will omit the details in this case. By results of Bayer and Macrì [BM], computing $\mathrm{Amp}(M(\xi))$ reduces to computing the Gieseker wall. We first guess the equation of the Gieseker wall W_{guess}. Using numerical properties of Bridgeland stability, we show that there cannot be larger walls. Then we explicitly construct sheaves that are destabilized along W_{guess} to show that it is the Gieseker wall.

The effective cone. Finally, in §7, we describe the effective cone of $M(\xi)$ following [CHW]. This will generalize the discussion of $\mathrm{Eff}(\mathbb{P}^{2[n]})$ in §3. There is a close connection between stable base loci of linear systems and interpolation problems. We introduce the higher rank interpolation problem in §7 and explain its solution in the following 3 cases:

(1) For complete intersection schemes in $\mathbb{P}^{2[n]}$,
(2) For monomial schemes in $\mathbb{P}^{2[n]}$,
(3) For a general sheaf F in $M(\xi)$.

The first two cases serve as warm up for the third case and they determine the linear systems on $\mathbb{P}^{2[n]}$ that contain a given complete intersection or monomial scheme in their stable base locus. The third case computes $\mathrm{Eff}(M(\xi))$.

The key to all these computations is to find a convenient resolution of the sheaf I_Z or F. Bridgeland stability determines the resolution to use. If a sheaf F is destabilized along a Bridgeland wall W by a subobject A, then the resolution

$$0 \to A \to F \to B \to 0$$

allows us to solve interpolation problems inductively. The main challenge then is to compute the Bridgeland wall where F is destabilized and determine the destabilizing sequence. In §7, we will completely describe these in the three cases following [CH] and [CHW].

Finally, we stress that none of the material presented in this survey is original. We intend it as a guide that highlights the key points of the arguments already in the literature.

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2. Preliminaries on birational geometry

In this section, we recall basic definitions and results of birational geometry necessary for this survey. We refer the reader to [De], [KM] and [La] for detailed information.

Let X be a normal projective variety. The variety X is called *factorial* if every Weil divisor is Cartier; X is called $\mathbb{Q}$-factorial if every Weil divisor has a multiple which is Cartier. For simplicity, we will always assume that X is $\mathbb{Q}$-factorial. Following the work of Kleiman, Kollár, Mori, Reid and others in the 1980s, it is customary to translate problems of birational geometry on X to problems of convex geometry.

Definition 2.1. Two Cartier divisors D_1, D_2 are *numerically equivalent* ($D_1 \equiv D_2$) if $D_1 \cdot C = D_2 \cdot C$ for every curve $C \subset X$. This definition easily extends to $\mathbb{Q}$ or $\mathbb{R}$ Cartier divisors $\mathrm{Pic}(X) \otimes \mathbb{Q}$ or $\mathrm{Pic}(X) \otimes \mathbb{R}$. The Néron-Severi space $N^1(X)$ is the $\mathbb{R}$-vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence $\mathrm{Pic}(X) \otimes \mathbb{R}/\equiv$.

The idea of Mori is to encode contractions and birational contractions of a variety by convex cones in the vector space $N^1(X)$.

Definition 2.2. A Cartier divisor D is *ample* if a sufficiently large multiple $\mathcal{O}_X(mD)$ defines an embedding of X into projective space. An $\mathbb{R}$-divisor is ample if it is a nonnegative $\mathbb{R}$-linear combination of ample divisors. A Cartier divisor D is *nef* if $D \cdot C \geq 0$ for every curve C on X.

Numerical nature of ampleness. The Nakai-Moishezon Criterion [La, 1.2.23] says that D is ample if and only if $D^{\dim(V)} \cdot V > 0$ for every subvariety $V \subseteq X$. Consequently, being ample is a numerical condition. The set of ample divisors in $N^1(X)$ forms an open convex cone called the *ample cone* $\mathrm{Amp}(X)$. Being nef is evidently numerical. The set of nef divisors in $N^1(X)$ forms a closed convex cone called the *nef cone* $\mathrm{Nef}(X)$. Kleiman's Theorem [La, 1.4.9] clarifies the relation between these two cones. The closure of $\mathrm{Amp}(X)$ is $\mathrm{Nef}(X)$ and the interior of $\mathrm{Nef}(X)$ is $\mathrm{Amp}(X)$. These cones record all the contractions of X and hence are among the most important invariants of X.

Computing the ample cone. If $f : X \to Y$ is a morphism, the pullback of an ample divisor D on Y is a nef divisor on X. By constructing birational morphisms from X to other projective varieties and pulling back ample divisors, we generate a subcone of $\mathrm{Nef}(X)$. Conversely, given any irreducible curve C on X, we obtain the inequality $C \cdot D \geq 0$ that must be satisfied by every nef divisor on X. Thus, by finding irreducible curves on X, we cut out a cone that contains $\mathrm{Nef}(X)$. One strategy for computing the nef cone is to construct enough morphisms from X and curves on X to make these two bounds coincide.

Definition 2.3. A divisor is *effective* if it is a nonnegative linear combination of codimension one subvarieties. A divisor D is *big* if for some $m > 0$ the linear system $|mD|$ defines a rational map on X whose image has the same dimension as X.

Numerical nature of bigness. Kodaira's Lemma [La, 2.2.7] gives a useful characterization of big divisors. A divisor is big if and only if it is numerically equivalent to the sum of an ample and an effective divisor. Hence, being big is a numerical condition. The set of big divisors in $N^1(X)$ forms an open convex cone called the *big cone* $\mathrm{Big}(X)$. Every big divisor has a positive multiple which is effective.

Being effective is not a numerical condition. For example, on an elliptic curve all degree zero divisors are numerically equivalent, but only torsion divisors have a positive multiple which is effective. Consequently, some care is needed when discussing numerical equivalence classes of effective divisors. The *effective cone* $\mathrm{Eff}(X)$ is the cone spanned by numerical classes of all effective divisors. The cone $\mathrm{Eff}(X)$ contains $\mathrm{Big}(X)$, but in general it is neither open nor closed (see [La, 1.5.1] for an example of a two-dimensional effective cone that contains one of its extremal rays but not the other). The closure of $\mathrm{Eff}(X)$ is called the *pseudo-effective cone* $\overline{\mathrm{Eff}}(X)$.

Computing the effective cone. On a $\mathbb{Q}$-factorial variety, one generates a subcone of $\mathrm{Eff}(X)$ by finding codimension one subvarieties. An irreducible curve C on X is called *moving* if deformations of C cover a Zariski dense subset of X. The intersection number of an effective divisor D and an irreducible curve C can be negative only if $C \subset D$. Consequently, given a moving curve C, we must have $C \cdot D \geq 0$ for every effective divisor. Hence, moving curve classes cut out a cone that contains $\mathrm{Eff}(X)$. In fact, a theorem of Boucksom, Demailly, Paun and Peternell [BDPP] characterizes $\overline{\mathrm{Eff}}(X)$ as the dual to the cone of moving curves. One strategy for computing $\mathrm{Eff}(X)$ is to construct enough effective divisors and moving curves to make the two bounds coincide.

Definition 2.4. The *base locus* of a divisor D is the intersection of all the divisors linearly equivalent to D

$$\mathrm{Bs}(D) = \bigcap_{E \in |D|} E.$$

The *stable base locus* of D is the intersection of base loci of positive multiples of D

$$\mathbb{B}\mathrm{s}(D) = \bigcap_{m\geq 1} \mathrm{Bs}(mD).$$

The rational map given by an effective divisor D is typically not everywhere defined on X. The indeterminacy locus of the map is contained in $\mathrm{Bs}(D)$. The stable base locus is related to the indeterminacy of the maps defined by sufficiently large and divisible multiples of D. Given a rational map $f : X \dashrightarrow Y$, the pullback of an ample divisor on Y has stable base locus contained in the indeterminacy of the map f.

Definition 2.5. A divisor is called *movable* if its stable base locus has codimension at least two. The *movable cone* $\mathrm{Mov}(X)$ is the cone in $N^1(X)$ spanned by the classes of movable divisors.

The movable cone encodes the birational contractions of X. We have the following containments among the cones we introduced

$$\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \cap \mathrm{Mov}(X) \cap \mathrm{Big}(X), \text{ and } \mathrm{Nef}(X), \mathrm{Big}(X), \mathrm{Mov}(X) \subset \overline{\mathrm{Eff}}(X).$$

These cones are the key players in the birational geometry of X and will be the focus of our attention in this survey.

Given an integral divisor D, the *section ring* is

$$R(D) = \bigoplus_{m\geq 0} H^0(X, mD).$$

If this ring is finitely generated, then the model associated to D is $M(D) = \mathrm{Proj}(R(D))$. It is important to have an explicit description of the rational map $X \dashrightarrow M(D)$ and decompose the map into a sequence of simple birational transformations called divisorial contractions, flips and Mori fibrations (see [De], [KM] for definitions and details).

The Cox ring and Mori dream spaces. Pick a basis $D_1, \ldots, D_r$ of $\mathrm{Pic}(X)$. The *Cox Ring* or *total coordinate ring* of X is

$$\bigoplus_{m_1,\ldots,m_r} H^0(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r).$$

A *Mori dream space* is a variety X where $N^1(X) \cong \mathrm{Pic}(X)\otimes\mathbb{R}$ and the Cox ring is finitely generated. This notion was introduced by Hu and Keel [HK] and distinguishes a nice class of varieties from the point of view of the Minimal Model Program. The Minimal Model Program runs and terminates for every effective divisor on a Mori dream space.

3. Preliminaries on the Hilbert scheme of points

The Hilbert scheme $\mathbb{P}^{2[n]}$ of points on $\mathbb{P}^2$ is the first example of a moduli space of sheaves on $\mathbb{P}^2$. As motivation for later sections, we begin by discussing the geometry of $\mathbb{P}^{2[n]}$. We refer the reader to [Go], [Le] and [N] for systematic developments of the basic geometry of the Hilbert scheme. For more information on the birational geometry of the Hilbert scheme of points on surfaces, the reader can refer to [ABCH], [BM], [BM2], [BC], [CH] and [H].

The configuration space and the symmetric product. Let S be a smooth projective surface. The configuration space $\mathrm{Config}_n(S)$ parameterizes unordered n-tuples of distinct points on S. The symmetric product $S^{(n)} = S^n/\mathfrak{S}_n$, which is the quotient of the n-fold product of S under the action of the symmetric group permuting the labelling of the points, gives a natural compactification of $\mathrm{Config}_n(S)$.

The symmetric product has many nice properties. For example, it has a stratification by the multiplicities of the points. Let $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ be a partition of n. Let $S_{m_1,\ldots,m_r} \subset S^{(n)}$

be the locus parameterizing r distinct points $(p_1, \dots, p_r)$ that occur with repetitions $(m_1, \dots, m_r)$ in $S^{(n)}$, respectively. Then for each partition $m_1 \geq \cdots \geq m_r$ of n, $S_{m_1,\dots,m_r}$ is a locally closed subset of $S^{(n)}$of dimension $2r$. The symmetric product $S^{(n)}$ has a stratification by these subsets. Consequently, the geometry of $S^{(n)}$ has a nice combinatorial structure. However, $S^{(n)}$ is singular.

Definition 3.1. The *Hilbert scheme* $S^{[n]}$ of n points is the scheme that parameterizes zero-dimensional subschemes of S of length n, or, equivalently, subschemes of S that have constant Hilbert polynomial n.

The Hilbert schemes $S^{[n]}$ play an important role in many branches of mathematics, including algebraic geometry, topology, combinatorics, representation theory and mathematical physics. For example, when S is a K3 surface, $S^{[n]}$ furnish some of the few known examples of compact holomorphic symplectic manifolds. The geometry of $S^{[n]}$ is crucial in Haiman's work on the $n!$ conjecture [Hai]. The cohomology of $S^{[n]}$ is closely tied to representations of the Heisenberg Lie algebra via Nakajima's work [N].

Basic properties of the Hilbert scheme. The union of n reduced points forms a scheme of length n. Hence, $\mathrm{Config}_n(S)$ is naturally a subset of $S^{[n]}$. By a theorem of Fogarty, $S^{[n]}$ provides a smooth, projective compactification of $\mathrm{Config}_n(S)$.

Theorem 3.2 (Fogarty [F1])**.** *Let S be a smooth projective surface. The Hilbert scheme of points $S^{[n]}$ is a smooth, irreducible, projective variety of dimension $2n$. The configuration space $\mathrm{Config}_n(S)$ is a dense Zariski open subset of $S^{[n]}$.*

Remark 3.3. Given a smooth projective variety X, the configuration space $\mathrm{Config}_n(X)$ of n points on X parameterizes unordered n-tuples of points on X. The symmetric product $X^{(n)} = X^n/\mathfrak{S}_n$ gives a compactification of $\mathrm{Config}_n(X)$. When $\dim(X) = 1$, $X^{(n)}$ is smooth and coincides with the Hilbert scheme parameterizing length n subschemes of X. When $\dim(X) \geq 3$, the Hilbert scheme of points in general is singular and has many irreducible components of different dimensions. This is the reason to concentrate on the case of surfaces.

There is a morphism $h : S^{[n]} \to S^{(n)}$ called the *Hilbert-Chow morphism* that associates to a scheme Z its support counted with multiplicity

$$Z \mapsto \sum_{p \in \mathrm{Supp}(Z)} l_p(Z)p$$

(see [Le]). A *resolution of singularities* is a birational morphism $f : X \to Y$, where X is smooth and $Y - Y^{\mathrm{sing}}$ is isomorphic to $X - f^{-1}(Y^{\mathrm{sing}})$. A resolution is *crepant* if $f^*(K_Y) = K_X$.

Theorem 3.4 (Fogarty [F1])**.** *The Hilbert-Chow morphism $h : S^{[n]} \to S^{(n)}$ is a crepant resolution of singularities.*

Theorem 3.4 allows us to compute the canonical bundle of $S^{[n]}$ and conclude the following corollary (see [BC]).

Corollary 3.5. *Let S be a surface with ample anticanonical bundle $-K_S$. Then $S^{[n]}$ is a log Fano variety. In particular, $S^{[n]}$ is a Mori dream space.*

The punctual Hilbert scheme. The stratification of $S^{(n)}$ by multiplicity induces a stratification of the Hilbert scheme $S^{[n]}$ via the Hilbert-Chow morphism. A scheme supported at one point is called a *punctual scheme*. To understand points of $S^{[n]}$, it suffices to understand punctual schemes since any zero-dimensional scheme naturally decomposes into punctual schemes along its support. The scheme parameterizing length n subschemes of S supported at a point x is called the *Briançon variety* B_n.

Up to isomorphism, the Briançon varieties B_n are independent of the choice of smooth surface or point. Therefore, we can consider the local model $\mathbb{C}^{2[n]}_{(0,0)}$. Let

$$I = (x^n, y - a_1x - a_2x^2 - \cdots - a_{n-1}x^{n-1}).$$

Then $\mathbb{C}[x,y]/I$ is spanned by

$$1, x, x^2, \cdots, x^{n-1},$$

hence I is a point of $\mathbb{C}^{2[n]}$. These zero-dimensional schemes are called *curvilinear schemes* since they are contained in the smooth curve defined by $y - a_1x - a_2x^2 - \cdots - a_{n-1}x^{n-1}$. They form an $(n-1)$-dimensional smooth locus in B_n. A Theorem of Briançon ([Br], see also [Go], [Le]) says that curvilinear schemes are dense in B_n.

Theorem 3.6 (Briançon [Br])**.** *The Briançon variety B_n on a surface is irreducible of dimension $n-1$ and contains the curvilinear schemes as a dense open subset.*

Consequently, the inverse image of $S_{m_1,\ldots,m_r}$ under the Hilbert-Chow morphism is irreducible and has dimension $r+n$. In particular, the locus of nonreduced schemes of length n forms an irreducible divisor B.

The Picard group of $\mathbb{P}^{2[n]}$. From now on we will restrict to the case $S = \mathbb{P}^2$. We can understand the Néron-Severi space of $\mathbb{P}^{2[n]}$ in terms of the Hilbert-Chow morphism. The exceptional locus of h is the irreducible divisor B parameterizing nonreduced schemes. Since the diagonal has codimension 2 in $\mathbb{P}^{2(n)}$ and $\mathbb{P}^{2(n)}$ is $\mathbb{Q}$-factorial, we conclude that the rational Picard groups of $\mathbb{P}^{2[n]} - B$ and $\mathbb{P}^{2(n)}$ are isomorphic.

The Picard group of $\mathbb{P}^{2(n)}$ is isomorphic to the $\mathfrak{S}_n$-invariant line bundles on $(\mathbb{P}^2)^n$ and is generated by one element $\mathcal{O}_{\mathbb{P}^{2(n)}}(1)$. We can pull it back via h. Geometrically, schemes whose support intersect a fixed line l in $\mathbb{P}^2$ give a section of this line bundle on $\mathbb{P}^{2[n]}$. We denote its class by H. Hence, the Néron-Severi space of $\mathbb{P}^{2[n]}$ is the two-dimensional $\mathbb{Q}$-vector space spanned by H and B. In fact, Fogarty computed the Picard group over $\mathbb{Z}$ for any surface. We state here the special case of $\mathbb{P}^2$.

Theorem 3.7 (Fogarty [F2])**.** *The Picard group of $\mathbb{P}^{2[n]}$ is isomorphic to $\mathbb{Z}H \oplus \mathbb{Z}\frac{B}{2}$.*

The class $\frac{B}{2}$ is not effective, so it is harder to make sense of it geometrically. Since we will work with cones of divisors, we will be able to scale the divisors. Hence, there is no harm in working with B.

The ample cone of $\mathbb{P}^{2[n]}$. Since $N^1(\mathbb{P}^{2[n]})$ is the two-dimensional vector space spanned by H and B, a convex cone in $N^1(\mathbb{P}^{2[n]})$ is determined by specifying its two extremal rays. We implement the strategy for computing $\mathrm{Amp}(\mathbb{P}^{2[n]})$ outlined in §2.

The Hilbert-Chow morphism h is a birational morphism which is not an isomorphism. It contracts the locus of nonreduced schemes. More concretely, fix $n-1$ distinct points $p_0, p_1, \ldots, p_{n-2}$. Choose coordinates so that $p_0 = [0:0:1]$. The ideals

$$I = (y - mx, y^2)$$

define double points at p_0 and are parameterized by the $\mathbb{P}^1$ of tangent directions at p_0. The curve parameterizing double points at p_0 union the points $p_1, \ldots, p_{n-2}$ is contracted by h to the point $2p_0 + \sum_{i=1}^{n-2} p_i$. Hence, H is base-point-free but not ample and defines an extremal edge of the nef cone.

Maps to Grassmannians. The other extremal edge of the nef cone is harder to describe. Let Z be a zero-dimensional scheme and let I_Z denote its ideal sheaf. Consider the standard exact sequence of sheaves

$$0 \to I_Z(k) \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{O}_Z(k) \to 0.$$

The associated long exact sequence of cohomology yields the inclusion

$$H^0(\mathbb{P}^2, I_Z(k)) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)).$$

The dimension of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$ is $N(k) = \binom{k+2}{2}$. To require a polynomial to vanish at a point imposes one linear condition on polynomials. If the conditions imposed by Z are independent, the vector space $H^0(\mathbb{P}^2, I_Z(k))$ has dimension $N(k) - n$. For a general set of points, these conditions are independent. However, for special sets of points, the conditions may fail to be independent.

Example 3.8. Four points impose independent conditions on conics in $\mathbb{P}^2$ if and only if the four points are not collinear.

By sending a scheme Z to the vector space $H^0(\mathbb{P}^2, I_Z(k))$, we get a rational map

$$\phi_k : \mathbb{P}^{2[n]} \dashrightarrow G(N(k) - n, N(k))$$

to the Grassmannian of $(N(k) - n)$-dimensional subspaces of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$. In general, ϕ_k is only a rational map because as Example 3.8 shows some schemes may fail to impose independent conditions on polynomials of degree k and do not determine an $(N(k) - n)$-dimensional subspace.

If $k \geq n - 1$, then every scheme of length n imposes independent conditions on polynomials of degree k and ϕ_k is a morphism. Every zero-dimensional scheme in $\mathbb{P}^2$ has a minimal free resolution of the form

$$0 \to \oplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \oplus_{i=1}^{m+1} \mathcal{O}_{\mathbb{P}^2}(-a_i) \to I_Z \to 0,$$

where $n + 1 \geq b_j$ for $1 \leq j \leq m$ and $n \geq a_i$ for $1 \leq i \leq m + 1$ (see [**?**]). Twisting by $\mathcal{O}_{\mathbb{P}^2}(k)$ and taking cohomology, one can see that $h^1(\mathbb{P}^2, I_Z(k)) = h^2(\mathbb{P}^2, I_Z(k)) = 0$ for $k \geq n - 1$. Since the Euler characteristic is constant, we conclude that $h^0(\mathbb{P}^2, I_Z(k))$ always has the expected dimension if $k \geq n - 1$.

The second extremal edge of the ample cone. Consider the morphism

$$\phi_{n-1} : \mathbb{P}^{2[n]} \to G = G(N(n-1) - n, N(n-1)),$$

where $N(n-1) = \binom{n+1}{2}$. The pullback $\phi_{n-1}^* \mathcal{O}_G(1)$ is a base-point-free divisor. Hence, it is nef. However, $\phi_{n-1}^* \mathcal{O}_G(1)$ is not ample. Every scheme of length n imposes independent conditions on polynomials of degree $n - 1$; however, polynomials of degree $n - 1$ do not suffice to cut out every scheme of length n. Suppose Z consists of n collinear points. Then any polynomial of degree $n - 1$ vanishing on Z vanishes along the line containing Z. Hence, the vector space of polynomials of degree $n - 1$ vanishing on Z is the vector space of polynomials of degree $n - 1$ that are divisible by the equation of the line. If we take any other n points on the same line, this vector space does not change. Hence, $\phi_{n-1}^* \mathcal{O}_G(1)$ has degree zero on positive-dimensional subvarieties of $\mathbb{P}^{2[n]}$ and is not ample. We conclude that $\phi_{n-1}^* \mathcal{O}_G(1)$ spans the other extremal ray of the nef cone.

Computing the class of $\phi_{n-1}^* \mathcal{O}_G(1)$. One can compute the class of $\phi_{n-1}^* \mathcal{O}_G(1)$ in terms of H and B using test curves. Fix $n - 1$ general points Γ and a general line l disjoint from the points. Given $p \in l$, let Z_p be the scheme $\Gamma \cup p$ of length n. Let A be the curve in $\mathbb{P}^{2[n]}$ obtained by varying the point p along l. Since all the schemes Z_p are reduced, the resulting curve is disjoint from B. Its degree with respect to H is one. Finally, fix $\binom{n+1}{2} - n$ general points Ω and consider the linear spaces W of polynomials of degree $n-1$ that vanish at these points. Then subspaces of codimension n that intersect W give a section of $\mathcal{O}_G(1)$. There is a unique curve of degree $n-1$ containing $\Gamma \cup \Omega$.

The line l intersects this curve in $n-1$ points. Consequently, we have the following intersection numbers

$$A \cdot H = 1,\ A \cdot B = 0,\ A \cdot \phi_{n-1}^* \mathcal{O}_G(1) = n-1.$$

Next, take a general pencil in $|\mathcal{O}_{\mathbb{P}^1}(n)|$ and consider the curve C induced in $\mathbb{P}^{2[n]}$. By the Riemann-Hurwitz formula, this pencil is ramified $2n-2$ times. The points in the pencil meet a general line once. Since the resulting map to G is constant it has degree zero on $\phi_{n-1}^* \mathcal{O}_G(1)$. We conclude that we have the following intersection numbers

$$C \cdot H = 1,\ C \cdot B = 2n-2,\ C \cdot \phi_{n-1}^* \mathcal{O}_G(1) = 0.$$

Hence, the class of $\phi_{n-1}^* \mathcal{O}_G(1)$ is $(n-1)H - \frac{1}{2}B$. We have proved the following theorem.

Theorem 3.9 ([LQZ], [ABCH])**.** *The nef cone of $\mathbb{P}^{2[n]}$ is the closed cone spanned by*

$$H \quad \textit{and} \quad (n-1)H - \frac{1}{2}B.$$

The first defines the Hilbert-Chow morphism and contracts the locus of nonreduced schemes. The latter defines the morphism

$$\phi_{n-1} : \mathbb{P}^{2[n]} \to G(N(n-1)-n, N(n-1)),$$

which contracts the locus of schemes supported on a fixed line.

The effective cone of $\mathbb{P}^{2[n]}$. The effective cone of $\mathbb{P}^{2[n]}$ is harder to compute and depends more subtly on arithmetic properties of n. We will now give examples and state the second author's theorem from [H] computing the cone for all n. In later sections, we will explain the proof and discuss generalizations of the theorem to the moduli spaces of sheaves.

The locus of nonreduced schemes B is the exceptional divisor of the Hilbert-Chow morphism. Consequently, it defines an extremal edge of the effective cone. The main problem is to describe the other extremal ray.

Example: n is a triangular number. In $\mathbb{P}^{2[3]}$ the locus of collinear schemes $D_{\mathcal{O}(1)}$ is a divisor with class $H - \frac{1}{2}B$. Following the strategy described in §2, to see that $D_{\mathcal{O}(1)}$ is extremal, we need to construct a dual moving curve. Given three noncollinear points p_1, p_2, p_3, let C be a smooth conic containing them. A pencil in the linear system $|\mathcal{O}_C(p_1+p_2+p_3)|$ induces a moving curve in $\mathbb{P}^{2[3]}$. This curve is disjoint from $D_{\mathcal{O}(1)}$ since none of the points in the pencil can be collinear by Bezout's Theorem. We conclude that $D_{\mathcal{O}(1)}$ is extremal.

More generally, let $n = \frac{k(k+1)}{2}$ be a triangular number. The schemes that lie on a curve of degree $k-1$ form an effective divisor $D_{\mathcal{O}(k-1)}$ with class $(k-1)H - \frac{1}{2}B$ (see [ABCH]). By exhibiting a disjoint moving curve, one can see that $D_{\mathcal{O}(k-1)}$ is extremal. Let $p_1, \ldots, p_n$ be a set of points that do not lie on a curve of degree $k-1$. Let C be a smooth curve of degree k that contains $p_1, \ldots, p_n$. Consider the curve in $\mathbb{P}^{2[n]}$ induced by a pencil in the linear system $|\mathcal{O}_C(\sum_{i=1}^n p_i)|$. Since this linear system is not equal to $|\mathcal{O}_C(k-1)|$ by assumption, none of the members of this pencil lie on a curve of degree $k-1$. Consequently, we obtain a moving curve disjoint from $D_{\mathcal{O}(k-1)}$. We conclude that $D_{\mathcal{O}(k-1)}$ spans the other extremal ray of the effective cone.

Example: n is one less or one more than a triangular number. Similar constructions work when n is one less or one more than a triangular number. When $n = \frac{k(k+1)}{2} + 1$, an extremal ray of the effective cone is spanned by the divisor of schemes of length n that have a subscheme of length $n-1$ that is contained in a curve of degree $k-1$. Similarly, when $n = \frac{k(k+1)}{2} - 1$, an extremal ray of the effective cone is spanned by the divisor of schemes that together with an auxiliary point p lie on a curve of degree $k-1$.

Example: $n = 12$**.** The first interesting case which cannot be reduced to the previous examples is $n = 12$. Let us reinterpret the previous examples. Three points are collinear if they fail to impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^2}(1)$. Similarly, $n = \frac{k(k+1)}{2}$ points lie on a curve of degree $k-1$ if they fail to impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^2}(k-1)$. In all these examples, $\chi(I_Z(k-1)) = 0$ and the locus of Z with $h^0(I_Z(k-1)) \neq 0$ is a divisor.

The key idea is to replace line bundles by higher rank vector bundles. Consider the bundle $T_{\mathbb{P}^2}(2)$. Using the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{O}_{\mathbb{P}^2}(3)^{\oplus 3} \to T_{\mathbb{P}^2}(2) \to 0,$$

we see that $h^0(\mathbb{P}^2, T_{\mathbb{P}^2}(2)) = 24$. Since $T_{\mathbb{P}^2}(2)$ has rank 2, we expect each point to impose 2 linear conditions on sections of $T_{\mathbb{P}^2}(2)$. Hence, for a general set of 12 points, we expect the map

$$H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(2)) \to H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(2) \otimes \mathcal{O}_Z)$$

to be an isomorphism. This is indeed the case. Consequently, the locus of schemes in $\mathbb{P}^{2[12]}$ where the map fails to be an isomorphism is a divisor $D_{T_{\mathbb{P}^2}(2)}$ with class $7H - B$.

Unlike in the line bundle case, proving the required cohomology vanishing is no longer trivial. When $n = 12$, the following argument works. By the Euler sequence, $T_{\mathbb{P}^2}(-2)$ has no cohomology. By Serre duality, $T_{\mathbb{P}^2}(-4)$ also has no cohomology. A general collection of 12 points has a resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3} \to I_Z \to 0.$$

Twisting this sequence by $T_{\mathbb{P}^2}(2)$ and taking cohomology, we conclude that $H^i(\mathbb{P}^2, T_{\mathbb{P}^2}(2) \otimes I_Z) = 0$ for all i since $T_{\mathbb{P}^2}(-2)$ and $T_{\mathbb{P}^2}(-4)$ have no cohomology.

Finally, to show that $D_{T_{\mathbb{P}^2}(2)}$ is extremal in $\mathbb{P}^{2[12]}$, we need to exhibit a dual moving curve. Vary the scheme Z by varying the map $\mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3}$ in the resolution. There exist complete curves in $\mathbb{P}^{2[12]}$ obtained this way and our cohomology computation shows that they are all disjoint from $D_{T_{\mathbb{P}^2}(2)}$. Since these curves are moving, we conclude that $D_{T_{\mathbb{P}^2}(2)}$ is extremal.

The strategy in general. Given a rank r bundle E, we expect that requiring a section to vanish at a point imposes r linear conditions on the space of sections. If we ask the sections to vanish at n points, we would expect to get a codimension rn vector subspace of the space of sections. Unlike the case of line bundles, even if the points are general, the conditions they impose may fail to be independent. If E is a vector bundle with $h^0(E) = rn$ and $h^i(E) = 0$ for $i = 1, 2$, we have $\chi(E \otimes I_Z) = 0$. Assuming that Z imposes independent conditions on sections of E, the map

$$H^0(\mathbb{P}^2, E \otimes I_Z) \to H^0(\mathbb{P}^2, E \otimes \mathcal{O}_Z)$$

is an isomorphism. The locus of schemes where the map is not an isomorphism is an effective divisor D_E. One can easily compute the class of D_E as $c_1(E)H - \mathrm{rk}(E)\frac{B}{2}$. This raises the question of finding vector bundles E such that a general set of n points imposes independent conditions on their sections. The following theorem completely answers this question. Define the slope of a vector bundle E by $\frac{\mathrm{ch}_1(E)}{\mathrm{rk}(E)}$.

Theorem 3.10 (Huizenga [H])**.** *Let Z be a general point in $\mathbb{P}^{2[n]}$. Let $\mu_{\min}$ be the minimal positive slope of a stable vector bundle E such that $\chi(E \otimes I_Z) = 0$. Let F be a general stable vector bundle of slope $\mu \geq \mu_{\min}$ and sufficiently large and divisible rank such that $\chi(F \otimes I_Z) = 0$. Then $F \otimes I_Z$ has no cohomology and D_F is an effective divisor. In particular, the effective cone of $\mathbb{P}^{2[n]}$ is the closed convex cone spanned by the rays*

$$B \text{ and } \mu_{\min} H - \frac{1}{2} B.$$

Given n, it is easy to compute $\mu_{\min}$ in practice. We will explain how to compute $\mu_{\min}$ in §4. We will discuss and prove a generalization of Theorem 3.10 to the moduli spaces of Gieseker semistable sheaves in §7.

4. Preliminaries on the Moduli spaces of sheaves

In this section, we review preliminaries on moduli spaces of Gieseker semistable sheaves on $\mathbb{P}^2$. More detailed developments of the theory can be found in [DLP], [HuL] and [LP].

4.1. **Basic definitions.** Let F be a coherent sheaf on $\mathbb{P}^2$. The dimension $\dim(F)$ of F is the dimension of the support of F. The sheaf F is *pure* or has *pure dimension* d if every nontrivial coherent subsheaf of F has dimension d. If F has dimension d, then the asymptotic Riemann-Roch Theorem [La] implies that the Hilbert polynomial $P_F(m) = \chi(F(m))$ has the form

$$P_F(m) = a_d \frac{m^d}{d!} + O(m^{d-1}).$$

The Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^2}$ will be denoted by P and will play a special role throughout our discussion. Recall that

$$P(m) = P_{\mathcal{O}_{\mathbb{P}^2}}(m) = \frac{1}{2}\left(m^2 + 3m + 2\right).$$

The *reduced Hilbert polynomial* p_F is defined by

$$p_F = \frac{P_F}{a_d}.$$

Definition 4.1. A coherent sheaf F is *(Gieseker) semistable* if F is pure and whenever $E \subseteq F$ then $p_E(m) \leq p_F(m)$ for $m \gg 0$. If the inequality is strict for every nontrivial proper subsheaf E, then F is called *(Gieseker) stable*.

In this survey, (semi)stable will always mean Gieseker (semi)stable. When we discuss several notions of stability simultaneously, we will specify Gieseker for clarity.

Logarithmic invariants. Fix a Chern character $\xi = (r, \mathrm{ch}_1, \mathrm{ch}_2)$ on $\mathbb{P}^2$. A character ξ is *(semi)stable* if it is the Chern character of a (semi)stable sheaf. If the rank $r > 0$, then the *slope* μ and the *discriminant* Δ are defined by

$$\mu = \frac{\mathrm{ch}_1}{r}, \quad \text{and} \quad \Delta = \frac{1}{2}\mu^2 - \frac{\mathrm{ch}_2}{r}.$$

The rank, slope and discriminant determine the Chern character. Hence, we can equivalently record the Chern character by $\xi = (r, \mu, \Delta)$. The advantage of the slope and the discriminant is that they are additive on tensor products

$$\mu(E \otimes F) = \mu(E) + \mu(F) \qquad \text{and} \qquad \Delta(E \otimes F) = \Delta(E) + \Delta(F).$$

The classification of stable vector bundles on $\mathbb{P}^2$ is more conveniently expressed in terms of these invariants, so we will primarily use them instead of ch_1 and ch_2. Consequently, we need to recast our notions and formulae in terms of these invariants. When $r > 0$, the Riemann-Roch formula reads

$$\chi(E) = r(P(\mu) - \Delta).$$

Given two sheaves E and F, their Euler characteristic is defined by the formula

$$\chi(E, F) = \sum_{i=0}^{2} (-1)^i \operatorname{ext}^i(E, F).$$

When both sheaves have non-zero rank, the Euler characteristic is computed by the Riemann-Roch formula

$$\chi(E,F) = r(E)r(F)\left(P(\mu(F)-\mu(E)) - \Delta(E) - \Delta(F)\right).$$

One can express the semistability of a pure sheaf F with $r > 0$ in terms of μ and Δ. The sheaf F is semistable if and only if for every nontrivial proper subsheaf $E \subset F$, $\mu(E) \leq \mu(F)$ with $\Delta(E) \geq \Delta(F)$ in case of equality. There is a different notion of stability that plays a central role.

Definition 4.2. A pure sheaf F of rank $r > 0$ is *μ-(semi)stable* or *slope-(semi)stable* if for every nontrivial subsheaf $E \subset F$ with $r(E) < r(F)$, we have $\mu(E) < \mu(F)$ ($\mu(E) \leq \mu(F)$).

Remark 4.3. If F is a pure sheaf of rank $r > 0$, then

$$\mu\text{-stability} \implies \text{Gieseker stability} \implies \text{Gieseker semistability} \implies \mu\text{-semistability}.$$

The reverse implications are all false in general. However, if the rank and ch_1 are relatively prime, then a μ-semistable sheaf is automatically stable since there cannot be a subsheaf of smaller rank and the same slope. In that case, all four notions coincide.

Example 4.4. If E and F are two μ-semistable sheaves of the same slope, then $E \oplus F$ is μ-semistable. Similarly, if E and F are two Gieseker semistable sheaves with the same slope and discriminant, then $E \oplus F$ is Gieseker semistable. The structure sheaf $\mathcal{O}_{\mathbb{P}^2}$ and the ideal sheaf of a point I_p are both trivially μ-stable since they have rank 1. While $F = \mathcal{O}_{\mathbb{P}^2} \oplus I_p$ is μ-semistable, it is not Gieseker semistable since the inclusion $\mathcal{O}_{\mathbb{P}^2} \to F$ destabilizes F.

Stability has the following immediate consequences.

Observation 4.5. If E and F are stable sheaves such that $\mu(E) \geq \mu(F)$ with $\Delta(E) < \Delta(F)$ when $\mu(E) = \mu(F)$, then $\mathrm{Hom}(E,F) = 0$. Suppose there is a nonzero homomorphism $\phi : E \to F$. Then the image I of ϕ is a subsheaf of F and the kernel K of ϕ is a subsheaf of E. If $K \neq 0$, then by stability of E and convexity of slope in exact sequences, $\mu(K) \geq \mu(E) \geq \mu(I)$. Otherwise, $E \cong I$. Hence, by stability of F, $\mu(E) \leq \mu(I) \leq \mu(F)$. Since $\mu(E) \geq \mu(F)$ by assumption, all the slopes have to be equal. Again by stability and convexity of Δ, $\Delta(K) \geq \Delta(E) \geq \Delta(I) \geq \Delta(F)$. This contradicts the assumption $\Delta(E) < \Delta(F)$.

Observation 4.6. If E is a stable bundle, then $\mathrm{Hom}(E,E) \cong \mathbb{C}$ is generated by homotheties. If ϕ is a homomorphism between two stable bundles with the same slope and discriminant, ϕ is either an isomorphism or zero. Given a nonzero homomorphism $\phi \in \mathrm{Hom}(E,E)$, let λ be an eigenvalue of ϕ restricted to a fiber. Then $\phi - \lambda I$ is not an isomorphism, hence must be identically zero. We conclude that the only endomorphisms of a stable bundle are homotheties.

Observation 4.7. If E and F are stable sheaves on $\mathbb{P}^2$ and $\mu(E) - \mu(F) < 3$, then $\mathrm{Ext}^2(E,F) = 0$. By Serre duality, $\mathrm{Ext}^2(E,F)$ is dual to $\mathrm{Hom}(F, E(-3))$. Since $\mu(E(-3)) = \mu(E) - 3 < \mu(F)$ and both F and $E(-3)$ are stable, the latter is zero. In particular, if E is stable, then $\mathrm{Ext}^2(E,E) = 0$.

Stability also has much more subtle consequences such as the Bogomolov inequality.

Theorem 4.8 (Bogomolov)**.** *If E is a μ-semistable sheaf on a smooth projective surface, then $\Delta(E) \geq 0$.*

Later in this section, we will see a much more precise inequality for $\mathbb{P}^2$.

Filtrations. Every torsion-free sheaf can be canonically filtered by semistable sheaves. The corresponding filtration is known as the *Harder-Narasimhan filtration*. Each semistable sheaf can be further filtered by stable sheaves. The corresponding *Jordan-Hölder filtration* is not canonical, but the associated graded object is canonical.

Theorem 4.9 (Harder-Narasimhan Filtration)**.** *Let E be a torsion free sheaf. Then there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that all quotients $F_i = E_i/E_{i-1}$ are μ-semistable and

$$\mu_{\max}(E) = \mu(F_1) > \mu(F_2) > \cdots > \mu(F_m) = \mu_{\min}(E).$$

The same statement holds if we replace μ by the reduced Hilbert polynomial and μ-semistability by Gieseker semistability.

Given a semistable sheaf with reduced Hilbert polynomial p (or slope μ), we can find an increasing filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the graded pieces $F_i = E_i/E_{i-1}$ are stable with the same reduced Hilbert polynomial (slope). Two semistable sheaves are *S-equivalent* if their associated Jordan-Hölder factors are isomorphic.

Stability and moduli spaces. Stability is the notion required to construct moduli spaces of sheaves via GIT. We now state Gieseker's fundamental theorem. Let X be a smooth, polarized algebraic surface and let P be a Hilbert polynomial. For any variety Y, let $\mathcal{M}_X(P)(Y)$ be the functor that associates to Y, the set of isomorphism classes of Y-flat families of semistable sheaves on X with Hilbert polynomial P that are parameterized by Y.

Theorem 4.10 (Gieseker)**.** *There exists a projective coarse moduli space $M_X(P)$ for the functor $\mathcal{M}_X(P)$. The points of $M_X(P)$ parameterize S-equivalence classes of Gieseker semistable sheaves on X with Hilbert polynomial P. The set of isomorphism classes of stable sheaves forms an open subset of $M_X(P)$.*

The properties of stability we have discussed so far generalize to any smooth projective variety X and the moduli space of Gieseker semistable sheaves exists on X [Ma], [Si]. On $\mathbb{P}^2$, these moduli spaces have many desirable properties.

Theorem 4.11 (Drézet-Le Potier)**.** *Let ξ be the Chern character of a stable sheaf on $\mathbb{P}^2$ and let $M(\xi)$ denote the moduli space of Gieseker semistable sheaves with Chern character ξ. Then $M(\xi)$ is an irreducible, normal, factorial projective variety of dimension $r^2(2\Delta - 1) + 1$.*

Remark 4.12. By deformation theory, the Zariski tangent space to the moduli space is given by $\mathrm{Ext}^1(E, E)$ [HuL]. If E is stable, then the moduli space is smooth if $\mathrm{Ext}^2(E, E) = 0$ [HuL]. Since $\mathrm{Ext}^2(E, E)$ automatically vanishes for stable bundles on $\mathbb{P}^2$, $M(\xi)$ is smooth along the locus of stable bundles. Typically $M(\xi)$ has singularities along the locus of strictly semistable sheaves.

We now begin to review the detailed classification of stable vector bundles on $\mathbb{P}^2$. We begin by studying vector bundles whose moduli spaces are zero-dimensional.

4.2. **Exceptional bundles.** A stable sheaf E on $\mathbb{P}^2$ is called *exceptional* if $\mathrm{Ext}^1(E, E) = 0$. An exceptional sheaf must be a homogeneous vector bundle. Given $g \in \mathrm{Aut}(\mathbb{P}^2)$, $g^*(E)$ is another sheaf with the same invariants. Since the tangent space to the moduli space at E is trivial, the map $g \mapsto g^*(E)$ is constant. Hence, $g^*(E) \cong E$ for every $g \in \mathrm{Aut}(\mathbb{P}^2)$ and E is a homogeneous vector bundle.

Line bundles $\mathcal{O}_{\mathbb{P}^2}(n)$ and the tangent bundle $T_{\mathbb{P}^2}$ are examples of exceptional bundles. If E is an exceptional bundle, then any twist $E(n)$ and the dual E^* are also exceptional.

If E is exceptional, Observations 4.6 and 4.7 imply that

$$\mathrm{Hom}(E, E) \cong \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^2(E, E) = 0.$$

Consequently,

$$\chi(E,E) = r(E)^2(1-2\Delta(E)) = 1.$$

Hence,

$$\Delta(E) = \frac{1}{2}\left(1 - \frac{1}{r(E)^2}\right) < \frac{1}{2}.$$

Expanding in terms of the Chern character, we see that

$$1 = r^2(E) - \operatorname{ch}_1^2(E) + 2r(E)\operatorname{ch}_2(E) = r(E)(r(E) + 2\operatorname{ch}_2(E)) - \operatorname{ch}_1^2(E).$$

Since $2\operatorname{ch}_2(E)$ and $\operatorname{ch}_1(E)$ are integers, $r(E)$ and $\operatorname{ch}_1(E)$ are relatively prime. Consequently, E is necessarily μ-stable.

Exceptional slopes. A rational number α is called an *exceptional slope* if it is the slope of an exceptional bundle. Denote the set of exceptional slopes by $\mathscr{E}$. If E and F are exceptional bundles of the same slope, then

$$\chi(E,F) = r(E)r(F)(1-\Delta(E)-\Delta(F)) > 0.$$

By Observation 4.7, $\operatorname{Ext}^2(E,F) = 0$. Hence, $\hom(E,F) > 0$. By Observation 4.6, we conclude that E and F are isomorphic. Hence, given an exceptional slope α, there is a unique exceptional bundle E_α with slope α. Since its rank and first Chern class are relatively prime integers, its rank is necessarily the smallest integer r_α such that $r_\alpha\alpha \in \mathbb{Z}$ and its discriminant is

$$\Delta_\alpha = \frac{1}{2}\left(1 - \frac{1}{r_\alpha^2}\right).$$

There is a one-to-one correspondence between the set of exceptional slopes $\mathscr{E}$ and dyadic integers $\varepsilon : \mathbb{Z}\left[\frac{1}{2}\right] \to \mathscr{E}$, defined inductively by $\varepsilon(n) = n$ for $n \in \mathbb{Z}$ and

$$\varepsilon\left(\frac{2p+1}{2^{q+1}}\right) = \varepsilon\left(\frac{p}{2^q}\right).\varepsilon\left(\frac{p+1}{2^q}\right),$$

where $\alpha.\beta$ is defined by

$$\alpha.\beta = \frac{\alpha+\beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3+\alpha-\beta}.$$

Define the *order* of an exceptional slope $\alpha \in \mathscr{E}$ to be the smallest natural number q such that $\alpha = \varepsilon(\frac{p}{2^q})$.

Example 4.13. The following table shows the exceptional slopes in the interval $[0, \frac{1}{2}]$ of order up to 4.

$\frac{p}{2^q}$	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$
$\varepsilon\left(\frac{p}{2^q}\right)$	0	$\frac{13}{34}$	$\frac{5}{13}$	$\frac{75}{194}$	$\frac{2}{5}$	$\frac{179}{433}$	$\frac{12}{29}$	$\frac{70}{169}$	$\frac{1}{2}$
$\operatorname{ord}\left(\varepsilon\left(\frac{p}{2^q}\right)\right)$	0	4	3	4	2	4	3	4	1

Given $\alpha = \varepsilon\left(\frac{p}{2^q}\right)$ and $\beta = \varepsilon\left(\frac{p+1}{2^q}\right)$, one can define new exceptional bundles via mutation. Mutations are defined via the following exact sequences:

$$0 \to E_\alpha \to E_\beta \otimes \operatorname{Hom}(E_\alpha, E_\beta)^* \to M \to 0, \text{ and}$$

$$0 \to M' \to E_\alpha \otimes \operatorname{Hom}(E_\alpha, E_\beta) \to E_\beta \to 0,$$

where the maps are natural coevaluation and evaluation maps. All exceptional bundles are obtained via mutations starting from line bundles [D2]. The correspondence between dyadic integers and exceptional slopes follows from the construction of exceptional bundles by mutations.

Example 4.14. Starting with $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{\mathbb{P}^2}(1)$, we obtain $T_{\mathbb{P}^2}$ via the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) \to T_{\mathbb{P}^2} \to 0.$$

4.3. **The classification of stable bundles.** Suppose E is a stable sheaf of slope $\mu < \alpha$ with $\alpha - \mu < 3$ and discriminant Δ, then $\mathrm{Hom}(E_\alpha, E) = 0$. Since $\mathrm{Ext}^2(E_\alpha E) = 0$, we conclude that

$$\chi(E_\alpha, E) = P(\mu - \alpha) - \Delta_\alpha - \Delta \leq 0.$$

Similarly, if $\mu > \alpha$ and $\mu - \alpha < 3$, then $\mathrm{Hom}(E, E_\alpha) = 0$ and we conclude that

$$\chi(E_\alpha, E) = P(\alpha - \mu) - \Delta_\alpha - \Delta \leq 0.$$

Hence, the invariants of stable sheaves must lie above the fractal-like curve δ in the (μ, Δ)-plane defined by

$$\delta(\mu) = \sup_{\{\alpha \in \mathscr{E} : |\mu - \alpha| < 3\}} (P(-|\mu - \alpha|) - \Delta_\alpha).$$

For each exceptional slope $\alpha \in \mathscr{E}$, there is an interval $I_\alpha = (\alpha - x_\alpha, \alpha + x_\alpha)$, where the sup is defined on I_α by

$$\delta(\mu) = P(-|\mu - \alpha|) - \Delta_\alpha, \quad \text{if} \quad \mu \in I_\alpha.$$

One may compute [DLP]

$$x_\alpha = \frac{3 - \sqrt{5 + 8\Delta_\alpha}}{2}.$$

The graph of $\delta(\mu)$ is an increasing concave up parabola on the interval $[\alpha - x_\alpha, \alpha]$ and a decreasing

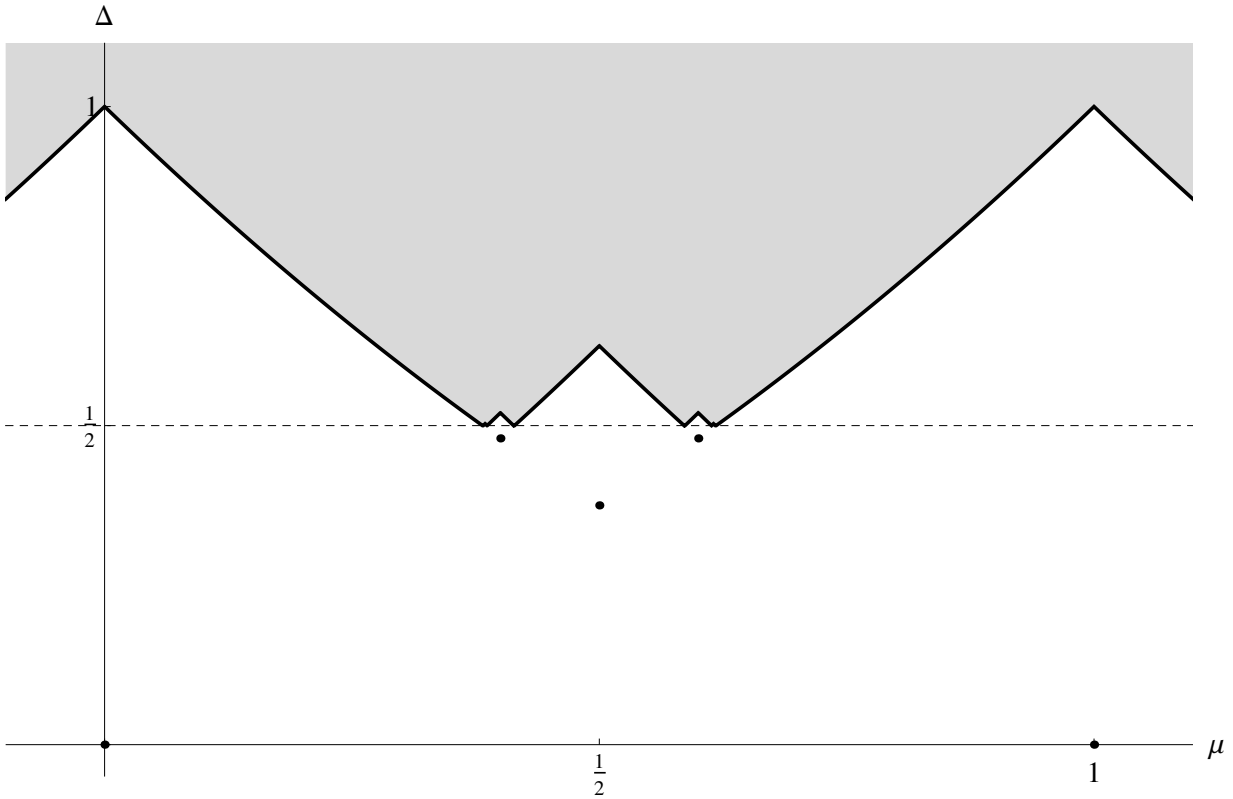

FIGURE 1. The curve $\delta(\mu) = \Delta$ occurring in the classification of stable bundles. The Chern characters corresponding to positive dimensional moduli spaces lie on or above the curve $\delta(\mu) = \Delta$.

concave up parabola on the interval $[\alpha, \alpha + x_\alpha]$ (see Figure 1). The graph of δ over I_α is symmetric across the vertical line $\mu = \alpha$ for $\alpha \in \mathscr{E}$ and is invariant under translation by integers. Furthermore, for every $\alpha \in \mathscr{E}$, $\delta(\alpha \pm x_\alpha) = \frac{1}{2}$.

The fundamental theorem on the existence of moduli spaces of semistable sheaves due to Drézet and Le Potier is the following.

Theorem 4.15 ([DLP], [LP])**.** *Let $\xi = (r, \mu, \Delta)$ be a Chern character of positive integer rank. There exists a positive dimensional moduli space of semistable sheaves $M(\xi)$ with Chern character ξ if and only if $c_1 = r\mu \in \mathbb{Z}$, $\chi = r(P(\mu) - \Delta) \in \mathbb{Z}$ and $\Delta \geq \delta(\mu)$.*

As we already remarked, when the moduli spaces $M(\xi)$ are nonempty, they are irreducible, normal, factorial varieties of dimension $r^2(2\Delta - 1) + 1$. The fact that $\mathbb{P}^{2[n]}$ is a Mori dream space generalizes to $M(\xi)$.

Theorem 4.16. [CHW] *Let ξ be the Chern character of a stable coherent sheaf. Then the moduli space $M(\xi)$ is a Mori dream space.*

The graph of δ intersects the line $\Delta = \frac{1}{2}$ in a generalized Cantor set

$$C := \mathbb{R} - \bigcup_{\alpha \in \mathscr{E}} I_\alpha.$$

Every rational number $q \in \mathbb{Q}$ lies in some interval I_α; equivalently, the Cantor set consists entirely of irrational numbers [D2, Theorem 1]. The fact that there are no rational numbers in C reflects the fact that all the exceptional bundles are obtained via a sequence of mutations starting with line bundles. In fact, the Cantor set C has the following remarkable property which will play a crucial role in the description of the effective cone.

Theorem 4.17. [CHW] *A point of C is either an end point of an I_α (hence a quadratic irrational) or transcendental.*

The proof depends on number theoretic properties of exceptional slopes. One can compute the continued fraction expansion of points in C to arbitrary accuracy by approximating the number by exceptional slopes. For an exceptional slope in $[0, \frac{1}{2}]$, the even length continued fraction expansion is a palindrome consisting of strings of 1 and 2 [H]. One shows that if a point of C is not an end point of I_α for some α, then the continued fraction expansion of the point is not repeating and begins in arbitrarily long palindromes. By a theorem of Adamczewski and Bugeaud [AB], this suffices to conclude that the number is transcendental.

Picard group of the moduli space. A theorem of Drézet determines the Picard group of $M(\xi)$. Semistable sheaves whose Chern characters satisfy $\delta(\mu) = \Delta$ are called *height zero* sheaves. Their moduli spaces are *height zero* moduli spaces. Their invariants lie on the δ-curve. Semistable sheaves whose Chern characters satisfy $\delta(\mu) > \Delta$ are *positive height* sheaves and their moduli spaces are *positive height* moduli spaces. Their invariants lie above the δ-curve.

The derived dual induces a homomorphism $K(\mathbb{P}^2) \to K(\mathbb{P}^2)$. We will write ξ^* for the dual Chern character. The Euler characteristic depends only on Chern characters, so it induces a bilinear pairing $(\xi, \zeta) = \chi(\xi^*, \zeta)$ on $K(\mathbb{P}^2) \otimes \mathbb{R}$. Correspondingly, $\xi^\perp$ denotes the orthogonal complement of ξ in $K(\mathbb{P}^2) \otimes \mathbb{R}$ with respect to this pairing.

Theorem 4.18 (Drézet)**.** *If $\Delta > \delta(\mu)$, then the Picard group of $M(\xi)$ is a free abelian group on two generators naturally identified with $\xi^\perp$ in $K(\mathbb{P}^2)$. If $\Delta = \delta(\mu)$, then the Picard group of $M(\xi)$ is an infinite cyclic group.*

In $M(\xi)$, linear equivalence and numerical equivalence coincide and the Néron-Severi space $N^1(M(\xi)) = \operatorname{Pic}(M(\xi)) \otimes \mathbb{R}$. When the Picard rank of a projective variety is one, then the

ample, effective and movable cones coincide and are equal to the half-space containing an effective or ample divisor. Therefore, when $M(\xi)$ is a height zero moduli space, there is nothing further to discuss. For the rest of the survey, we will always assume that ξ is a Chern character of positive height. In this case, $N^1(M(\xi))$ is a two-dimensional vector space. Hence, the cones $\mathrm{Amp}(M(\xi))$, $\mathrm{Mov}(M(\xi))$ and $\mathrm{Eff}(M(\xi))$ are determined by specifying their two extremal rays.

Elementary modifications. Let $p \in \mathbb{P}^2$ be a point. Let E be a coherent sheaf and let $E \to \mathcal{O}_p$ be a surjective homomorphism. Then the kernel

$$0 \to E' \to E \to \mathcal{O}_p \to 0$$

is called an *elementary modification* of E. It is easy to see that E' and E have the same rank and slope, whereas

$$\Delta(E') = \Delta(E) + \frac{1}{\mathrm{rk}(E)} \quad \text{and} \quad \chi(E') = \chi(E) - 1.$$

If E is μ-(semi)stable, then E' is μ-(semi)stable. We will use elementary modifications to increase Δ and decrease χ of semistable sheaves.

Singular sheaves. Some of the sheaves parameterized by $M(\xi)$ may fail to be locally free. We call semistable sheaves that are not locally free *singular sheaves*. The locus of singular sheaves will play an important role in our discussion of the ample cone. The following theorem determines when the locus is empty.

Theorem 4.19. [CH2] *Let $\xi = (r, \mu, \Delta)$ be an integral Chern character with $r > 0$ and $\Delta \geq \delta(\mu)$. The locus of singular sheaves in $M(\xi)$ is empty if and only if $\Delta - \delta(\mu) < \frac{1}{r}$ and μ is not an exceptional slope.*

The singular sheaves in Theorem 4.19 may be constructed using appropriate elementary modifications. When the locus of singular sheaves is not empty, then its codimension in $M(\xi)$ is equal to $r - 1$ [LP].

4.4. **Theorem 3.10 revisited.** Let Z be a scheme of length n. If $\chi(E \otimes I_Z) = 0$, then the Riemann-Roch formula implies that the invariants (μ, Δ) of E lie on the parabola $\Delta = P(\mu) - n$. In order to compute $\mu_{\min}$ in Theorem 3.10, we first check if there are any exceptional points (α, Δ_α) with $\alpha > 0$ that satisfy the equation $\Delta = P(\mu) - n$. If so, $\mu_{\min} = \alpha$. Otherwise, we find the intersection (μ_0, Δ_0) of $\Delta = P(\mu) - n$ with the δ-curve in the half-space $\mu > 0$. Then $\mu_{\min} = \mu_0$. By Theorem 4.17, the intersection of the parabola $\Delta = P(\mu) - n$ with the line $\Delta = \frac{1}{2}$, which is a quadratic irrational, occurs along an interval I_α. Hence, the parabola intersects the δ-curve along the same interval. Note that since the pieces of the δ-curve and $\Delta = P(\mu) - n$ are both rational translates of the same rational parabola, the intersection is always a rational point and easy to compute.

5. Preliminaries on Bridgeland stability

Bridgeland stability conditions were introduced by Bridgeland in [Br1]. Bridgeland [Br2] and Arcara and Bertram [AB] constructed Bridgeland stability conditions on smooth projective surfaces. Since then they have revolutionized the study of the birational geometry of the moduli spaces of sheaves on surfaces. In this section, we recall the definition of Bridgeland stability conditions and some basic facts concerning Bridgeland stability conditions on $\mathbb{P}^2$ developed in [ABCH] and [CH]. For a thorough discussion of Bridgeland stability conditions, one needs to introduce many technical tools such as t-structures on a triangulated category. However, since we are interested only in very specific examples, the reader can concentrate only on these examples and omit the generalities.

The definition of Bridgeland stability. Let $\mathcal{A}$ be an abelian category. A *central charge* Z on $\mathcal{A}$ is a group homomorphism

$$Z : K(\mathcal{A}) \to \mathbb{C}.$$

The central charge is *positive* if for every $0 \neq E \in \mathcal{A}$, $Z(E)$ is in the extended upper half-plane:

$$Z(E) \in \{re^{i\pi\theta} | r > 0, 0 < \theta \leq 1\}.$$

A positive charge allows one to define a notion of stability on $\mathcal{A}$.

Definition 5.1. Given a positive central charge Z on $\mathcal{A}$, the *Z-slope* of a nonzero object $E \in \mathcal{A}$ is

$$\mu_Z(E) = -\frac{\Re(Z(E))}{\Im(Z(E))}.$$

An object E of $\mathcal{A}$ is called *Z-(semi)stable* if for every proper subobject $F \subset E$ in $\mathcal{A}$ we have $\mu_Z(F) < \mu_Z(E)$ ($\mu_Z(F) \leq \mu_Z(E)$).

The Harder-Narasimhan filtration plays a crucial role in studying the geometry of moduli spaces of sheaves. We say that Z has the *Harder-Narasimhan property* if every $0 \neq E \in \mathcal{A}$ has a finite Harder-Narasimhan filtration, i.e., a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

where $F_i = E_i/E_{i-1}$ are semistable with respect to Z and

$$\mu_Z(F_1) > \mu_Z(F_2) > \cdots > \mu_Z(F_n).$$

Definition 5.2. A *Bridgeland stability condition* σ on the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on a smooth projective variety X is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}^b(X)$ and Z is a positive central charge that has the Harder-Narasimhan property.

One imposes an additional technical assumption on Bridgeland stability conditions. Fix a norm $|| \cdot ||$ on $K_{\text{num}}(X) \otimes \mathbb{R}$. The Bridgeland stability condition is called *full* if it satisfies the *support property*: There is a constant $C > 0$ such that for all Z-semistable objects $E \in \mathcal{A}$, $||E|| \leq C|Z(E)|$. The main theorem of [Br1] is the following.

Theorem 5.3. [Br1] *The set of full Bridgeland stability conditions* $\text{Stab}(X)$ *has the structure of a complex manifold. Furthermore, the map*

$$\text{Stab}(X) \to \text{Hom}(K_{num}(X), \mathbb{C}), \quad (\mathcal{A}, Z) \mapsto Z$$

is a local homeomorphism.

Wall and chamber structure. Fix a class $\xi \in K_{\text{num}}(X)$ and consider the set of σ-semistable objects $E \in \mathcal{D}^b(X)$ of class ξ as σ varies. Then there exists a locally finite set of walls (real codimension one submanifolds with boundary) in $\text{Stab}(X)$ depending only on ξ such that within a chamber the set of σ-(semi)stable objects of class ξ do not change (see [BM]). At each wall, there exists an object which is stable on one side of the wall and gets destabilized on the other side. Along the wall the object is semistable. These walls are called *Bridgeland walls*.

Given a stability condition $\sigma = (\mathcal{A}, Z)$, one can form moduli spaces parameterizing S-equivalence classes of σ-semistable objects with class ξ [AP]. When $X = \mathbb{P}^2$, for each ξ, there are only *finitely many* Bridgeland walls. Moreover, the corresponding moduli spaces are *projective schemes*, which can be constructed via GIT [ABCH]. We will next explain the Bridgeland stability conditions on $\mathbb{P}^2$ in greater detail.

The main example. In the case of $\mathbb{P}^2$, the Bridgeland stability conditions constructed by Bridgeland [Br2] and Arcara and Bertram [AB] have the following form. Any torsion-free coherent sheaf E on $\mathbb{P}^2$ has a Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with respect to the Mumford slope with semistable factors $F_i = E_i/E_{i-1}$ such that

$$\mu_{\max}(E) = \mu(F_1) > \cdots > \mu(F_n) = \mu_{\min}(E).$$

Let $s \in \mathbb{R}$.

- Define $\mathcal{Q}_s$ to be the full subcategory of $\mathrm{coh}(\mathbb{P}^2)$ consisting of sheaves such that their quotient by their torsion subsheaf have $\mu_{\min}(Q) > s$.
- Define $\mathcal{F}_s$ to be the full subcategory of $\mathrm{coh}(\mathbb{P}^2)$ consisting of torsion-free sheaves F with $\mu_{\max}(F) \leq s$.

Then the abelian category

$$\mathcal{A}_s := \{E \in \mathcal{D}^b(\mathbb{P}^2) : \mathrm{H}^{-1}(E) \in \mathcal{F}_s, \mathrm{H}^0(E) \in \mathcal{Q}_s, H^i(E) = 0 \text{ for } i \neq -1, 0\}$$

obtained by tilting the category of coherent sheaves with respect to the torsion pair $(\mathcal{F}_s, \mathcal{Q}_s)$ is the heart of a bounded t-structure on $\mathcal{D}^b(\mathbb{P}^2)$. Let

$$Z_{s,t}(E) = -\int_{\mathbb{P}^2} e^{-(s+it)H} \,\mathrm{ch}(E),$$

where H is the hyperplane class on $\mathbb{P}^2$. The pair $(\mathcal{A}_s, Z_{s,t})$ is a Bridgeland stability condition for every $(s,t) \in \mathbb{R}^2$ with $t > 0$. We thus obtain a half-plane of Bridgeland stability conditions for $\mathbb{P}^2$.

Bridgeland walls. Fix a Chern character $\xi \in K(\mathbb{P}^2)$. An object E with Chern character ξ is destabilized along a wall $W(E,F)$ by F if E is semistable on one side of $W(E,F)$ but $F \subset E$ in the category $\mathcal{A}_s$ satisfies $\mu_{s,t}(F) > \mu_{s,t}(E)$ on the other side of $W(E,F)$. Along the wall $W(E,F)$, we have $\mu_{s,t}(F) = \mu_{s,t}(E)$. This relation gives us the equation for $W(E,F)$. The equation of the wall depends only on the Chern character $\mathrm{ch}(E) = \xi$ and $\mathrm{ch}(F) = \zeta$.

(1) If $\mu(\xi) = \mu(\zeta)$ (where the Mumford slope is interpreted as ∞ if the rank is 0), then the wall $W(\xi,\zeta)$ is the vertical line $s = \mu(\xi)$ (interpreted as the empty set when the slope is infinite).

(2) Otherwise, we may assume $\mu(\xi)$ is finite, so that $r \neq 0$. The walls $W(\xi,\zeta)$ and $W(\xi,\xi+\zeta)$ are equal, so we may further reduce to the case where both ξ and ζ have nonzero rank. Then the wall $W(\xi,\zeta)$ is the semicircle centered at the point $(s,0)$ with

$$s = \frac{1}{2}(\mu_1 + \mu_2) - \frac{\Delta_1 - \Delta_2}{\mu_1 - \mu_2}$$

and having radius ρ satisfying

$$\rho^2 = (s - \mu_1)^2 - 2\Delta_1.$$

Observe that the distinct semicircular walls are all *disjoint* and *nested*. The semicircles are centered along the s-axis, with smaller semicircles having centers closer to the vertical wall $s = \mu$.

Quivers and Moduli spaces via GIT. The moduli spaces of Bridgeland semistable objects on $\mathbb{P}^2$ with Chern character ξ are projective and can be constructed via GIT (see [ABCH]). A lemma of Macrì [Mac, Lemma 3.16] guarantees that if the heart of a t-structure $\mathcal{A}$ contains the objects

$$\mathcal{O}_{\mathbb{P}^2}(k-2)[2], \mathcal{O}_{\mathbb{P}^2}(k-1)[1], \text{ and } \mathcal{O}_{\mathbb{P}^2}(k),$$

then $\mathcal{A}$ is the extension-closure of these three objects. Then one can obtain a stability condition with the category equal to $\mathcal{A}$ by assigning the three objects $\mathcal{O}_{\mathbb{P}^2}(k-2)[2], \mathcal{O}_{\mathbb{P}^2}(k-1)[1], \mathcal{O}_{\mathbb{P}^2}(k)$ any

three complex numbers in the extended upper-half-plane. We call such stability conditions *quiver stability conditions*. The objects of $\mathcal{A}$ are complexes of the form

$$\mathbb{C}^{m_0} \otimes \mathcal{O}_{\mathbb{P}^2}(k-2) \to \mathbb{C}^{m_1} \otimes \mathcal{O}_{\mathbb{P}^2}(k-1) \to \mathbb{C}^{m_2} \otimes \mathcal{O}_{\mathbb{P}^2}(k).$$

The corresponding moduli space of Bridgeland stable objects can then be identified with a moduli space of quiver representations, which is constructed via GIT by a theorem of King [Ki].

One can check that given any Chern character ξ and a stability condition σ that does not lie on a Bridgeland wall, one can find a path between σ and a quiver stability condition $(\mathcal{A}, Z)$ without crossing any walls (see [ABCH]). Consequently, one deduces the following theorem.

Theorem 5.4. [ABCH] *The moduli spaces of Bridgeland semistable objects for $\mathbb{P}^2$ are isomorphic to certain moduli spaces of quiver representations and can be constructed via GIT.*

There are only finitely many possible numerical invariants for subobjects in the case of quiver stability conditions. Consequently, given a Chern character ξ, the number of Bridgeland walls in the (s,t)-plane is finite. Hence, there exists a largest semicircular Bridgeland wall $W_{\max}$ to the left of the vertical line $s = \mu$ that contains all other semicircular walls. We call $W_{\max}$ the *Gieseker wall*. We call the smallest Bridgeland wall in $W_{\max}$, the *collapsing wall* W_{collapse}. No Gieseker semistable sheaf is Bridgeland stable inside W_{collapse}.

For every (s,t) with $s < \mu$ contained outside the Gieseker wall $W_{\max}$, Bridgeland stability coincides with Gieseker stability. In the same region, the corresponding Bridgeland moduli spaces are isomorphic to $M(\xi)$ [ABCH].

The correspondence between Bridgeland walls and Mori Walls. It was observed in [ABCH] that there is a one-to-one correspondence between the Bridgeland walls in the (s,t)-plane and the stable base locus decomposition of $\text{Eff}(\mathbb{P}^{2[n]})$.

Theorem 5.5 ([ABCH], [LZ])**.** *A scheme $Z \in \mathbb{P}^{2[n]}$ is in the stable base locus of a linear system $aH - \frac{B}{2}$ precisely when $a < \mu$ if and only if the ideal sheaf I_Z is destabilized along the Bridgeland wall W with center $-\mu - \frac{3}{2}$.*

The correspondence between the Bridgeland walls and Mori walls inspires many of the constructions we will see in the rest of this survey. If F is destabilized along the wall $W(\xi, \zeta)$, then there is a one-dimensional subspace in $K(\mathbb{P}^2)$ orthogonal to the span of ξ and ζ. Since $\text{Pic}(M(\xi))$ is identified with $\xi^{\perp}$, this subspace determines a line in $N^1(M(\xi))$. We expect this line to intersect $\text{Eff}(M(\xi))$ in a ray R which exactly determines when F lies in the stable base locus of a linear system.

6. The ample cone of the moduli space of sheaves

In this section, following [CH2], we describe how to calculate the ample cone of $M(\xi)$ when $\text{ch}_1(\xi)$ and $\text{rk}(\xi)$ are coprime or when $\text{rk}(\xi)$ is small. We follow the strategy outlined in §2. Let ξ be the Chern character of a positive height stable sheaf.

The Donaldson-Uhlenbeck-Yau compactification. The moduli space $M(\xi)$ admits a morphism analogous to the Hilbert-Chow morphism. There is a surjective, birational morphism $j : M(\xi) \to M^{DUY}(\xi)$ to the Donaldson-Uhlenbeck-Yau compactification $M^{DUY}(\xi)$ of the moduli space of stable vector bundles constructed by Jun Li (see [Li] and [HuL]). The Donaldson-Uhlenbeck-Yau compactification first arose in the context of gauge theory. Jun Li later gave an algebraic interpretation. Let M^{μ} be the moduli space of μ-semistable sheaves, which can be constructed via GIT. Then there is a natural morphism $j : M(\xi) \to M^{\mu}$. Jun Li showed that the image of j is homeomorphic to the Donaldson-Uhlenbeck-Yau moduli space.

Given a μ-semistable singular sheaf F, we have the exact sequence

$$0 \to F \to F^{**} \to T_F \to 0.$$

The double dual F^{**} is reflexive, hence on a surface is automatically locally free and μ-semistable. The sheaf T_F is torsion supported on finitely many points. Two μ-semistable sheaves F and G parameterized by M^μ correspond to the same closed point of M^μ if and only if $F^{**} \cong G^{**}$ and the supports of T_F and T_G define the same point in $\mathbb{P}^{2(n)}$, where n is the length of T_F and T_G (see [HuL]).

If $r > 1$ and the singular locus in $M(\xi)$ is nonempty, then one can construct positive dimensional non-isomorphic Gieseker stable sheaves with the same double dual and singular support using elementary modifications [CH2]. Consequently, the morphism j is not an isomorphism and contracts curves. The line bundle $\mathcal{L}_1$ defining j is base-point-free but not ample (see [HuL]). It corresponds to a Chern character $u_1 \in \xi^\perp \cong \operatorname{Pic} M(\xi)$ and spans an extremal ray of $\operatorname{Amp}(M(\xi))$. We conclude the following proposition.

Proposition 6.1. *Assume that $\xi = (r, \mu, \Delta)$ is a positive height Chern character such that either μ is exceptional or $\Delta \geq \delta(\mu) + \frac{1}{r}$. Then the morphism*

$$j : M(\xi) \to M^{DUY}(\xi)$$

is a birational morphism that contracts positive dimensional varieties. Hence, the line bundle defining j spans an extremal ray of $\operatorname{Amp}(M(\xi))$.

In the rest of this section, we will discuss the other extremal ray of $\operatorname{Amp}(M(\xi))$, which we call the *primary* extremal edge.

Remark 6.2. If the locus of singular sheaves in $M(\xi)$ is empty, then j is an isomorphism. In that case, the map induced by $E \mapsto E^*$ is an isomorphism that interchanges the two extremal rays of $\operatorname{Amp}(M(\xi))$.

6.1. **A nef divisor on $M(\xi)$.** In [BM], Bayer and Macrì construct nef divisors on moduli spaces of Bridgeland semistable objects and describe the curves that have zero intersection with their divisors. Since for stability conditions outside the Gieseker wall, the Bridgeland moduli space is isomorphic to $M(\xi)$, the Bayer-Macrì divisors provide nef divisors on $M(\xi)$. In fact, we will see that computing the primary edge of $\operatorname{Nef}(M(\xi))$ amounts to computing the Gieseker wall.

Let X be a smooth projective variety. Given a stability condition $\sigma = (\mathcal{A}, Z) \in \operatorname{Stab}(X)$, a choice of numerical invariants ξ and a family $\mathcal{E} \in \mathcal{D}^b(S \times X)$ of σ-semistable objects of class ξ parameterized by a proper, finite type scheme S, Bayer and Macrì define a linear functional on curves in S. Let p and q denote the projections from $S \times X$ to S and X, respectively. $\Phi_{\mathcal{E}}(-)$ be the Fourier-Mukai transform mapping $F \in D^b(S)$ to $q_*(p^*(F) \otimes \mathcal{E})) \in D^b(X)$. Given a curve $C \subset S$, the Bayer-Macrì functional is defined by

$$\ell_{\sigma,\mathcal{E}}([C]) = \Im\left(-\frac{\Phi_{\mathcal{E}}(\mathcal{O}_C)}{Z(\xi)}\right).$$

This linear functional corresponds to a Cartier divisor class on S. Bayer and Macrì prove the following fundamental theorem.

Theorem 6.3 (Bayer, Macrì [BM])**.** *The divisor class $\ell_{\sigma,\mathcal{E}}$ is nef. A curve C has $\ell_{\sigma,\mathcal{E}} \cdot C = 0$ if and only if the objects parameterized by two general points of C are S-equivalent.*

Now we can implement the strategy described in §2 for computing the nef cone. Let $(\mathcal{A}, Z) = \sigma_0 \in W_{\max}$ be a stability condition on the Gieseker wall. Let ℓ_{σ_0} be the Bayer-Macrì divisor. To compute the class of ℓ_{σ_0}, consider the functional $N^1(M(\xi)) \to \mathbb{R}$ defined by

$$\xi' \mapsto \Im\left(-\frac{Z(\xi')}{Z(\xi)}\right).$$

Since the pairing $(\xi, \zeta) = \chi(\xi \otimes \zeta)$ is nondegenerate, we can write this functional as $(\zeta, -)$ for a unique $\zeta \in \xi^\perp$. In terms of the isomorphism $\xi^\perp \cong \operatorname{Pic} M(\xi)$, then $\zeta = [\ell_{\sigma_0}]$. Considering $(\zeta, \operatorname{ch} \mathcal{O}_p)$

shows that ζ has negative rank. Furthermore, if $W_{\max} = W(\xi', \xi)$ (so that $Z(\xi')$ and $Z(\xi)$ are real multiples of one another), then ζ is a negative rank character in $(\xi')^{\perp}$. The ray in $N^1(M(\xi))$ determined by σ_0 depends only on $W_{\max}$, and not the particular choice of σ_0.

The method for computing $\mathrm{Amp}(M(\xi))$**.** In order to compute the ample cone, we guess the Gieseker wall. We first show that under suitable numerical assumptions, the rank of the first destabilizing object can be at most the rank of ξ. Consequently, we only need to consider walls $W(\xi, \xi')$ where $\mathrm{rk}(\xi') \leq \mathrm{rk}(\xi)$. The equation for the center of a wall $W(\xi, \xi')$ is given by

$$s = \frac{\mu(\xi') + \mu(\xi)}{2} - \frac{\Delta(\xi') - \Delta(\xi)}{\mu(\xi') - \mu(\xi)}.$$

If all the invariants are fixed and $\Delta(\xi')$ decreases, the center moves left and the wall becomes larger. Similarly, if $\Delta(\xi)$ is sufficiently large, then the second term will dominate and the wall will increase in size as $\mu(\xi')$ is closer to $\mu(\xi)$. This allows us to guess the Gieseker wall $W_{\max}$. We then need to show that the guess is correct.

First, using our numerical assumptions, we show that there can be no bigger walls. Then by the construction of Bayer-Macrì, we obtain a nef divisor ℓ_{σ_0} corresponding to our guess. To show that ℓ_{σ_0} is extremal in $\mathrm{Amp}(M(\xi))$, we need to construct a curve C of Gieseker semistable sheaves that become S-equivalent with respect to the Bridgeland stability condition σ_0. Then according to Theorem 6.3, C has intersection zero with ℓ_{σ_0}. Therefore, ℓ_{σ_0} is nef but not ample.

The Gieseker Wall. Let $\xi = (r, \mu, \Delta)$ be a stable Chern character. Let $\xi' = (r', \mu', \Delta')$ be the stable Chern character satisfying the following defining properties:

- $0 < r' \leq r$ and $\mu' < \mu$,
- Every rational number in the interval (μ', μ) has denominator greater than r,
- The discriminant of any stable bundle of slope μ' and rank at most r is at least Δ',
- The minimal rank of a stable Chern character with slope μ' and discriminant Δ' is r'.

The character ξ' is easily computed using Drézet and Le Potier's classification of stable bundles described in §4. The next theorem describes the Gieseker wall when r and ch_1 are coprime and Δ is sufficiently large.

Theorem 6.4. [CH2] *Let $\xi = (r, \mu, \Delta)$ be a positive height Chern character such that r and c_1 are coprime. Suppose Δ is sufficiently large, depending on r and μ. The Gieseker wall for $M(\xi)$ is the wall $W(\xi', \xi)$ where ξ and ξ' have the same Bridgeland slope.*

As a consequence of Theorem 6.4, we compute the ample cone.

Theorem 6.5. [CH2] *Let ξ be a Chern character satisfying the assumptions of Theorem 6.4. Then $\mathrm{Amp}(M(\xi))$ is spanned by u_1 and a negative rank character in $(\xi')^{\perp}$.*

It is possible to give explicit lower bounds on Δ in the theorems. The proof of Theorem 6.4 has two parts. First, we need to show that $W(\xi, \xi')$ is the largest possible wall. Then we need to construct an object that is destabilized at the wall $W(\xi, \xi')$. The fundamental estimate that allows us to reduce the problem to a finite problem is the following.

Proposition 6.6. *Suppose there is a wall $W(\xi, \theta)$ larger than $W(\xi, \xi')$ with $\mathrm{rk}(\theta) > \mathrm{rk}(\xi)$. Let ρ_θ be the radius of $W(\xi, \theta)$. Then*

$$\rho_\theta^2 \leq \frac{\mathrm{rk}(\xi)^2}{2(\mathrm{rk}(\xi) + 1)} \Delta(\xi).$$

Proof. Let

$$0 \to K^k \to F^f \to E^e \to C^c \to 0$$

be a sequence, where $E \in M(\xi)$ is destabilized by F with Chern character θ with kernel K of rank k and cokernel C of rank c. Using the fact that $F \in \mathcal{Q}_s$ along $W(\xi,\theta)$ and $K \in \mathcal{F}_s$ along $W(\xi,\theta)$ [ABCH], we obtain the following inequalities on the center s_θ and radius ρ_θ of $W(\xi,\theta)$:

$$f(s_\theta + \rho_\theta) \leq f\mu(F) = c_1(F) = c_1(K) + c_1(E) - c_1(C).$$

Since $\mu(K) \leq s_\theta - \rho_\theta$, we can rearrange to obtain the inequality

$$(k+f)\rho_\theta \leq (k-f)s_\theta + e\mu(E) - c_1(C).$$

Since $c_1(C) \geq 0$, we can drop that term. Squaring both terms and rearranging, we get

$$\rho_\theta^2 \leq \frac{(k-f)^2}{2kf}\Delta(E).$$

The inequality is as weak as possible when $f = e+1$ and $k = 1$, which yields the desired inequality. □

Given Proposition 6.6, one can show that if Δ is sufficiently large and $W(\xi,\xi')$ is a Bridgeland wall, then $W(\xi,\xi')$ is the Gieseker wall. To conclude the proof of Theorems 6.4 and 6.5 one needs to construct curves with zero intersection with ℓ_{σ_0}. Define a character $\xi'' = \xi - \xi'$. One can show that if $\Delta(\xi)$ is sufficiently large or $\mathrm{rk}(\xi) \leq 6$, ξ'' is a stable character. The following theorem gives the desired curves.

Theorem 6.7. *Let ξ be a Chern character satisfying the assumptions of Theorem 6.4. Fix general sheaves $F \in M(\xi')$ and $Q \in M(\xi'')$. Then the general sheaf E given by an extension*

$$0 \to F \to E \to Q \to 0$$

is Gieseker stable. Furthermore, we obtain curves in $M(\xi)$ by varying the extension class.

If E is a Gieseker stable extension as in the theorem, then E is strictly semistable with respect to a stability condition σ_0 on $W(\xi',\xi)$, and not semistable with respect to a stability condition below $W(\xi',\xi)$. Thus $W(\xi',\xi)$ is an actual wall for $M(\xi)$, and it is the Gieseker wall. Any two Gieseker stable extensions of Q by F are S-equivalent with respect to σ_0, so any curve C in $M(\xi)$ obtained by varying the extension class satisfies $\ell_{\sigma_0}.C = 0$. Therefore, ℓ_{σ_0} spans an edge of the ample cone. Dually, C spans an edge of the Mori cone of curves.

One can prove an analogous theorem when $\mathrm{rk}(\xi) \leq 6$.

Theorem 6.8. [CH2] *Let $\xi = (r,\mu,\Delta)$ be a positive height Chern character with $r \leq 6$.*

1. *If ξ is not a twist of $(6,\frac{1}{3},\frac{13}{18})$, then $\mathrm{Amp}(M(\xi))$ is spanned by u_1 and a negative rank character in $(\xi')^\perp$.*
2. *If $\xi = (6,\frac{1}{3},\frac{13}{18})$, then $\mathrm{Amp}(M(\xi))$ is spanned by u_1 and a negative rank character in $(\mathrm{ch}\,\mathcal{O}_{\mathbb{P}^2})^\perp$.*

The natural analogs of Theorems 6.4 and 6.7 are almost true when $\mathrm{rk}(\xi) \leq 6$. Adjustments are necessary for certain small discriminant cases since a different Chern character may actually maximize the wall. When the rank and the first Chern class are not relatively prime, proving the stability of the extension also becomes considerably harder. As the rank increases beyond 6, the exceptions become more common, and many more ad hoc arguments are required when using current techniques.

The ample cone of $M(\xi)$ was computed earlier for some special Chern characters. Strømme computed $\mathrm{Amp}(M(\xi))$ when the rank of ξ is two and either c_1 or $c_2 - \frac{1}{4}c_1^2$ is odd [St]. Similarly, when the slope is $\frac{1}{r}$, Yoshioka [Y] computed the ample cone of $M(\xi)$ and described the first flip. Theorems 6.5 and 6.8 contain these as special cases. Bridgeland stability has also been effectively used to compute ample cones of moduli spaces of sheaves on other surfaces. For example, see [AB], [BM], [BM2], [MYY1], [MYY2] for K3 surfaces, [MM], [Y2], [YY] for abelian surfaces, [Ne] for

Enriques surfaces, and [BC] for the Hilbert scheme of points on Hirzebruch surfaces and del Pezzo surfaces.

7. Effective divisors on moduli spaces of sheaves

Let $\xi = (r, \mu, \Delta)$ with $r > 0$ be a positive height stable Chern character on $\mathbb{P}^2$. In this section, we discuss the effective cone of $M(\xi)$ following [CHW].

Brill-Noether divisors. Brill-Noether divisors give a natural way of constructing effective divisors on $M(\xi)$. First, if $\chi(\xi) = 0$ and $\mu \geq 0$, then $M(\xi)$ has a theta divisor

$$\Theta := \{F \in M(\xi) | h^0(F) \neq 0\}.$$

By a theorem of Göttsche and Hirschowitz [GH], a general sheaf in $M(\xi)$ has no cohomology. Hence, the locus of sheaves that do have cohomology is a codimension one determinantal locus and we obtain an effective divisor.

Given ξ, let ζ be a stable Chern character such that $\chi(\xi \otimes \zeta) = 0$ and $\mu(\xi \otimes \zeta) \geq 0$. Then we obtain a rational map

$$M(\xi) \times M(\zeta) \dashrightarrow M(\xi \otimes \zeta) \text{ mapping } (F, E) \mapsto F \otimes E.$$

We can then pullback the theta divisor and restrict to the fibers of either projection. More explicitly, given $E \in M(\zeta)$, consider the locus

$$D_E = \{F \in M(\xi) | h^0(E \otimes F) \neq 0\}.$$

If this locus is not the entire moduli space, then it is a codimension one determinantal locus called the *Brill-Noether divisor* associated to E. This raises the problem of determining when Brill-Noether divisors are effective or equivalently when for a general sheaf $G \in M(\xi)$, $E \otimes G$ has no cohomology.

The interpolation problem. A sheaf E *satisfies interpolation* with respect to a coherent sheaf F on $\mathbb{P}^2$ if $h^i(E \otimes F) = 0$ for every i (in particular, $\chi(E \otimes F) = 0$). The stable base locus decomposition of $M(\xi)$ is closely tied to the higher rank interpolation problem.

Problem 7.1 (Higher rank interpolation)**.** *Given $F \in M(\xi)$ determine the minimal slope $\mu \in \mathbb{Q}$ with $\mu + \mu(\xi) \geq 0$ for which there exists a vector bundle E of slope μ satisfying interpolation with respect to F.*

If E satisfies interpolation with respect to F, then the Brill-Noether divisor

$$D_E := \{G \in M(\xi) | h^1(E \otimes G) \neq 0\}$$

is an effective divisor that does not contain F in its base locus. The interpolation problem in general is very hard, but has been solved in the following cases:

(1) $F = I_Z$, where Z is a complete intersection, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(2) $F = I_Z$, where Z is a monomial, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(3) $F = I_Z$, where Z is a general, zero-dimensional scheme in $\mathbb{P}^2$ [H].
(4) $F \in M(\xi)$ is a general stable sheaf [CHW].

These theorems depend on finding a good resolution of F. If F were unstable, then the maximal destabilizing object would yield an exact sequence

$$0 \to A \to F \to B \to 0.$$

The idea is to destabilize F via Bridgeland stability and use the exact sequence arising from the Harder-Narasimhan filtration just past the wall where F is destabilized. If a bundle E satisfies interpolation with respect to A and B, then E satisfies interpolation for F by the long exact sequence for cohomology. One may hope that for an interpolating bundle E with minimal slope, E

satisfies interpolation for F because it does so for both A and B. Because the Harder-Narasimhan filtrations of A and B are "simpler," we can try to prove interpolation inductively. This strategy works in all 4 cases listed above.

A few basic facts. A vector bundle on $\mathbb{P}^2$ is called *prioritary* if $\mathrm{Ext}^2(E, E(-1)) = 0$. Semistable sheaves are prioritary. Let ξ be the Chern character of a prioritary sheaf, then the stack $P(\xi)$ of prioritary sheaves is irreducible and contains the stack of semistable sheaves as an open substack [HiL].

Theorem 7.2. [CH] *Let Z be a zero-dimensional subscheme of $\mathbb{P}^2$. Suppose there exists a vector bundle E of slope μ such that $H^1(\mathbb{P}^2, I_Z \otimes E) = 0$. Then for each rational slope $\mu' > \mu$, there exists a prioritary bundle that satisfies interpolation with respect to I_Z. In fact, if $\xi' = (r', \mu', \Delta')$ is a Chern character orthogonal to I_Z and r' is sufficiently large and divisible, then the general prioritary sheaf with Chern character ξ' satisfies interpolation with respect to I_Z.*

Sketch of proof. Using the Harder-Narasimhan and Jordan-Hölder filtrations, one may assume that E is stable. If

$$\mu + k \leq \mu' < \mu + k + 1,$$

then $F = E(k)^a \oplus E(k+1)^b$ is a prioritary sheaf with slope μ' for appropriate choices of integers a and b. Then it is easy to see that $H^i(\mathbb{P}^2, F \otimes I_Z) = 0$ for $i > 0$. Applying a sequence of elementary modifications yields a prioritary sheaf of slope μ' that satisfies interpolation with respect to I_Z. Since satisfying interpolation is an open condition, then the general prioritary sheaf in the moduli stack satisfies interpolation. In particular, if there are stable sheaves with Chern character ξ', then the general stable sheaf satisfies interpolation with respect to I_Z. □

It may happen that even though there are prioritary sheaves of slope μ' satisfying interpolation, the point (μ', Δ') is below the δ-curve so that there are no stable sheaves satisfying interpolation. Finally, the following easier variant holds for pure one-dimensional sheaves.

Theorem 7.3. [CH] *Let F be a semistable pure one-dimensional sheaf. Suppose E is a prioritary bundle with Chern character (r, μ, Δ) that satisfies interpolation with respect to F. Let $\xi' = (r', \mu, \Delta')$ be a Chern character with $\Delta' > \Delta$ and r' sufficiently large and divisible. Then the general prioritary sheaf with Chern character ξ' satisfies interpolation with respect to F.*

Complete intersections. We now explain our strategy in the simplest case of complete intersections. Let Z be the complete intersection of two curves of degrees $a \leq b$. Then we have the resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a-b) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \to I_Z \to 0.$$

It is not possible to find a vector bundle E such that the twists of E with respect to $-a, -b$ and $-a-b$ all have vanishing cohomology. Consequently, proving interpolation for I_Z with respect to this resolution amounts to proving that a map on global sections is an isomorphism. This in general is a hard problem to answer.

Instead, let us consider the Bridgeland resolution of I_Z. We claim that I_Z is destabilized at the Bridgeland wall corresponding to the map $\mathcal{O}_{\mathbb{P}^2}(-a) \to I_Z$. The corresponding Bridgeland sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to I_Z \to \mathcal{O}_C(-b) \to 0,$$

where C is the curve defined by the degree a equation. By the Theorem of Göttsche and Hirschowitz [GH], for a general prioritary vector bundle with $\chi(E(-a)) = 0$ all cohomology of $E(-a)$ vanishes. The set of invariants $\zeta = (r, \mu, \Delta)$ that are the Chern characters of E such that $\chi(E(-a)) = 0$ lie on the parabola $P(\mu - a) = \Delta$. Similarly, the set of invariants of E that are orthogonal to $\mathcal{O}_C(b)$ so that $\chi(E \otimes \mathcal{O}_C(b)) = 0$ lie on the line $\mu = -b$. We are reduced to checking that for invariants where the two curves intersect, a general prioritary vector bundle of sufficiently large and divisible

rank satisfies $h^i(E \otimes \mathcal{O}_C(-b)) = 0$ for all i. Then, by the long exact sequence of cohomology, E will satisfy interpolation for I_Z.

Next, we analyze $\mathcal{O}_C(-b)$. The Bridgeland destabilizing sequence for $\mathcal{O}_C(-b)$ is given by

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-b) \to \mathcal{O}_C(-b) \to \mathcal{O}_{\mathbb{P}^2}(-a-b)[1] \to 0.$$

Since both of these objects are line bundles, by the theorem of Göttsche and Hirschowitz, for a general prioritary bundle with $\chi(E(-b)) = \chi(E(-a-b)) = 0$, $E(-b)$ and $E(-a-b)$ will have no cohomology. Consequently, $E \otimes O_C(-b)$ will have no cohomology.

By Theorems 7.2 and 7.3, to conclude the proof all we need to check is that the intersection of the three curves $P(\mu - a - b) = P(\mu - b) = \Delta$ and the line $\mu = -b$ occurs below the intersection of $P(\mu - a) = \Delta$ and $\mu = -b$. This is an easy consequence of our assumption $a \leq b$. We conclude that for $\mu \geq b + \frac{a-3}{2}$, there exists bundles of slope μ that satisfy interpolation with respect to I_Z.

On the other hand, by varying the curve of degree b and fixing the curve of degree a, we obtain a one parameter family of complete intersection schemes in $\mathbb{P}^{2[ab]}$. The resulting curve has zero intersection with the linear system $\left(b + \frac{a-3}{2}\right) H - \frac{1}{2}B$. Therefore, no bundle of positive slope less than $b + \frac{a-3}{2}$ can satisfy interpolation with respect to I_Z. We conclude the following theorem.

Theorem 7.4. [CH, Theorem 5.1] *Let Z be a zero-dimensional complete intersection scheme of curves of degrees $a \leq b$. Then there exists a vector bundle of slope μ satisfying interpolation with respect to I_Z if and only if*

$$\mu \geq \mu_{\min} = b + \frac{a-3}{2}.$$

In particular, I_Z is in the stable base locus of a linear system $\mu H - \frac{1}{2}B$ if and only if $\mu < \mu_{\min}$.

Monomial schemes. Next we sketch the case of monomial schemes. A *monomial scheme Z* is a zero-dimensional scheme in $\mathbb{P}^2$ whose ideal is generated by monomials. We can assume that I_Z is generated by

$$x^{a_1}, x^{a_2}y^{b_2}, \cdots, x^{a_{r-1}}y^{b_{r-1}}, y^{b_r},$$

where $a_1 > a_2 > \cdots > a_{r-1}$ and $b_2 < b_3 \cdots < b_r$. For convenience, set $b_1 = a_r = 0$. Then I_Z has resolution

$$0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}(-a_i - b_{i+1}) \xrightarrow{M} \bigoplus_{i=1}^{r} \mathcal{O}(-a_i - b_i) \to \mathcal{I}_Z \to 0,$$

where M is the $r \times (r-1)$ matrix with entries $m_{i,i} = y^{b_{i+1}-b_i}$, $m_{i+1,i} = -x^{a_i - a_{i+1}}$ and $m_{i,j} = 0$ otherwise.

A monomial scheme Z can be represented by a box diagram D_Z recording the monomials that are nonzero in $\mathbb{C}[x,y]/I_Z$. Let h_i be the number of boxes in the ith row counting from the bottom and let v_i be the number of boxes in the ith column counting from the left. Define

$$\mu_j = -1 + \frac{1}{j}\sum_{i=1}^{j}(h_i + i - 1), \quad \nu_k = -1 + \frac{1}{k}\sum_{i=1}^{k}(v_i + i - 1), \quad \mu_Z = \max_{j,k}(\mu_j, \nu_k).$$

Assume that the maximum is achieved by μ_h, (i.e., $\mu_Z = \mu_h$). Let D_U be the portion of D_Z lying above the hth horizontal line and let D_V be the potion of D_Z lying below this line. The diagrams D_U and D_V correspond to monomial zero-dimensional schemes U, V. We then have the following theorem.

Theorem 7.5. [CH] *Let Z be a zero-dimensional monomial scheme with Chern character ξ. There exists a vector bundle E of slope $\mu \in \mathbb{Q}$ satisfying interpolation for I_Z if and only if $\mu \geq \mu_Z$. We may take E to be prioritary. Furthermore, if there exists stable bundles of slope μ along Q_ξ, we may take E to be stable.*

The Bridgeland destabilizing sequence is given by

$$0 \to I_U(-h) \to I_Z \to I_{V \subset hL} \to 0,$$

where L is the line defined by $y = 0$. One proves the theorem by inducting on the complexity of Z. In fact, one computes the entire Harder-Narasimhan filtration of I_Z for different Bridgeland stability conditions, inductively decomposing the box diagram of the monomial scheme into pieces until each piece is a rectangle. As a corollary, one determines when monomial schemes are in the stable base loci of linear systems on the Hilbert schemes of points.

Corollary 7.6. *Let $Z \in \mathbb{P}^{2[n]}$ be a monomial scheme. Then Z is in the stable base locus of a linear system $aH - \frac{1}{2}B$ if and only if $a < \mu_Z$.*

Fix a term order. By passing to the generic initial ideal, which defines a monomial scheme, we obtain bounds on when a general scheme $Z \in \mathbb{P}^{2[n]}$ is in the stable base locus of a linear system.

Remark 7.7. Every Betti diagram of a zero-dimensional scheme occurs as the Betti diagram of a monomial scheme. We will shortly see that general schemes may have a different interpolating slope than the monomial schemes with the same Betti diagram. It is an interesting problem to find a complete set of invariants of a scheme that will determine the best interpolating slope.

The effective cone of the moduli space of sheaves. We now describe the effective cone of $M(\xi)$ in general. First, we will state the answer. Then we will explain the main steps in proving the answer.

Recall that Q_ξ is the parabola of orthogonal invariants in the (μ, Δ)-plane. It is given by $\chi(\xi \otimes \zeta) = 0$ and consequently defined by the quadratic equation

$$P(\mu(\xi) + \mu) - \Delta(\xi) - \Delta = 0.$$

Since the intersection of Q_ξ with $\Delta = \frac{1}{2}$ is a quadratic irrational, by Theorem 4.17, Q_ξ intersects $\Delta = \frac{1}{2}$ along some I_α and determines an exceptional bundle E_α. This bundle controls the effective cone of $M(\xi)$. Let ξ_α be the Chern character of E_α.

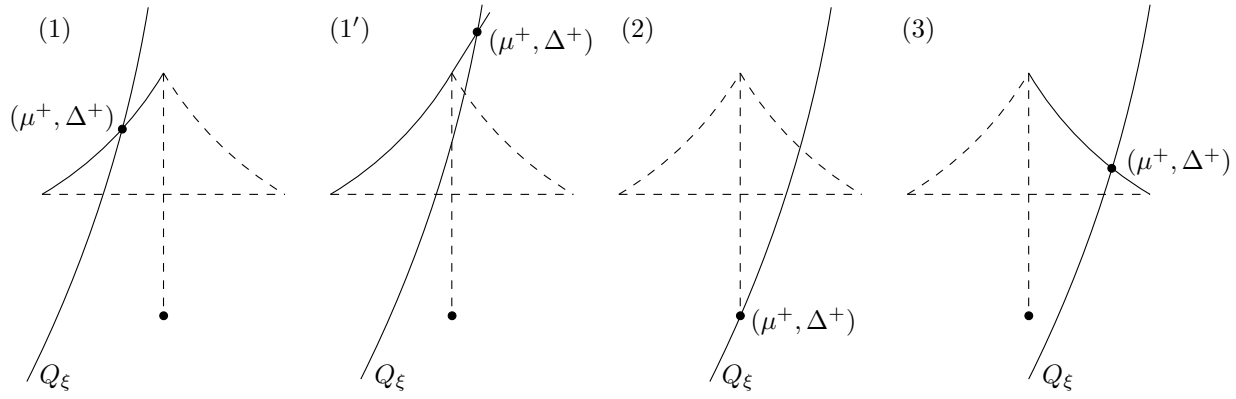

FIGURE 2. Possible relative positions of the curve $\delta(\mu) = \Delta$ and the parabola Q_ξ.

The main theorem is in terms of the following invariants (see Figure 2):

(1) If $\chi(\xi, \xi_\alpha) > 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha}$.
(2) If $\chi(\xi, \xi_\alpha) = 0$, let $(\mu^+, \Delta^+) = (\alpha, \Delta_\alpha)$.
(3) If $\chi(\xi, \xi_\alpha) < 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha-3}$.

Theorem 7.8. [CHW] *Let F be a general point of $M(\xi)$ and let r^+ be sufficiently large and divisible. Let ζ be the Chern character with rank r^+, slope μ^+ and discriminant Δ^+. Then the general point E of $M(\zeta)$ satisfies interpolation with respect to F. Furthermore, the Brill-Noether divisor D_E spans an extremal ray of the effective cone of $M(\xi)$. If $\chi(\xi, \xi_\alpha) \neq 0$, then D_E also spans an extremal ray of the movable cone.*

Remark 7.9. The Case $(1')$ in Figure 2 is interesting. There are stable bundles on the orthogonal parabola Q_ξ with slopes $\mu < \mu^+$. Let ζ be the Chern character of a stable bundle on Q_ξ such that $\mu(\zeta) < \mu^+$ and $\mu(\zeta) + \mu(\xi) > 0$. Then the image of the tensor product map

$$M(\xi) \times M(\zeta) \dashrightarrow M(\xi \otimes \zeta)$$

is entirely contained in the theta divisor of $M(\xi \otimes \zeta)$. It is easy to give explicit examples where this phenomenon occurs. For example, let ξ be the Chern character with $r = 2, \mu = 0, \Delta = \frac{11}{2}$. Then $\mu^+ = \frac{9}{4}$, whereas the intersection of Q_ξ with the δ-curve has $\mu = \frac{21}{10}$. For rational orthogonal invariants with $\frac{21}{10} \leq \mu < \frac{9}{4}$, the image of the tensor product map lies in the theta divisor. In the case of the Hilbert scheme of points, Case $(1')$ does not occur for integrality reasons and the effective cone is always defined by the intersection of Q_ξ with the δ-curve (see [H]).

The other extremal ray of the effective cone. Theorem 7.8 describes one extremal ray of the effective cone. When the rank of ξ is one, then $M(\xi)$ is isomorphic to a Hilbert scheme of points. For $\mathbb{P}^{2[n]}$, we have already described the second extremal edge of the effective cone. The divisor B parameterizing nonreduced schemes spans the other extremal ray. When the rank of ξ is two, then the locus of singular sheaves has codimension one and defines an extremal effective divisor.

If the rank of ξ is at least three, then the locus of singular sheaves has codimension at least 2 in the moduli space. Consequently, the Serre duality map $E \mapsto E^*(-3)$ defines a rational map between $M(\xi)$ and $M(\xi^* \otimes \xi_K)$ which is an isomorphism in codimension one. This map interchanges the two extremal rays of the effective cone. Alternatively, one can view the other extremal ray of the effective cone by considering the intersection of the left half of the parabola Q_ξ with the δ-curve and defining the analogous h^2-Brill-Noether divisor.

The associated exceptional collection. We will now sketch the proof of Theorem 7.8. The key is to find convenient resolutions of E and F in terms of a well chosen exceptional collection. The resolution is given in terms of exceptional bundles determined by the associated exceptional slope.

A *triad* is a triple (E, G, F) of exceptional bundles where the slopes are of the form $(\alpha, \alpha.\beta, \beta)$, $(\beta - 3, \alpha, \alpha.\beta)$, or $(\alpha.\beta, \beta, \beta + 3)$ for some exceptional slopes α, β of the form

$$(*) \qquad \alpha = \varepsilon\left(\frac{p}{2^q}\right) \qquad \beta = \varepsilon\left(\frac{p+1}{2^q}\right).$$

Corresponding to the triad (E, G, F) is a fourth exceptional bundle M defined as the cokernel of the canonical map $\mathrm{ev}^* : G \to F \otimes \mathrm{Hom}(G, F)^*$. The collection $(E^*(-3), M^*, F^*)$ is another triad.

The bundles of a triad form a *strong exceptional collection* for the bounded derived category $D^b(\mathbb{P}^2)$. In other words, if A, B are two bundles in a triad with A listed before B, then $\mathrm{Ext}^i(A, B) = 0$ for $i > 0$ and $\mathrm{Ext}^i(B, A) = 0$ for all i. Furthermore, a triad generates the derived category. This fact is a consequence of the generalized Beilinson spectral sequence.

Theorem 7.10 ([D])**.** *Let U be a coherent sheaf, and let (E, G, F) be a triad. Write*

$$\begin{aligned} G_{-2} &= E & F_{-2} &= E^*(-3) \\ G_{-1} &= G & F_{-1} &= M^* \\ G_0 &= F & F_0 &= F^*, \end{aligned}$$

and put $G_i = F_i = 0$ *if* $i \notin \{-2, -1, 0\}$. *There is a spectral sequence with* $E_1^{p,q}$*-page*

$$E_1^{p,q} = G_p \otimes H^q(U \otimes F_p)$$

which converges to U *in degree* 0 *and to* 0 *in all other degrees.*

Recall that the orthogonal parabola Q_ξ intersects the line $\Delta = \frac{1}{2}$ along an interval I_α. The *associated exceptional slope* is α. The exceptional slope α can be expressed as $\mu.\nu$ for two exceptional slopes μ and ν.

(1) If $(\xi, \xi_\alpha) > 0$, then we find a resolution of the general sheaf in terms of the exceptional collection $(E_{-\mu-3}, E_{-\nu}, E_{-\alpha})$.
(2) If $(\xi, \xi_\alpha) < 0$, then we find a resolution of the general sheaf in terms of the exceptional collection $(E_{-\alpha}, E_{-\mu-3}, E_{-\nu})$.
(3) If $(\xi, \xi_\alpha) = 0$, then either triad works and gives the same result.

The Beilinson spectral sequence implies the following resolution.

Theorem 7.11. [CHW] *Let* $(\xi, \xi_\alpha) > 0$ *and let* $U \in M(\xi)$ *be general. Let* $W \in D^b(\mathbb{P}^2)$ *be the mapping cone of the canonical evaluation map*

$$E_{-\alpha} \otimes \mathrm{Hom}(E_{-\alpha}, U) \to U,$$

so that there is a distinguished triangle

$$E_{-\alpha} \otimes \mathrm{Hom}(E_{-\alpha}, U) \to U \to W \to \cdot.$$

Then W *is isomorphic to a complex of the form*

$$E_{-\mu-3} \otimes \mathbb{C}^{m_1} \to E_{-\nu} \otimes \mathbb{C}^{m_2}$$

sitting in degrees -1 *and* 0. *Any two complexes of this form which are isomorphic to* W *are in the same orbit under the natural action of* $\mathrm{GL}(m_1) \times \mathrm{GL}(m_2)$ *on the space of such complexes.*

There is an analogous theorem when $(\xi, \xi_\alpha) < 0$ using the exceptional collection $(E_{-\alpha}, E_{-\mu-3}, E_{-\nu})$. In that case, the resolution has the form

$$W \to U \to E_{-\alpha-3}[1] \otimes \mathrm{Hom}(U, E_{-\alpha-3}[1])^* \to \cdot,$$

where W has a resolution of the form

$$0 \to E_{-\mu-3} \otimes \mathbb{C}^{m_1} \to E_{-\nu} \otimes \mathbb{C}^{m_2} \to W \to 0.$$

The map $E_{-\alpha} \otimes \mathrm{Hom}(E_{-\alpha}, U) \to U$ given in Theorem 7.11 defines the Bridgeland wall where U is destabilized. Similarly, when $(\xi, \xi_\alpha) < 0$ the map $U \to E_{-\alpha-3}[1] \otimes \mathrm{Hom}(U, E_{-\alpha-3}[1])^*$ defines the Bridgeland wall. This is the content of the following theorem. We state the case $(\xi, \xi_\alpha) > 0$ and leave the necessary modifications when $(\xi, \xi_\alpha) < 0$ to the reader.

Theorem 7.12. [CHW] *Let* $U \in M(\xi)$ *be general. Let* $\Lambda = W(U, E_{-\alpha})$ *be the wall in the* (s,t)*-plane of stability conditions where* U *and* $E_{-\alpha}$ *have the same* $\mu_{s,t}$*-slope. There is an exact sequence*

$$0 \to E_{-\alpha} \otimes \mathrm{Hom}(E_{-\alpha}, U) \to U \to W \to 0$$

in the corresponding categories $\mathcal{A}_s$. *The kernel and cokernel objects are* $\sigma_{s,t}$*-semistable outside of walls nested inside* Λ, *so* U *is* $\sigma_{s,t}$*-semistable along and outside* Λ, *but not stable inside* Λ. *The wall* Λ *has center* $(s_0, 0)$ *satisfying* $s_0 = -\mu^+ - \frac{3}{2}$, *where* μ^+ *is the corresponding orthogonal slope to* ξ.

When $(\xi, \xi_\alpha) = 0$, the general sheaf $U \in M(\xi)$ has a resolution of the form

$$0 \to E_{-\mu-3} \otimes \mathbb{C}^{m_1} \to E_{-\nu} \otimes \mathbb{C}^{m_2} \to U.$$

The locus of sheaves in $M(\xi)$ that do not have this resolution is an irreducible divisor of the form D_{E_α}. Furthermore, by varying the extension class, one obtains complete moving curves deducing that D_{E_α} spans an extremal ray of $\mathrm{Eff}(M(\xi))$.

When $(\xi, \xi_\alpha) \neq 0$, we have the resolution for the general sheaf in $M(\xi)$ provided by Theorem 7.11. We look for a resolution of the orthogonal bundle that should satisfy interpolation.

The Kronecker module. A *Kronecker quiver* Kr_N is a quiver consisting of two vertices v_1, v_2 and N arrows from v_1 to v_2. A representation of the quiver Kr_N of type (m_1, m_2) consists of the data of two vector spaces of dimension m_1 and m_2 and a collection of N morphisms between them. Two representations are equivalent if there exists a change of basis of the two vector spaces by an element of $\mathrm{GL}(m_1) \times \mathrm{GL}(m_2)$ that takes one to the other. A quiver representation is (semi)stable if the point is GIT (semi)stable for the action of $\mathrm{GL}(m_1) \times \mathrm{GL}(m_2)$. There is a moduli space of (semi)stable Kronecker modules $Kr_N(m_1, m_2)$ of dimension $m_1 m_2 N - m_1^2 - m_2^2 + 1$ provided this number is nonnegative.

By Theorem 7.11, we can associate a Kronecker module W to a general sheaf $U \in M(\xi)$. One can check that the general such Kronecker module is stable, hence gives a well-defined point of the moduli space $Kr_N(m_1, m_2)$, where $N = \hom(E_{-\mu-3}, E_{-\nu})$. We thus obtain a rational map

$$M(\xi) \dashrightarrow Kr_N(m_1, m_2).$$

When $(\xi, \xi_\alpha) \neq 0$, this map has positive dimensional fibers and contains complete curves in its fibers. Consequently, the pullback of the ample generator of $Kr_N(m_1, m_2)$ spans an extremal ray of the effective cone. Finally, this ray can be identified with D_E where E is a general bundle with rank r^+ sufficiently large and divisible, slope μ^+ and discriminant Δ^+. A general member V of this moduli space has a resolution

$$0 \to E_{\nu-3}^{n_1} \to E_{\mu}^{n_2} \to V \to 0$$

and satisfies interpolation with respect to U.

Remark 7.13. In our discussion, we have ignored moduli spaces $M(\xi)$ where the rank of ξ is zero. A very similar but easier theorem holds in that case. The main difference is that when $r(\xi) = 0$, the set of orthogonal invariants form a vertical line L_ξ rather than a parabola. The line L_ξ either contains the exceptional character $(\alpha, \Delta_\alpha) = (\mu^+, \Delta^+)$ or intersects the δ-curve at a point (μ^+, Δ^+). With these modifications, the analogues of Theorems 7.11 and 7.12 hold.

8. Examples

In this section, we work out several explicit examples of the theory described in the previous sections. We will describe the effective and ample cones and the stable base locus decomposition of the effective cone in several examples. For concreteness, we will concentrate on the Hilbert scheme of points.

Example 8.1. The simplest example is the Hilbert scheme $\mathbb{P}^{2[2]}$. Theorems 3.9 and 3.10 specialize to

$$\mathrm{Nef}(\mathbb{P}^{2[2]}) = \left[H - \frac{1}{2}B, H\right], \quad \mathrm{Eff}(\mathbb{P}^{2[2]}) = \left[H - \frac{1}{2}B, B\right].$$

We use the convention $[D_1, D_2]$ to denote the cone spanned by D_1, D_2. If a cone does not contain one of its extremal rays, we write a parenthesis instead of a bracket such as in $(D_1, D_2]$.

- The divisor H defines the Hilbert-Chow morphism and realizes $\mathbb{P}^{2[2]}$ as the blowup of the symmetric product $\mathbb{P}^{2(2)}$ along the diagonal.
- The divisor $H - \frac{1}{2}B$ defines the morphism $\phi_1 : \mathbb{P}^{2[2]} \to \mathbb{P}^{2*}$ sending $Z \in \mathbb{P}^{2[2]}$ to the line that contains Z. The morphism ϕ_1 realizes $\mathbb{P}^{2[2]}$ as a $\mathbb{P}^2$ bundle over $\mathbb{P}^{2*}$. This is the first example of the Kronecker fibration discussed in §7.

Example 8.2. Next, we consider the Hilbert scheme $\mathbb{P}^{2[3]}$. Theorems 3.9 and 3.10 specialize to

$$\mathrm{Nef}(\mathbb{P}^{2[3]}) = \left[2H - \frac{1}{2}B, H\right], \quad \mathrm{Eff}(\mathbb{P}^{2[3]}) = \left[D_{\mathcal{O}_{\mathbb{P}^2}}(1) = H - \frac{1}{2}B, B\right].$$

The stable base locus decomposition of the effective cone decomposes the effective cone into the following three chambers

$$\left[H - \frac{1}{2}B, 2H - \frac{1}{2}B\right) \left[2H - \frac{1}{2}B, H\right] (H, B].$$

- The morphism defined by H is the Hilbert-Chow morphism and the corresponding model is the symmetric product $\mathbb{P}^{2(3)}$. The exceptional locus of the Hilbert-Chow morphism is the divisor B, consequently B is in the base locus for divisor in the cone $(H, B]$. Since H is base point free, B is the only base locus in this chamber. This is a general feature for all $\mathbb{P}^{2[n]}$. The divisor H defines the Hilbert-Chow morphism and in the chamber $(H, B]$ the base locus is divisorial equal to B.
- The morphism defined by $2H - \frac{1}{2}B$ is the map

  $$\phi_2 : \mathbb{P}^{2[3]} \to G(3, H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))), \quad \phi_2(Z) = H^0(\mathbb{P}^2, I_Z(2)).$$

  This map is a divisorial contraction contracting the divisor of collinear points $D_{\mathcal{O}_{\mathbb{P}^2}(1)}$. Consequently, the base locus in the chamber $[H - \frac{1}{2}B, 2H - \frac{1}{2}B)$ is divisorial equal to $D_{\mathcal{O}_{\mathbb{P}^2}(1)}$. The orthogonal parabola to I_Z passes through the exceptional $\mathcal{O}_{\mathbb{P}^2}(1)$. One can interpret the divisor $D_{\mathcal{O}_{\mathbb{P}^2}(1)}$ as the locus of schemes that do not admit the resolution

  $$0 \to \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \to I_Z \to 0.$$

Example 8.3. As a final example of a Hilbert scheme of points on $\mathbb{P}^2$, we consider $\mathbb{P}^{2[4]}$. We have

$$\mathrm{Nef}(\mathbb{P}^{2[4]}) = \left[3H - \frac{1}{2}B, H\right], \quad \mathrm{Eff}(\mathbb{P}^{2[4]}) = \left[D_{T_{\mathbb{P}^2}} = 3H - B, B\right].$$

The effective cone decomposes into the following regions according to the stable base locus:

- In the cone $(H, B]$, the stable base locus is B.
- In the cone $\left[3H - \frac{1}{2}B, H\right]$, the stable base locus is empty. The divisor H defines the Hilbert-Chow morphism. The divisor $3H - \frac{1}{2}B$ defines the map $\phi_3 : \mathbb{P}^{2[4]} \to G(6, 10)$ sending Z to $H^0(\mathbb{P}^2, I_Z(3))$. The morphism ϕ_3 contracts the locus of collinear schemes.
- In the cone $\left[2H - \frac{1}{2}B, 3H - \frac{1}{2}B\right)$, the stable base locus consists of collinear schemes. The divisor $2H - \frac{1}{2}B$ defines the rational map $\phi_2 : \mathbb{P}^{2[4]} \to G(2, 6)$ and shows that $\mathbb{P}^{2[4]}$ is birational to $G(2, 6)$. The indeterminacy locus of this map is precisely the locus of collinear schemes. The map contracts the divisor of schemes that have a collinear subscheme of length 3. This divisor can equivalently be described as the locus of schemes whose resolution is not

  $$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to I_Z \to 0.$$

  These schemes are also precisely the schemes that fail to impose independent conditions on sections of $T_{\mathbb{P}^2}$.
- In the cone $\left[3H - B, 2H - \frac{1}{2}B\right)$, the stable base locus consists of the divisor $D_{T_{\mathbb{P}^2}}$.

As n increases the decomposition of the effective cone into chambers becomes more and more complicated. We refer the reader to [ABCH] for a complete description for $n \leq 9$.

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